Solutions to problems for Part 3

Sample Quiz Problems

Quiz Problem 1. Draw the phase diagram of the Ising Ferromagnet in an applied magnetic field. Indicate the critical point. Plot the magnetization as a function of the applied field for three temperatures \( T < T_c, \ T = T_c, \ T > T_c \).

Solution. See lecture notes

 Quiz Problem 2. Plot the behavior of the magnetization of the Ising ferromagnet as a function of the temperature, for three applied field cases: \( h < 0, \ h = 0, \ h > 0 \). Indicate the critical point.

Solution. See lecture notes

 Quiz Problem 3. Write down the definition of the critical exponents \( \alpha, \beta, \gamma, \delta, \eta \) and \( \nu \). What values do these exponents take within mean field theory.

Solution.

\[
C_v \sim t^{-\alpha}; \quad m \sim t^\beta; \quad \chi \sim t^{-\gamma}; \quad m(T_c) \sim h^{1/\delta}; \quad c(r) \sim e^{-r/\xi}/r^{d-2+\eta}
\]

where \( \xi = t^{-\nu} \) and \( t = |T - T_c| \). Within mean field theory \( \alpha = 0, \beta = 1/2, \gamma = 1, \delta = 3, \eta = 0, \nu = 1/2 \)

 Quiz Problem 4. Write down the mean field equation for the Ising ferromagnet in an applied field, on a lattice with co-ordination number \( z \) and exchange constant \( J \). From this equation find the critical exponent \( \delta \) for the Ising ferromagnet within mean field theory.

Solution.

\[
m = \text{Tanh}(\beta J z m + \beta h) \sim (\beta J z m + \beta h) - \frac{1}{3}(\beta J z m + \beta h)^3
\]

at the critical point \( \beta J z = 1 \), so \( m \sim h^{1/3} \) and hence \( \delta = 3 \).

 Quiz Problem 5. Write down the van der Waals equation of state. Draw the \( P,v \) phase diagram of the van der Waals gas and indicate the critical point.

Solution.

\[
P = \frac{k_b T}{v - b} - \frac{a}{v^2}
\]

 Quiz Problem 6. Make plots of the van der Waals equation of state isotherms, for \( T > T_c, \ T < T_c \) and for \( T = T_c \). For the case \( T < T_c \) explain why the non-convex part of the curve cannot occur at equilibrium and the Maxwell construction to obtain a physical \( P,v \) isotherm.

Solution. The Maxwell construction requires that,

\[
P'(v_g - v) = \int_{v_l}^{v_g} Pdv
\]

The is required as the oscillation in the Van der Waals equations occur in a non-convex region of the Helmholtz free energy graph of this model. The non-convex region is not an equilibrium or thermodynamic state as it is possible to lower the free energy by constructing a tie line below the non-convex region which corresponds to a mixture of the two phases at the end points of the tie line.
Quiz Problem 7. Write down the Landau free energy for the Ising and fluid-gas phase transitions. Explain the correspondences between the quantities in the magnetic and classical gas problems.

Solution.

\[ F = a(T - T_c)y^2 + by^4 + cy \]  

For the Ising model \( y = m, c = h \), for the van der Waals gas, \( y = v_g - v_l, c = P \).

Quiz Problem 8. Explain the meaning of second quantization. Discuss the way that it can be used in position space and in the basis of single particle wavefunctions. Write down the commutation relations for Bose and Fermi second quantized creation and annihilation operators.

Solution.

Second quantization is a formulation of quantum mechanics and of quantum field theory that is expressed in terms of creation and annihilation operators. In many body quantum physics creation and annihilation operators create and destroy particles in many body basis sets constructed from single particle wave functions. In the case of Fermions a many body basis function is a determinant, while for Bosons it is a permanent. The commutation relations for Fermions and Bosons are similar, except that for fermions we have anticommutators and for Bosons we have commutators. In many body quantum mechanics we have,

\[ [a_i, a_j^\dagger] = \delta_{ij}; \quad \text{for bosons; and} \quad \{a_i, a_j^\dagger\} = \delta_{ij}; \quad \text{for fermions;} \]  

while for quantum fields, we have,

\[ \{\psi(x), \psi^\dagger(x')\} = \delta(x - x'); \quad \text{for bosons; and} \quad \{\psi(x), \psi^\dagger(x')\} = \delta(x - x'); \quad \text{for fermions;} \]  

These days, new theories are often formulated using creation and annihilation operators rather than the Heisenberg or Schrödinger formulations of quantum theory.

Quiz Problem 9. Write down the Hamiltonian for BCS theory, and the decoupling scheme used to reduce it to a solvable form. Explain the physical reasoning for the decoupling scheme that is chosen.

Solution.

In the s-wave BCS theory a singlet state is assumed, so that,

\[ H_{\text{pair}} - \mu N = \sum_{k\sigma} (\epsilon_{k\sigma} - \mu) a_{k\sigma}^\dagger a_{k\sigma} + \sum_{kl} V_{kl} a_{k\uparrow}^\dagger a_{l\downarrow} - a_{l\uparrow} a_{k\downarrow}, \]  

where \( N = \sum_{k\sigma} n_{k\sigma} \) is the number of electrons in the Fermi sea. We carry out an expansion in the fluctuations,

\[ a_{-l\uparrow} a_{l\uparrow} = b_l + (a_{-l\uparrow} a_{l\uparrow} - b_l); \quad a_{-l\uparrow}^\dagger a_{l\downarrow}^\dagger = b_l^* + (a_{-l\uparrow}^\dagger a_{l\downarrow} - b_l^*) \]  

The mean field Hamiltonian keeps only the leading order term in the fluctuations so that,

\[ H_{\text{MF}} - \mu N = \sum_{k\sigma} (\epsilon_{k\sigma} - \mu) a_{k\sigma}^\dagger a_{k\sigma} + \sum_{kl} V_{kl} (a_{l\uparrow}^\dagger a_{k\downarrow}^\dagger b_l^* - b_k^* a_{l\uparrow} a_{k\downarrow} - b_k^* a_{-l\uparrow} a_{l\uparrow}^\dagger) \]  

This is the Hamiltonian that leads to the BSC solution.
Quiz Problem 10. Consider the inverse Bogoliubov-Valatin transformation,
\[ \gamma_{\vec{k} \sigma} = u_{\vec{k}}^* a_{\vec{k} \sigma} - \sigma v_{\vec{k}} a_{-\vec{k} - \sigma}^\dagger. \]  
(11)
Show that if the operators \( a, a^\dagger \) obey standard fermion anti-commutator relations, then the operators \( \gamma, \gamma^\dagger \) also obey these relations, provided,
\[ |u_{\vec{k}}|^2 + |v_{\vec{k}}|^2 = 1 \]  
(12)
Solution The anticommutator, 
\[ \{ \gamma_{\vec{k} \sigma}, \gamma_{\vec{l} \alpha} \} = \{ u_{\vec{k}}^* a_{\vec{k} \sigma} - \sigma v_{\vec{k}} a_{-\vec{k} - \sigma}^\dagger, u_{\vec{l}}^* a_{\vec{l} \alpha} - \sigma v_{\vec{l}} a_{-\vec{l} - \alpha}^\dagger \}. \]  
(13)
Expanding the anticommutator gives,
\[ \{ \gamma_{\vec{k} \sigma}, \gamma_{\vec{l} \alpha} \} = (u_{\vec{k}}^* a_{\vec{k} \sigma} - \sigma v_{\vec{k}} a_{-\vec{k} - \sigma}^\dagger)(u_{\vec{l}}^* a_{\vec{l} \alpha} - \sigma v_{\vec{l}} a_{-\vec{l} - \alpha}^\dagger) + (u_{\vec{l}}^* a_{\vec{l} \alpha} - \sigma v_{\vec{l}} a_{-\vec{l} - \alpha}^\dagger)(u_{\vec{k}}^* a_{\vec{k} \sigma} - \sigma v_{\vec{k}} a_{-\vec{k} - \sigma}^\dagger) \]  
(14)
which reduces to,
\[ \{ \gamma_{\vec{k} \sigma}, \gamma_{\vec{l} \alpha} \} = u_{\vec{k}}^* u_{\vec{l}}^* \{ a_{\vec{k} \sigma}, a_{\vec{l} \alpha} \} + \sigma \alpha v_{\vec{k}}^\dagger v_{\vec{l}}^\dagger \{ a_{\vec{l} \alpha}^\dagger, a_{\vec{k} \sigma}^\dagger \} - \alpha u_{\vec{k}}^* v_{\vec{l}} \{ a_{\vec{l} \alpha}, a_{\vec{k} \sigma} \} - \sigma v_{\vec{l}}^\dagger u_{\vec{k}} \{ a_{\vec{k} \sigma}^\dagger, a_{\vec{l} \alpha}^\dagger \} \]  
(15)
The first two anticommutators are zero. The second two anticommutators are finite when the conditions \( \delta(k, l)\delta(\sigma, -\alpha) \) hold. However when this condition holds, the two commutators are equal and opposite, so they sum to zero. Taking the adjoint of Eq. (15) shows that \( \{ \gamma_{\vec{k} \sigma}, \gamma_{\vec{l} \alpha} \} = 0 \). Now consider,
\[ \{ \gamma_{\vec{k} \sigma}, \gamma_{\vec{l} \alpha}^\dagger \} = (u_{\vec{k}}^* a_{\vec{k} \sigma} - \sigma v_{\vec{k}} a_{-\vec{k} - \sigma}^\dagger)(u_{\vec{l}} a_{\vec{l} \alpha}^\dagger - \sigma v_{\vec{l}} a_{-\vec{l} - \alpha}^\dagger) + (u_{\vec{l}} a_{\vec{l} \alpha}^\dagger - \sigma v_{\vec{l}} a_{-\vec{l} - \alpha}^\dagger)(u_{\vec{k}}^* a_{\vec{k} \sigma} - \sigma v_{\vec{k}} a_{-\vec{k} - \sigma}^\dagger) \]  
(16)
which reduces to
\[ \{ \gamma_{\vec{k} \sigma}, \gamma_{\vec{l} \alpha}^\dagger \} = u_{\vec{k}}^* u_{\vec{l}}^\dagger \{ a_{\vec{k} \sigma}, a_{\vec{l} \alpha}^\dagger \} + \sigma \alpha v_{\vec{k}}^\dagger v_{\vec{l}} \{ a_{\vec{l} \alpha}^\dagger, a_{\vec{k} \sigma} \} - \alpha u_{\vec{k}}^\dagger v_{\vec{l}} \{ a_{\vec{k} \sigma}, a_{\vec{l} \alpha}^\dagger \} - \sigma v_{\vec{l}} u_{\vec{k}} \{ a_{\vec{k} \sigma}^\dagger, a_{\vec{l} \alpha} \} = 0 \]  
(17)
This anticommutator thus requires Eq. (12) in order for the \( \gamma \) operators to obey Fermion anti-commutator relations.

Quiz Problem 11. Given that the energy of quasiparticle excitations from the BCS ground state have the spectrum,
\[ E = \sqrt{(\epsilon - \epsilon_F)^2 + \Delta^2}, \]  
(19)
where \( \Delta \) is the superconducting gap and \( \epsilon_F \) is the Fermi energy, show that the quasiparticle density of states if given by,
\[ D(E) = \frac{N(\epsilon_F)E}{(E^2 - \Delta^2)^{1/2}} \]  
(20)
Solution We use the relation,
\[ N(\epsilon_F)d\epsilon = D(E)dE; \quad \text{and} \quad dE = \frac{\epsilon - \epsilon_F}{(\epsilon - \epsilon_F)^2 + \Delta^2} \frac{d\epsilon}{(\epsilon - \epsilon_F)^{1/2}} \]  
(21)
to find,
\[ D(E) = \frac{N(\epsilon_F)(\epsilon - \epsilon_F)^2 + \Delta^2)^{1/2}}{\epsilon - \epsilon_F} \]  
(22)
and using,
\[ (\epsilon - \epsilon_F)^2 = E^2 - |\Delta|^2 \]  
(23)
yields,
\[ D(E) = \frac{N(\epsilon_F)E}{(E^2 - |\Delta|^2)^{1/2}} \]  
(24)
Quiz Problem 12. Describe the physical meaning of the superconducting gap, and the way in which BCS theory describes it.

Solution
The superconducting gap is the energy required to generate a quasiparticle excitation from the superconducting ground state. In BCS theory, the quasiparticles behave like non-interacting fermions and the energy required to generate a quasiparticle is at least $2\Delta(T)$.

Quiz Problem 13. (Omit this question) BCS theory works very well even quite near the superconducting transition, despite the fact that it is a mean field theory. Use the Ginzburg criterion to rationalize this result.

Solution

Quiz Problem 14. Given the general solutions to the BCS mean field theory

$$\Delta_{\vec{k}} = -\sum_{\vec{l}} V_{\vec{k}\vec{l}} \frac{\Delta_{\vec{l}}}{2E_{\vec{l}}}, \quad E_{\vec{k}} = ((\epsilon_{\vec{k}} - \mu)^2 + |\Delta_{\vec{k}}|^2)^{1/2}$$  \hspace{1cm} (25)

Describe the assumptions that are made in deducing that,

$$1 = \frac{N(\epsilon_F)V}{2} \int_{\epsilon_F - \hbar\omega_c}^{\epsilon_F + \hbar\omega_c} \frac{d\epsilon}{((\epsilon - \epsilon_F)^2 + |\Delta|^2)^{1/2}} =$$

$$N(\epsilon_F)V \int_0^{\hbar\omega_c/\Delta} \frac{dx}{(1+x^2)^{1/2}} = N(\epsilon_F)V \sinh^{-1}(\frac{\hbar\omega_c}{\Delta})$$ \hspace{1cm} (26)

and hence,

$$\Delta = 2\hbar\omega_c \exp[-\frac{1}{N(\epsilon_F)V}]$$ \hspace{1cm} (27)

Solution
We assume an isotropic gap, and that the attractive coupling between electrons is constant $-V$, over the range $\epsilon_F - \hbar\omega_c < \epsilon < \epsilon_F + \hbar\omega_c$. The density of states is assumed constant with value $N(\epsilon_F)$.

Assigned problems

Assigned Problem 1. Consider the Ising ferromagnet in zero field, in the case where the spin can take three values $S_i = 0, \pm 1$. a) Find equations for the mean field free energy and magnetization. b) Find the critical temperature and the behavior near the critical point. Are the critical exponents ($\beta, \gamma, \alpha, \delta$) the same as for the case $S = \pm 1$? Is the critical point at higher or lower temperature than the spin $\pm 1$ case? c) Is the free energy for the the spin $0, \pm 1$ case higher or lower than the free energy of the $\pm 1$ case? Why?

Solution. The partition function and Helmholtz free energy are,

$$Z = [1 + 2\cosh(\beta Jzm)]^N; \quad F = -k_B TNln[1 + 2\cosh(\beta Jzm)]$$ \hspace{1cm} (28)

The mean field equation is,

$$m = \frac{2\sinh(\beta Jzm)}{1 + 2\cosh(\beta Jzm)} \approx \frac{2}{3}\beta Jzm - \frac{1}{3}(\beta Jzm)^3 + ... \hspace{1cm} (29)$$

The critical point is at $\beta Jz = 3/2$, so the critical temperature is at $k_B T_c = 2Jz/3$ which is lower than that for spin $1/2$ due to the additional entropy of the spin one system. The critical exponents $\beta$ and $\delta$ are clearly the same as for the spin $1/2$ case. The free energy is lower for the spin 1 case due to the higher entropy.
**Assigned Problem 2.** (i) Starting from Eq. (36) of the lecture notes, prove Eq. (39) of the lecture notes. (ii) From Eq. (44) or (45) of the lecture notes prove Eq. (46) in three dimensions.

**Solution.** To fill in the details from Eq. (36) to (38), substitute the definitions of the Fourier transform and use translational invariance, as illustrated in the notes. To prove equation (39), note that the matrix $J_{il}$ is diagonalized by the Fourier transform, and assuming only nearest neighbors, we then have,

$$J(\vec{k}) = e^{ikx} + e^{-ikx} + e^{iky} + e^{-iky} + ...$$

which reduces to Eq. (39). The approximate expression is found by expanding for small $k$.

To show Eq. (46) from Eq. (45), we use the integral,

$$\int_0^\infty \frac{e^{-ax-b/x}}{x^d/2} dx = e^{-2(ab)^{1/2} (\pi b)^{1/2}}.$$  

with $a = 1/\xi^2$, $b = r^2/4$, $d = 3$, we find,

$$C(r) \approx \frac{1}{r} e^{-r/\xi}$$

**Assigned Problem 3.** The Dieterici equation of state for a gas is,

$$P = \frac{k_B T}{v} - \frac{b}{e} - \frac{a}{k_B T v^2}$$

where $v = V/N$. Find the critical point and the values of the exponents $\beta, \delta, \gamma$.

**Solution.** The critical point is found by solving,

$$P_c = -\frac{k_B T_c}{v_c} - \frac{b}{e} - \frac{a}{k_B T_c v_c^2};$$

$$\frac{\partial P}{\partial v} = 0 = -\frac{k_B T_c}{(v_c - b)^2} e^{-a/(k_B T v_c)} + \frac{k_B T_c}{v_c - b} \frac{a}{k_B T v_c^2} e^{-a/(k_B T v_c)}$$

Which simplify to,

$$-\frac{1}{v_c - b} + \frac{a}{k_B T v_c^2} = 0; \quad \text{so} \quad \frac{a}{k_B T_c} = \frac{v_c^2}{v_c - b}$$

The second derivative yields,

$$\frac{\partial^2 P}{\partial v^2} = 0 = 2 \frac{k_B T_c}{(v_c - b)^3} e^{-a/(k_B T v_c)} - 2 \frac{k_B T}{v_c - b} \frac{a}{k_B T v_c^2} e^{-a/(k_B T v_c)}$$

$$-2 \frac{k_B T_c}{v_c - b} \frac{a}{k_B T v_c^3} e^{-a/(k_B T v_c)} + \frac{k_B T_c}{v_c - b} \left( \frac{a}{k_B T v_c^2} \right)^2 e^{-a/(k_B T v_c)}$$

so that,

$$2 \frac{1}{(v_c - b)^2} - 2 \frac{1}{v_c - b} \frac{a}{k_B T v_c^2} - 2 \frac{a}{k_B T v_c^3} + \left( \frac{a}{k_B T v_c^2} \right)^2 = 0$$

and using Eq. (37),

$$-2 \frac{1}{v_c} - \frac{1}{v_c - b} + \left( \frac{1}{v_c - b} \right)^2 = 0; \quad \text{so} \quad v_c = 2b$$
so we find that,

\[ v_c = 2b; \quad k_B T_c = \frac{a}{4b}; \quad P_c = \frac{a}{4e^2 b^2} \]  

(40)

To find the critical exponents we write \( v = v_c + \delta v, T = T_c + \delta T, P = P_c + \delta P \), so that,

\[ P_c + \delta P = \frac{k_B (T_c + \delta T)}{v_c + \delta v - b} e^{-a/[k_B(T_c + \delta T)(v_c + \delta v)]} = \frac{k_B T_c}{b} \frac{(1 + \frac{\delta T}{T_c})}{1 + \frac{\delta v}{v_c}} e^{-a/[k_B(T_c + \delta T)(1 + \delta v/v_c)]} \]  

(41)

This reduces to,

\[ 1 + p = e^2 \frac{1 + t}{1 + 2x} Exp[-\frac{2}{(1 + t)(1 + x)}] \]  

(42)

where \( p = \delta P/P_c, t = \delta T/T_c, x = \delta v/v_c \). To third order this expansion gives,

\[ p = 3t + 2t^2 - \frac{2}{3} t^3 + (-2t - 4t^2)x + 2tx^2 - \frac{2}{3} x^3 \]  

(43)

Taking a derivative with respect to \( v \) at setting \( x = 0 \) leads to,

\[ \kappa_T = (-V(\frac{\partial P}{\partial V}))^{-1} \approx |T - T_c|^{-1} \]  

(44)

so \( \gamma = 1 \). The exponent \( \delta \) is found by setting \( t \) to zero so that \( p \approx x^3 \), so that \( \delta = 3 \). To find \( \beta \) we assume that \( p_l = p_g, x_l = -x_g \), so that,

\[ p_l = 3t + 2t^2 - \frac{2}{3} t^3 + (-2t - 4t^2)x_l + 2tx_l^2 - \frac{2}{3} x_l^3 \]  

(45)

\[ p - g = 3t + 2t^2 - \frac{2}{3} t^3 - (-2t - 4t^2)x_l + 2tx_l^2 + \frac{2}{3} x_l^3 \]  

(46)

Setting these equations to be equal yields,

\[ 2(-2t - 4t^2)x_l = \frac{2}{3} x_l^3, \quad \text{so that} \quad x_l \approx |T - T_c|^{1/2} \]  

(47)

where we dropped the \( t^2 \) term as it is higher order. In the above analysis the signs of the \( t \) and \( x \) are consistent but have to be checked each time.

---

**Assigned Problem 4.** Consider the Landau free energy,

\[ F = \frac{1}{2} am^2 + \frac{1}{4} bm^4 + \frac{1}{6} cm^6 \]  

(48)

where \( c > 0 \) as required for stability. Sketch the possible behaviors for \( a, b \) positive and negative, and show that the system undergoes a first order transition at some value \( a, b, c \). Find the discontinuity in \( m \) at the transition.

**Solution.**

The mean field equation is,

\[ \frac{\delta F}{\delta m} = am + bm^3 + cm^4 = 0; \]  

(49)

Note that in general we do a variation with respect to \( m \), so when we add fluctuations later, we need to use the Euler-Lagrange equation. Here the variation is the same as a partial derivative with respect to \( m \). Solving the mean field equation, we find five solutions.

\[ m = 0, \quad m = \pm m_\pm; \quad m_\pm^2 = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2c} \]  

(50)
Though there are always five solutions, only the real solutions are physical. Analysis of the behavior of the model reduces to identifying the real solutions, and finding which real solution has the lowest free energy. We can understand the nature of the solutions by looking at the second derivative,

$$\frac{\partial^2 F}{\partial m^2} = a + 3bm^2 + 5cm^4,$$  \tag{51}

which enables us to distinguish between maxima and minima. We also use the fact that $F$ is symmetric in $m$ and that at large $m$, because $c$ is positive, $F$ is large and positive for large $|m|$. Finally, without loss of generality, we can divide through by $c$, or equivalently set $c = 1$. Now consider the four cases for $a, b$.

(i) $a > 0, b > 0$. In this case $b^2 - 4ac < b$, so $m^2_\pm$ is always negative. Therefore the solutions $m_\pm$ are always imaginary. The only real solution is $m = 0$, which is a minimum having $F(0) = 0$.

(ii) $a < 0, b > 0$. In this regime, $b^2 + 4|a|c > b^2$, so $m^2_+ > 0$, so that $m_+$ is real. However $-b - (b^2 + 4|a|c)^{1/2}$ remains negative, so $m_-$ remains imaginary. The real solutions are thus a maximum at $m = 0$ and two symmetric minima at $m^2_\pm = \pm \frac{1}{2}(b^2 + 4|a|c)^{1/2} - b$.

There is a phase transition between states (i) and (ii) that occurs at $a = 0$ where two new solutions emerge and the extremum at $m = 0$ changes from a maximum for $a < 0$ to a minimum for $a > 0$. The nature of the transition is found by making a small $|a|$ expansion of the solutions $m_+$, which leads to $m \approx |a|^{1/2}$. This is the Ising/Van der Waals universality class we have studied using mean field theory, where we found $|a| \approx |T - T_c|$.

(iii) $a < 0, b < 0$. In this regime, $b^2 + 4|a|c > b^2$, so $m_+$ is real but $m_-$ remains imaginary. Therefore, as in case (ii), there is a maximum at $m = 0$, and minima at $\pm m_+$.

(iv) $a > 0, b < 0$. In this regime there are several things going on. First, the discriminant $b^2 - 4a$ is negative for $b^2 < 2a$, so in this regime there is only one real solution, a minimum at $m = 0$. For $b^2 > 2a$, there are five real solutions because $|b| \pm (b^2 - 4a)^{1/2} > 0$. Moreover, we know that the solution at $m = 0$ is a minimum, so we know that $\pm m_-$ are maxima, while $\pm m_+$ are minima.

The final issue we have to resolve is the behavior of the minima $m_+$ as a function of $a, b$, in particular we need to know if $F(m_+)$ is greater than or less than $F(0)$. If the lowest free energy state changes it corresponds to a phase transition. We can solve this problem by evaluating $F(m_+)$, or we can find solutions where $F(m_0) = 0$ and then solve $m_0 = m_+$. The later leads to,

$$m_0^2 = \frac{3}{2} \left[ \frac{|b|}{2} \pm \left( \frac{b^2}{4} - \frac{4a}{3} \right)^{1/2} \right]$$  \tag{52}

and setting $m_0^2 = m_+^2$, we find,

$$\frac{3}{2} \left[ \frac{|b|}{2} \pm \left( \frac{b^2}{4} - \frac{4a}{3} \right)^{1/2} \right] = \frac{1}{2} |b| + (b^2 - 4a)^{1/2}$$  \tag{53}

which has solution

$$|b|_* = 4 \left( \frac{a}{3} \right)^{1/2}; \quad \text{with} \quad m_0^2 = m_+^2(|b|_*, a) = 2 \left( \frac{a}{3} \right)^{1/2}$$  \tag{54}

$m_*$ is the magnetization on the phase boundary defined by $|b|_*$. There is then a first order phase transition from magnetization $m_*$ for $|b| < |b|_*$, to magnetization $m = 0$ for $|b| > |b|_*$. The behavior of the magnetization on the phase boundary is $m_* \approx a^{1/4} \approx |T - T_c|^{1/4}$, which is the mean field result for the order parameter near a tricritical point where a line of second order phase transitions meets a line of first order phase transitions.

**Assigned Problem 5.** The BCS pairing Hamiltonian is a simplified model in which only pairs with zero center of mass momentum are included in the analysis. We also assume that the fermion pairing that leads to superconductivity occurs in the singlet channel. The BCS Hamiltonian is then,

$$H_{\text{pair}} - \mu N = \sum_{k\sigma} (\epsilon_k - \mu) a_{k\sigma}^\dagger a_{k\sigma} + \sum_{k\ell} V_{\ell k} a_{k\uparrow}^\dagger a_{\ell\downarrow}^\dagger a_{-\ell\downarrow} a_{-k\uparrow},$$  \tag{55}

where $N = \sum_{k\sigma} n_{k\sigma}$ is the number of electrons in the Fermi sea. Defining,

$$b_k = \langle a_{-k\downarrow} a_{k\uparrow} \rangle, \quad \text{and} \quad b_k^* = \langle a_{k\uparrow}^\dagger a_{-k\downarrow} \rangle.$$

\[ b_k = \langle a_{-k\downarrow} a_{k\uparrow} \rangle, \quad \text{and} \quad b_k^* = \langle a_{k\uparrow}^\dagger a_{-k\downarrow} \rangle. \]  \tag{56}
carry out a leading order expansion in fluctuations, leading to,

\[ H_{MF} - \mu N = \sum_{\vec{k}} (\epsilon_{\vec{k}} - \mu) a_{\vec{k} \uparrow}^\dagger a_{\vec{k} \uparrow} + \sum_{\vec{k}\ell} V_{\vec{k}\ell} (a_{\vec{\ell} \downarrow}^\dagger a_{\vec{k} \downarrow} + b_{\vec{k}}^\ast a_{\vec{k} \uparrow} - b_{\vec{k}}^\ast b_{\vec{k} \downarrow}) \]  

(57)

This is the Hamiltonian that we will solve to find the thermodynamic behavior of superconductors, using an atomistic model.

**Solution.** The mean-field Hamiltonian corresponding to Eq. (38) can be considered as a first order expansion in the fluctuations i.e.

\[ a_{\vec{k} \downarrow} a_{\vec{k} \uparrow} = <\vec{k} \downarrow a_{\vec{k} \uparrow} > + [a_{\vec{k} \downarrow} a_{\vec{k} \uparrow} - <\vec{k} \downarrow a_{\vec{k} \uparrow}>] \]  

(58)

Substitution of this into (38), along with the definitions (39) lead to \( H_{MF} \).

---

**Assigned Problem 6.** Using the Bogoliubov-Valatin transformation (Eq. 107), show that the mean field BCS Hamiltonian (Eq. (106)) reduces to Eq. (110), provided Equations (111) and (112) are true.

**Solution.** With this transformation (ie. plug (6) into (3)), the mean field Hamiltonian looks messy,\n
\[ H_{MF} - \mu N = \sum_{\vec{k}} (\epsilon_{\vec{k}} - \mu) (a_{\vec{k} \uparrow}^\dagger a_{\vec{k} \uparrow} + a_{\vec{k} \downarrow}^\dagger a_{\vec{k} \downarrow}) - \sum_{\vec{k}} (\Delta_{\vec{k}} a_{\vec{k} \uparrow}^\dagger a_{\vec{k} \downarrow} - b_{\vec{k}}^\ast \Delta_{\vec{k}}) = \sum_{\vec{k}} (\epsilon_{\vec{k}} - \mu) \left( [u_{\vec{k} \gamma_{\vec{k} \uparrow}}^\dagger + v_{\vec{k} \gamma_{\vec{k} \downarrow}}] u_{\vec{k} \gamma_{\vec{k} \uparrow}} + v_{\vec{k} \gamma_{\vec{k} \downarrow}} \right) \]  

Expanding this yields,

\[ \sum_{\vec{k}} (\epsilon_{\vec{k}} - \mu) [u_{\vec{k} \gamma_{\vec{k} \uparrow}}^\dagger + v_{\vec{k} \gamma_{\vec{k} \downarrow}}] u_{\vec{k} \gamma_{\vec{k} \uparrow}} + v_{\vec{k} \gamma_{\vec{k} \downarrow}} \gamma_{\vec{k} \uparrow}^\dagger ] \]  

\[ + (\epsilon_{\vec{k}} - \mu) [u_{\vec{k} \gamma_{\vec{k} \uparrow}}^\dagger + v_{\vec{k} \gamma_{\vec{k} \downarrow}}] u_{\vec{k} \gamma_{\vec{k} \uparrow}} + v_{\vec{k} \gamma_{\vec{k} \downarrow}} \gamma_{\vec{k} \uparrow}^\dagger ] \]  

\[ - \Delta_{\vec{k}} [u_{\vec{k} \gamma_{\vec{k} \uparrow}}^\dagger + v_{\vec{k} \gamma_{\vec{k} \downarrow}}] u_{\vec{k} \gamma_{\vec{k} \uparrow}} + v_{\vec{k} \gamma_{\vec{k} \downarrow}} \gamma_{\vec{k} \uparrow}^\dagger ] \]  

We now collect the terms in this expression into three categories: Those which have no operators in them; those which can be reduced to diagonal form i.e. those which are of the form \( \gamma_{\vec{k}} \gamma_{\vec{k}} \) and; those that are off diagonal (e.g. \( \gamma_{\vec{k}} \gamma_{\vec{k}}^\dagger \)). The first stage is to collect together the terms which look to be in these three categories. First the constant term,

\[ H_{MF} - \mu N = \sum_{\vec{k}} b_{\vec{k}}^\ast \Delta_{\vec{k}} \]

The following terms can be converted to diagonal form,

\[ + (\epsilon_{\vec{k}} - \mu) \left( [u_{\vec{k} \gamma_{\vec{k} \uparrow}}^2 + v_{\vec{k} \gamma_{\vec{k} \downarrow}}^2] \gamma_{\vec{k} \uparrow}^\dagger \right) \]  

\[ - \Delta_{\vec{k}} [u_{\vec{k} \gamma_{\vec{k} \uparrow}}^\dagger + v_{\vec{k} \gamma_{\vec{k} \downarrow}}] u_{\vec{k} \gamma_{\vec{k} \uparrow}} + v_{\vec{k} \gamma_{\vec{k} \downarrow}} \gamma_{\vec{k} \uparrow}^\dagger ] \]
Finally the off-diagonal terms are,
\[ + (\epsilon_k - \mu) \left( u_k^* \psi_{\gamma_{k_{\downarrow}} \gamma_{k_{\uparrow}}} + u_k^* \psi_{\gamma_{k_{\uparrow}} \gamma_{k_{\downarrow}}} \right) - \Delta_k \left[ (u_k^* \psi_{\gamma_{k_{\downarrow}} \gamma_{k_{\uparrow}}} + v_k^* \psi_{\gamma_{k_{\uparrow}} \gamma_{k_{\downarrow}}} \right) \]

We have to rearrange the terms categorized as diagonal above, as we need them in the form $\gamma^\dagger \gamma$. We do this using the commutation relation, i.e. $\gamma^\dagger \gamma = 1 - \gamma^\dagger \gamma$. This yields,
\[ H_{MF} - \mu N = \sum_k b_k^* \Delta_k + 2(\epsilon_k - \mu)|\psi_k|^2 - \Delta_k u_k^* u_k - \Delta_k^* u_k^* u_k^* \]

Finally the off-diagonal terms are (as before),
\[ + (\epsilon_k - \mu) \left( u_k^* \psi_{\gamma_{k_{\downarrow}} \gamma_{k_{\uparrow}}} + u_k^* \psi_{\gamma_{k_{\uparrow}} \gamma_{k_{\downarrow}}} \right) - \Delta_k \left[ (u_k^* \psi_{\gamma_{k_{\downarrow}} \gamma_{k_{\uparrow}}} + v_k^* \psi_{\gamma_{k_{\uparrow}} \gamma_{k_{\downarrow}}} \right) \]

We note that $n_{k_{\sigma}} = n_{-k_{\sigma}}$, and that the expectation of other terms that transform into one another through the transformation $\hat{k} \to -\hat{k}$, are equivalent. Collecting terms then leads to Eqs. (110)-(112) of the lecture notes.

---

**Assigned Problem 7.** We define
\[ \psi_k = \frac{g_k}{(1 + |g_k|^2)^{1/2}} \]

show that Eq. (111) reduces to (117).

**Solution** Starting with,
\[ 2(\epsilon_k - \mu)(1 - |\psi_k|^2)^{1/2} \psi_k + \Delta_k \psi_k^2 \]

We have,
\[ |u_k|^2 = 1 - |v_k|^2 = \frac{1}{1 + |g_k|^2}; \quad u_k = \frac{1}{(1 + |g_k|^2)^{1/2}}. \]

Substitution into Eq. () leads to,
\[ 2(\epsilon_k - \mu)|g_k|^2 + \Delta_k |g_k|^2 - \Delta_k^* = 0. \]

**Assigned Problem 8.** Show that $E_k$ as defined in Eq. (118) is in agreement with Eq. (119).

**Solution** We need to show that the definitions,
\[ E_k = (\epsilon_k - \mu)(|u_k|^2 - |v_k|^2) + \Delta_k u_k^* v_k + \Delta_k^* v_k^* u_k, \]

and
\[ E_k^* = [(\epsilon_k - \mu)^2 + |\Delta_k|^2]^{1/2} \]

are the same. To do this we use the results of Problem 9 in Eq. (53) to write,
\[ E_k = (\epsilon_k - \mu)\left[ \frac{[E_k + (\epsilon_k - \mu)]^2}{4E_k^2} - \frac{[E_k - (\epsilon_k - \mu)]^2}{4E_k^2} \right] + \Delta_k \frac{\Delta_k^*}{2E_k} + \frac{\Delta_k^*}{2E_k}. \]

which reduces to Eq. (54) as required.
Assigned Problem 9. Prove the relations Eq. (119-121).

Solution We have,
\[ g_k = \frac{E_k - (\epsilon_k - \mu)}{\Delta_k}; \quad E_k = [(\epsilon_k - \mu)^2 + |\Delta_k|^2]^{1/2} \] (66)
so that,
\[ |\Delta_k|^2 = E_k^2 - (\epsilon_k - \mu)^2; \] (67)
so that,
\[ |g_k|^2 = \frac{E_k^2 - (\epsilon_k - \mu)^2}{E_k^2 - (\epsilon_k - \mu)^2} = \frac{E_k - (\epsilon_k - \mu)}{E_k + (\epsilon_k - \mu)} \] (68)
We then find,
\[ |v_k|^2 = \frac{|g_k|^2}{1 + |g_k|^2} = \frac{E_k - (\epsilon_k - \mu)}{2E_k}; \quad |u_k|^2 = \frac{E_k + (\epsilon_k - \mu)}{2E_k} \] (69)
and hence,
\[ u_k v_k^* = \frac{g_k}{1 + |g_k|^2} = g_k |u_k|^2 = \frac{\Delta_k^*}{2E_k} \] (70)


Solution We start with,
\[ b_k^- = < a_{-k\downarrow} a_{k\uparrow} > = < (u_{k\uparrow} \gamma_{k\downarrow} - v_{k\uparrow} \gamma_{k\downarrow}^\dagger)(u_{k\uparrow}^* \gamma_{k\downarrow} + v_{k\uparrow}^* \gamma_{k\downarrow}^\dagger) > \] (71)
To evaluate this expression at finite temperature, we must evaluate,
\[ < O > = \frac{\text{tr}(\hat{O}e^{-\beta H})}{\text{tr}(e^{-\beta H})} \] (72)
The only terms that survive are the terms that are diagonal, such as \( n_{\epsilon_\sigma} \). We thus find,
\[ b_k^- = < a_{-k\downarrow} a_{k\uparrow} > = u_k v_k^* \gamma_{-k\downarrow} \gamma_{k\uparrow}^\dagger - u_k^* v_k \gamma_{-k\downarrow}^\dagger \gamma_{k\uparrow} > = u_k v_k^* [1 - < n_{\epsilon_\uparrow} > - < n_{-\epsilon_\downarrow} >] \] (73)
Finally we use the fact that \( < n_{\epsilon_\sigma} > = < n_{-\epsilon_\sigma} > \) to find Eq. (135).

Assigned Problem 11. Starting from Eq. (137), prove the relation (145).

Solution In the case of the weak-coupling s-wave model Eq. (137) reduces to,
\[ \frac{1}{N(\epsilon_F)V} = \int_0^{\hbar \omega_c} d\omega \frac{Tanh(\frac{\omega}{2}(\epsilon^2 + \Delta^2)^{1/2})}{(\epsilon^2 + \Delta^2)^{1/2}} \] (74)
As shown in lectures, the critical point is found from the equation,
\[
\frac{1}{N(\epsilon_F) V} = \ln(\beta \hbar \omega_c/2) \tanh(\infty) + a_1; \quad \text{where,} \quad a_1 = - \int_0^\infty \ln(x) \text{sech}^2(x) dx = 0.8188. \tag{75}
\]

The critical temperature is then,
\[
k_B T_c = 1.13 \hbar \omega_c \exp\left[-\frac{1}{N(\epsilon_F) V}\right]. \tag{76}
\]

To find the behavior of the gap near \( T_c \), we carry out a first order Taylor expansion of Eq. (64) using \( \Delta^2 \) as the small quantity. This leads to,
\[
\frac{\tanh(\beta \epsilon/2)^{1/2}}{(\epsilon^2 + |\Delta|^2)^{1/2}} \approx \frac{\tanh(\beta \epsilon/2) (1 + \Delta^2/\epsilon^2)}{\epsilon (1 + \Delta^2/\epsilon^2)} \approx \frac{\beta \epsilon}{\epsilon} + \frac{\beta^2 |\Delta|^2 \text{sech}^2(\beta \epsilon/2)}{4 \epsilon^2} - \frac{|\Delta|^2 \tanh(\beta \epsilon/2)}{2 \epsilon^3}. \tag{77}
\]

Taking the gap to be real, we find,
\[
\frac{1}{N(\epsilon_F) V} = \int_0^{\hbar \omega_c} d\epsilon \frac{\tanh(\beta \epsilon/2)}{\epsilon} + \Delta^2 \int_0^{\hbar \omega_c} d\epsilon \left[ \frac{\beta \epsilon}{4 \epsilon^2} \text{sech}^2(\beta \epsilon/2) - \frac{1}{2} \frac{\tanh(\beta \epsilon/2)}{\epsilon^3} \right]. \tag{78}
\]

This expression may be written in the form,
\[
\frac{1}{N(\epsilon_F) V} = \ln(\frac{1}{2} \beta \hbar \omega_c) + a_1 + \Delta^2 \frac{\beta^2}{8} a_2 \tag{79}
\]
where,
\[
a_1 = 0.8188; \quad \text{and} \quad a_2 = \int_0^\infty dx \left[ \frac{\text{sech}(x)}{x^2} - \frac{\tanh(x)}{x^3} \right] = -0.853 \tag{80}
\]

We expand \( \beta \) about \( \beta_c \), but we only need to consider carry out this expansion in the first term on the right hand side, so that,
\[
\frac{1}{N(\epsilon_F) V} = \ln\left(\frac{\hbar \omega_c}{2k_B T_c (1 - t)}\right) + a_1 + \Delta^2 \frac{\beta^2}{8} a_2; \quad \text{where} \quad t = (T_c - T)/T_c. \tag{81}
\]

We then find that,
\[
\ln(1 - t) \approx -t = -\Delta^2 \frac{\beta^2}{8} |a_2|; \quad \text{so that} \quad \frac{\Delta(T)}{k_B T_c} \approx \left(\frac{8}{|a_2|}\right)^{1/2} t^{1/2} \tag{82}
\]
so that,
\[
\frac{\Delta(T)}{k_B T_c} \approx 3.06(1 - T/T_c)^{1/2}; \quad T \to T_c. \tag{83}
\]

This is correct near \( T_c \). Also note that Eq. (127) and (141) imply that,
\[
\frac{2\Delta(0)}{k_B T_c} \approx \frac{4}{1.13} \approx 3.52, \quad \text{for} \quad T \to T_c. \tag{84}
\]

**Assigned Problem 12.** Consider a ferromagnetic nearest neighbor, spin 1/2, square lattice Ising model where the interactions along the x-axis, \( J_x \), are different than those along the y-axis, \( J_y \). Extend the low and high temperature expansions Eq. (150) and Eq. (152) to this case. Does duality still hold? From your expansions, find the internal energy and the specific heat.

**Solution**
\[
Z = \sum_{\{S_i = \pm 1\}} \prod_{<ij>} e^{K_{ij} S_i S_j} = (\cosh(K_x))^N(\cosh(K_y))^N \sum_{\{S_i = \pm 1\}} \prod_{<ij>} (1 + t_{ij} S_i S_j). \tag{85}
\]
where the first virial coefficient $t_{ij} = t_x$ for horizontal bonds and $t_{ij} = t_y$ for vertical bonds, where $t_x = \tanh(K_x), t_y = \tanh(K_y)$. The diagrams used are similar, but now we have to treat subclasses with different numbers of horizontal and vertical bonds, so that,

$$-\beta F = \ln(Z) = \frac{zN}{2} \ln(\cosh(K)) + N\ln(2) + \ln[1 + Nt^4 + 2Nt^6 + N(N + \frac{9}{2})t^8 + 0(t^{10})]$$  \hspace{1cm} (86)$$

becomes

$$-\beta F = N[\ln(2) + \ln(\cosh(K_x)) + \ln(\cosh(K_y))] + t_x^2t_y^2 + t_x^2t_y^2(t_x^2 + t_y^2) + \frac{5}{2}(t_x^4t_y^4) + t_x^2t_y^6 + t_x^6t_y^2 + ...]$$ \hspace{1cm} (87)$$

Similarly, the extension of the low temperature expansion,

$$Z = \sum_{\{S_i = \pm 1\} <ij>} e^{KS_iS_j} = e^{KzN[1 + Ns^4 + 2Ns^6 + N(N + \frac{9}{2})s^8 + O(s^{10})]}$$  \hspace{1cm} (88)$$

to the anisotropic case leads to,

$$-\beta F = N[K_x + K_y + s_x^2s_y^2 + s_x^2s_y^2(s_x^2 + s_y^2) + \frac{5}{2}s_x^4s_y^4 + s_x^2s_y^6 + s_x^6s_y^2]$$ \hspace{1cm} (89)$$

where $s_x = e^{-2K_x}, s_y = e^{-2K_y}$. Duality holds for both $x$ and $y$ directions.

**Assigned Problem 13.** Find the second virial coefficient for six cases: (i) the classical hard sphere gas; (ii) Non-interacting Fermions; (iii) Non-interacting Bosons; (iv) The van der Waals gas.

**Solution**

$$\frac{Pv}{k_BT} = \sum_{l=1}^{\infty} a_l(T)(\frac{\lambda^3}{v})^{l-1}$$ \hspace{1cm} (90)$$

where the first virial coefficient $a_1(T) = 1$, and the second virial coefficient is $a_2(T)$. The virial expansion is most often carried out in the grand canonical ensemble, where we may write,

$$\frac{P}{k_BT} = \frac{1}{\lambda^3} \sum_{l=1}^{\infty} b_l z^l; \hspace{0.5cm} \frac{N}{V} = \frac{1}{\lambda^3} \sum_{l=1}^{\infty} l b_l z^l$$ \hspace{1cm} (91)$$

so that,

$$\frac{Pv}{k_BT} = \sum_{l=1}^{\infty} b_l z^l / \sum_{l=1}^{\infty} l b_l z^l = \sum_{l=1}^{\infty} a_l(T)(\frac{\lambda^3}{v})^{l-1}$$ \hspace{1cm} (92)$$

which gives relations between the quantities $a_l(T)$ and $b_l(T)$. For the second virial coefficient the relationship is $a_2(T) = -b_2(T)$. We have already calculated the coefficients $b_l$ for the ideal Bose and Fermi gases, with the results,

$$b_l = (-1)^{l+1} \frac{1}{l^{5/2}}; \hspace{0.5cm} (\text{Ideal Fermi}) \hspace{0.5cm} b_l = \frac{1}{l^{5/2}}; \hspace{0.5cm} (\text{Ideal Bose})$$ \hspace{1cm} (93)$$

so we have,

$$\frac{Pv}{k_BT} = 1 - \frac{1}{2^{5/2}}(\frac{\lambda^3}{v}) + ...; \hspace{0.5cm} (\text{Ideal Bose})$$ \hspace{1cm} (94)$$

and

$$\frac{Pv}{k_BT} = 1 + \frac{1}{2^{5/2}}(\frac{\lambda^3}{v}) + ...; \hspace{0.5cm} (\text{Ideal Fermi})$$ \hspace{1cm} (95)$$

The van der Waals equation of state may be expanded to find the second virial coefficient,

$$P = k_BT \frac{v-b}{v} \frac{a^2}{v}, \hspace{0.5cm} \text{so that} \hspace{0.5cm} \frac{Pv}{k_BT} = \frac{1}{(1-b/v)} - \frac{a}{k_BT} \approx 1 + \frac{b-a/k_BT}{v} + ...$$ \hspace{1cm} (96)$$
For the classical interacting gas, $b_2$ is given by,

$$b_2 = \frac{1}{2V\lambda^3} \int d_1^3 d^3 r_2 \left[ e^{-\beta u(\vec{r}_1 - \vec{r}_2)} - 1 \right] = \frac{1}{2\lambda^3} \int d^3 r \left[ e^{-\beta u(\vec{r})} - 1 \right]$$  \hspace{1cm} (97)

For the hard sphere problem, with a hard core radius of $R$, we then find that $b_2 = -2\pi R^3/(3\lambda^3)$, so the virial expansion for this case is,

$$\frac{P_v}{k_BT} = 1 + \frac{2\pi R^3}{3v} + ....$$  \hspace{1cm} (98)

The behavior of the second virial coefficient as a function of temperature can be used to deduce the interaction potential, and the importance of quantum effects as they have different temperature dependences.

PHY831 - Quiz 5, Friday November 11, 2011
Answer all questions. Time for quiz - 25 minutes

Name:

1. (i) Plot the behavior of the magnetization of the Ising ferromagnet as a function of the temperature, for three applied field cases: $h < 0$, $h = 0$, $h > 0$. Indicate the critical point.
(ii) Write down the definition of the critical exponents $\alpha$, $\beta$, $\gamma$, $\delta$, $\eta$ and $\nu$ for the Ising ferromagnet critical point. What values do these exponents take within mean field theory?
(iii) Write down (or derive) the mean field equation for the spin 1/2 Ising ferromagnet in an applied field, on a lattice with co-ordination number $z$ and exchange constant $J$. From this equation find the critical exponent $\delta$ for the Ising ferromagnet within mean field theory.

2. (i) Write down (or derive) the van der Waals equation of state. Make plots of the van der Waals equation of state isotherms, for $T > T_c$, $T < T_c$ and for $T = T_c$.
(ii) For the case $T < T_c$, explain why the non-convex part of the curve cannot occur at equilibrium and show the Maxwell construction to obtain a physical $P, v$ isotherm. Write the mathematical statement of the Maxwell construction.
(iii) Write down the definition of the critical exponents $\alpha$, $\beta$, $\gamma$, $\delta$, $\eta$ and $\nu$ for the liquid-gas critical point. What values do these exponents take within Van der Waals theory.

3. Write down the Hamiltonian for BCS theory, and the decoupling scheme used to reduce it to a solvable form. Explain the physical reasoning for the decoupling scheme that is chosen.

PHY831 - Quiz 6, Friday November 11, 2011
Answer all questions. Time for quiz - 25 minutes

Name:

1. (i) Describe the physical meaning of the superconducting gap, and the way in which BCS theory describes it.
(ii) Given that the energy of quasiparticle excitations from the BCS ground state have the spectrum,

$$E = \left[ (\epsilon - \epsilon_F)^2 + |\Delta|^2 \right]^{1/2},$$  \hspace{1cm} (99)

where $\Delta$ is the superconducting gap and $\epsilon_F$ is the Fermi energy, show that the quasiparticle density of states is given by,

$$D(E) = \frac{N(\epsilon_F)E}{(E^2 - |\Delta|^2)^{1/2}}$$  \hspace{1cm} (100)

Draw this density of states and compare it to the density of states of the system in the absence of the pairing term in the Hamiltonian.
2. (i) Explain the importance of “linked-cluster” theorems in the perturbation theory of many particle systems.
(ii) Draw the low temperature series expansion diagrams to order $s^8$ (where $s = \text{Exp}[−2\beta J]$) for the square lattice, nearest neighbor, spin half Ising ferromagnet partition function. What is the degeneracy of each of these diagrams? Use these terms to write down an expansion for the Helmholtz free energy and give a physical reason why only the terms of order $N$ are kept.

3. (i) Write down the mathematical form of the virial expansion for many particle systems and explain why it is important. What physical properties can be extracted from the second virial coefficient?
(ii) Given that,

$$b_l = \frac{1}{l! \lambda^{l-3} V} \text{ (sum over all } l \text{ - connected - cluster integrals)}$$

find the second virial coefficient for the classical gas with hard core repulsive interactions, where the hard core radius is $R$. 