PHY481 - Lecture 24: Energy in the magnetic field, Maxwell's term
Griffiths: Chapter 7

Energy stored in inductors

An external voltage source is used to provide the energy required to establish a magnetic field in an inductor. The rate at which work is done by the external source is,

$$P = \frac{dW}{dt} = \frac{dU}{dt} = VI \tag{1}$$

We have shown that the voltage across an inductor is $-LdI/dt$, so we have,

$$\frac{dU}{dt} = LI\frac{dI}{dt} \tag{2}$$

The total energy stored in an inductor is the integral of the work so that,

$$U = \int_0^\infty \frac{dU}{dt}dt = \int_0^i LI\frac{dI}{dt}dt = \frac{1}{2}Li^2 \tag{3}$$

where i is the final current.

Energy stored in magnetic fields

This energy is stored in the magnetic field of the inductor. By considering a solenoid we can find the energy density in the magnetic field, through

$$U = \frac{1}{2}Li^2 = \frac{1}{2}\mu_0 n^2 Ali^2 = \frac{1}{2\mu_0}B^2 Al \tag{4}$$

using $B = \mu_0 ni$, $L = \mu_0 N^2 A/l$ and $Al = volume$, the energy density in the magnetic field is then found to be,

$$u = \frac{1}{2\mu_0}B^2. \tag{5}$$

Energy in terms of current sources

Consider the flux in a single loop,

$$\phi_B = \int \vec{B}\cdot d\vec{a} = \int (\vec{\nabla}\wedge\vec{A}).d\vec{a} = \oint \vec{A}\cdot d\vec{l} = Li \tag{6}$$

To find an expression for the energy in terms of distributed current sources, we write,

$$U = \frac{1}{2}i\int \vec{A}\cdot d\vec{l} \to \frac{1}{2}\int \vec{A}\cdot\vec{i}dl \to \frac{1}{2}\int \vec{A}\cdot\vec{j}d\tau. \tag{7}$$

Where the last expression on the right hand side may be considered to be a postulate. To show that it is the same as the expression in terms of the energy density in the magnetic field, we write,

$$\frac{1}{2}\int \vec{A}\cdot\vec{j}d\tau = \frac{1}{2\mu_0}\int \vec{A}\cdot(\vec{\nabla}\wedge\vec{B})d\tau \tag{8}$$

where we used Ampere's law, $\vec{\nabla}\wedge\vec{B} = \mu_0\vec{j}$. Now use the vector identity,

$$\vec{\nabla}\cdot(\vec{A}\wedge\vec{B}) = \vec{A}\cdot(\vec{\nabla}\wedge\vec{B}) - \vec{B}\cdot(\vec{\nabla}\wedge\vec{A}) \tag{9}$$

so that,

$$\frac{1}{2\mu_0}\int \vec{A}\cdot(\vec{\nabla}\wedge\vec{B})d\tau = -\frac{1}{2\mu_0}\int \vec{\nabla}\cdot(\vec{A}\wedge\vec{B})d\tau + \frac{1}{2\mu_0}\int \vec{B}\cdot(\vec{\nabla}\wedge\vec{A})d\tau = \frac{1}{2\mu_0}\int B^2 d\tau \tag{10}$$

where the term $\int \vec{\nabla}\cdot(\vec{A}\wedge\vec{B})d\tau$ is removed by using the divergence theorem to convert it to a surface integral and then taking the volume to infinity.

The energy to set up a magnetic field may then be expressed in two ways,

$$\frac{1}{2}\int \vec{A}\cdot\vec{j}d\tau = \frac{1}{2\mu_0}\int_{all\ space} B^2 d\tau \tag{11}$$

Recall that the energy to set up an electric charge distribution is

$$\frac{1}{2}\int \rho V d\tau = \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau \tag{12}$$

which look very similar.

The energy stored in magnetic fields can be large and has been proposed as an alternative to batteries, provided superconducting wires are used in order to minimize the resistive losses. Large amounts of energy are stored in the earth's magnetic field and in the magnetic fields of galaxies.

Example: Comparison of the energy stored in the earth's electric and magnetic fields

The energy stored in the electric field of the earth is,

$$U_{electric} \approx Volume * \frac{\epsilon_0}{2}(110V/m)^2 = (4\pi(6400km)^2 2km)\frac{\epsilon_0}{2}(110V/m)^2 \approx 4 * 10^{10} J \tag{13}$$

Is this a lot of energy? One gallon of gasolene has energy content $1.26 \times 10^8 J$, so its relatively small In the above we use the fact that the earth's electric field reduces to about one half of its sea level value at an altitude of $2km$. There are considerable variations from place to place on the earth's surface and this is a rough average value, but good enough for an estimate correct to an order of magnitude or so.

The magnetic field of the earth extends to far greater distances than the earth's electric field, so we have to take into account the decay of the field with distance. The magnetic field at the earth surface varies in magnitude, from place to place, in the range $20\mu T$ to $70\mu T$. Lets take $50\mu T$ as a reasonable estimate. We may then take the magnitude of the earth field with distance to be approximately $50\mu T(R_e/r)^3$, where $R_e = 6400km$ is the radius of the earth. An estimate of the energy stored in the earth's magnetic field is then,

$$U_{magnetic} \approx \frac{1}{2\mu_0}\int_{R_e}^{\infty}(50\mu T)^2 R_e^6 \frac{4\pi r^2}{r^6} dr = \frac{2\pi(50\mu T)^2 R_e^3}{3\mu_0} \approx 10^{17} J, \tag{14}$$

which is much larger than the energy stored in the electric field. A typical power station is a giga watt, and the number of seconds in a year is about 3×10^7 so a power station running for about 3 years could generate the energy stored in the earth's field.

Maxwell term

Maxwell noticed that there is a logical inconsistency in Ampere's law that is easily illustrated in the case of charging capacitor. In that case Ampere's law may be used to calculate the magnetic field near the wire and using a simple closure of the contour we get the usual result $B(r) = \frac{\mu_o i}{2\pi s}$. However we may use any area to calculate the enclosed current, for example a contour that passes between the plates of the capacitor. In that no current is enclosed so Ampere's law says that the magnetic field is zero. Therefore Ampere's law cannot apply to the case of a time varying current. Maxwell resolved this difficulty by adding a new term which includes the effect of the electric field which builds up between the capacitor plates. His idea was to relate this electric field to the current flowing the circuit. From Gauss' law, we have,

$$\frac{d\phi_E}{dt} = \frac{1}{\epsilon_0}\frac{dq}{dt} \tag{15}$$

or the "Maxwell displacement current" is,

$$i_d = \frac{dq}{dt} = \epsilon_0 \frac{d\phi_E}{dt} \tag{16}$$

This relates the current flowing into the capacitor to the electric field between the plates so that Ampere's law is modified to,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0(i + i_d) = \mu_0 i + \mu_0\epsilon_0 \frac{d\phi_E}{dt} \tag{17}$$

This extra term is called the displacement current as it has the same dimensions as the true current in the circuit.

Example 30-7

A parallel plate capacitor is being charged at $i = 1C/s$. If the plates are circular with radius, $R = 0.1m$, and are separated by $d = 1cm$, find the magnetic field as a function of distance from the central axis of the capacitor.

Consider a circular loop or radius r centered on the axis of the capacitor and lying parallel to the plates. From Ampere's law we have,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i + \mu_0 \epsilon_0 \frac{d\phi_E}{dt} \tag{18}$$

The electric flux through the loop is given by,

$$\phi_E = \pi r^2 E \tag{19}$$

so the rate of change of the electric flux is,

$$\frac{d\phi_E}{dt} = \pi r^2 \frac{dE}{dt} = \frac{\pi r^2}{d}\frac{dV}{dt} = \frac{\pi r^2}{Cd}\frac{dQ}{dt} = \frac{\pi r^2}{\epsilon_0 A}\frac{dQ}{dt} \tag{20}$$

Here we have used the relation for a capacitor $Q = CV$ and the relation between electric field and voltage for a parallel plate capacitor $E = V/d$, and the expression for the capacitance of a parallel plate capacitor $C = \epsilon_0 A/d$. Evaluating the path integral for the magnetic field and equating it to the displacement current term, we then have,

$$2\pi r B(r) = \mu_0 \epsilon_0 \frac{\pi r^2}{\epsilon_0 A}\frac{dQ}{dt} \tag{21}$$

or

$$B(r) = \frac{r\mu_0}{2A}\frac{dQ}{dt} = \frac{\mu_0 r}{2\pi R^2}\frac{dQ}{dt} = \frac{\mu_0 i r}{2\pi R^2} \qquad r < R \tag{22}$$

For $r > R$, the magnetic field is $\frac{\mu_0 i}{2\pi r}$.