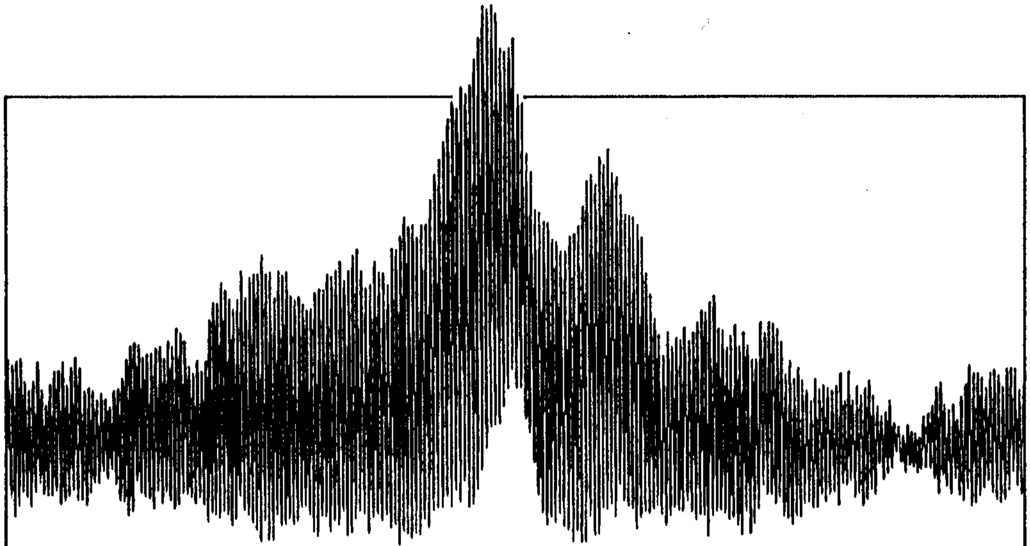

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Chapter 2

Fluctuations in Nonlinear Systems Driven by Colored Noise

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Abstract

We consider the spectral density of the fluctuations as well as rare large fluctuations in nonlinear systems driven by colored Gaussian noise. Special emphasis is placed on the review of recent results on the application of the method of optimal paths to the analysis of large fluctuations. We formulate the variational problems for the optimal paths along which the system moves with an overwhelming probability in the course of a fluctuation that brings the system to a given point in the phase space, and also in the course of the escape from a metastable state. The formulation relies on knowledge of the shape of the power spectrum of the noise, which can usually be determined experimentally. The solutions of the variational equations are considered for various shapes of the power spectrum, including the case of a spectrum with a sharp peak at finite frequency (quasi-monochromatic noise) where qualitative features of large fluctuations related to the noise color are distinctly pronounced (e.g., the occurrence of multiple crossings of a saddle point by the *optimal path* without a transition to another stable state). We analyze the problem of the probability density for the system to pass a given point at a given time before arrival at another point in the course of a large fluctuation (the pre-history problem). The maximum of this probability density lies on the optimal path. The results of recent analog simulations of large fluctuations in systems driven by Gaussian noise, including ones where the optimal paths have been visualized via the analysis of the pre-history, are discussed.

2.1. Introduction

The understanding of the pattern of fluctuations in dynamical systems driven by noise poses one of the fundamental problems of physical kinetics. The problem was formulated originally by Einstein [1] and Smoluchowski [2] in the description of the Brownian motion of a macroparticle. It is of immediate current interest in the context of a vast array of physical phenomena, starting with transport phenomena in solids (for instance, the kinetics of electrons interacting with phonons and/or impurities) [3], [4], and including kinetics of laser modes [5], [6] and kinetics of Josephson junctions [7]. The problem of noise-induced fluctuations is also immediately related to physical measurements: a physical instrument is a dynamical system driven by fluctuations from various sources, including the very quantity being measured.

The features of the fluctuations in a system depend on the character and intensity of the driving noise $f(t)$ and the way in which it couples to the system. In general, $f(t)$ may depend on the state of the system, i.e., the properties of the noise may be different in different states. Most noise of physical relevance can be described fully by its correlation functions $\langle f(t)\rangle$, $\langle f(t_1)f(t_2)\rangle, \ldots, \langle f(t_1)\cdots f(t_n)\rangle, \ldots$, where the brackets $\langle\cdots\rangle$ represent an average over an ensemble of statistically equivalent realizations of the noise. In many cases the correlation functions are mutually interrelated. A common situation arises when the noise driving a system originates from its coupling to a macroscopic system of N dynamical degrees of freedom, with $N \gg 1$ (e.g., a thermal bath). Such noise is typically a superposition of a large number of "elementary" fields or forces,

$$f(t) = \sum_{i=1}^{N} f_i(t). \tag{2.1}$$

In the simplest cases the $f_i(t)$ with different i refer to different elementary excitations in the bath and are mutually uncorrelated. If this is the case, then under some conditions (e.g., if the $f_i(t)$ are each of order $N^{-1/2}$), in the limit $N \to \infty$, $f(t)$ is Gaussian according to the central limit theorem of probability theory [8], [9]. In this case all the correlation functions can be expressed in terms of the two lowest-order ones. We shall deal only with stationary noise, i.e., noise whose statistical distribution does not change with time. The only characteristic of a stationary zero-mean ($\langle f(t)\rangle = 0$) Gaussian noise is the time correlation function $\phi(t)$ or, equivalently, its Fourier transform $\Phi(\omega)$,

$$\phi(t) = \langle f(t)f(0)\rangle, \qquad \Phi(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\phi(t). \tag{2.2}$$

All odd-order correlators vanish, while all the even-order ones are expressed in terms of $\phi(t)$. For example, the fourth-order correlator is

$$\begin{aligned}\langle f(t_1)f(t_2)f(t_3)f(t_4)\rangle &= \phi(t_1 - t_2)\phi(t_3 - t_4)\\ &\quad + \phi(t_1 - t_3)\phi(t_2 - t_4) + \phi(t_1 - t_4)\phi(t_2 - t_3).\end{aligned} \tag{2.3}$$

(It is easy to understand (2.3) if $f(t)$ is equal to the sum of a large number N of weak uncorrelated forces $f_i(t) \propto N^{-1/2}$ by noting that the omission of N terms in the sums with more than two coinciding i (or, for that matter, of any N terms) introduces only small errors of $O(N^{-1})$ in the fourth-order correlator.)

The function $\Phi(\omega)$ in (2.2) is called the power spectrum of the noise $f(t)$—it is precisely $\Phi(\omega)$ that is often measured to characterize the noise. The measurements usually assume that the noise is ergodic, i.e., that an ensemble average is equivalent to a time average. If this assumption is valid (as is usually the case in physical systems), then according to the Wiener–Khintchine theorem [5], [6]

$$\Phi(\omega) = \lim_{t_o \to \infty} \frac{1}{2t_o} \left| \int_{-t_o}^{t_o} dt\, f(t) \exp(i\omega t) \right|^2, \tag{2.4}$$

and measurement of $\Phi(\omega)$ simply involves recording $f(t)$ over a sufficiently long time $2t_o$ and calculating a Fourier transform.

Since $\Phi(\omega)$ is a physical observable it is advantageous to express the characteristics of a noise-driven system in terms of it. The shape of the power spectrum depends on the source of the noise. For example, if the noise results from coupling to a thermal bath then $\Phi(\omega)$ is determined by the density of states of the elementary excitations of the bath, the coupling constants between the bath and the system of interest, and the temperature. The shape of $\Phi(\omega)$ is used to differentiate, very roughly, between two types of noise: white and colored. White noise is characterized by a totally flat spectrum $\Phi(\omega) = \text{const.}$, in analogy with white radiation where all spectral components are of the same intensity. Noise whose spectrum deviates from a constant value is called "colored" (an additional reason for using optical terms in the context of noise is that "normal" incoherent radiation is itself noisy; the electric and magnetic fields fluctuate about their zero-average values).

Strictly speaking, any physical noise is colored: $\Phi(\omega)$ must necessarily vanish as $|\omega| \to \infty$, since otherwise the time correlation function $\phi(t)$ would diverge as $t \to 0$, i.e., the mean-square value of the noise would be infinite. However, if all the characteristic eigenfrequencies and reciprocal relaxation times of a noise-driven system are small compared to the frequencies over which $\Phi(\omega)$ changes, the effects of the color of the noise are expected to be minor and the noise can be assumed to be white. A macroscopic Brownian particle in a liquid is a typical example where the latter approximation holds; the characteristic reciprocal duration of the collisions with solvent molecules that give rise to the noisy forces acting on the Brownian particle is of order 10^{13} sec^{-1} and exceeds by many orders of magnitude the typical frequencies of mechanical motion of the particle.

Fluctuations in systems driven by white noise have been investigated for nearly 90 years (see [1], [2], [6], [9]–[15], and references therein), and the principal aspects of the problem are generally fairly well understood (although there are still problems to be solved, both numerical and qualitative, as discussed subsequently). A "massive attack" on the effects of the color of noise started only about a decade ago. Although several topical reviews on the subject have already appeared [15]–[17], a number of important new results have been obtained since. Some of these results and their underlying concepts are outlined herein. We primarily consider the effects of weak noise, i.e., the situation where the characteristic noise intensity D given by the maximum value of the power spectrum of the noise,

$$D = \Phi_{\max}(\omega), \tag{2.5}$$

is small (in all states of the system if the noise is state-dependent), so that the root-mean-square fluctuational deviations of a noise-driven system from its attractor (or attractors, if several of them coexist) are small compared to the distances between the attractors and to other characteristic length scales of the phase space of the system. Our aim is to demonstrate mainly the qualitative features of the pattern of fluctuations; in this spirit we shall avoid rigorous mathematical proofs and limit ourselves to physical ideas and to "physical rigor." We note that in many papers the terminology "colored noise" is reserved for exponentially correlated noise,

$$\phi(t) = \frac{D}{2t_c} e^{-|t|/t_c}. \tag{2.6}$$

The analysis below is *not* limited to this particular type of colored noise.

A function that characterizes much of the behavior of a fluctuating system is the statistical distribution, i.e., the stationary probability density (the probability density achieved as $t \to \infty$). The shape of this distribution depends substantially on whether dissipation and fluctuations in the system are both due to its coupling to a thermal bath (i.e., whether the noise is of thermal origin), or whether the driving noise is of nonthermal origin. In the first case there is a relation between the noise driving the system and the dissipative forces that extract energy from the system (cf. §2.2). As a consequence, the shape of the stationary distribution of the system (which in this case is an equilibrium distribution in the thermodynamic sense) for sufficiently weak coupling is Gibbsian regardless of the shape of the power spectrum $\Phi(\omega)$. If the power spectrum is not flat, i.e., if the noise is colored, the relaxation *toward* the equilibrium state occurs via an equation of motion that is not time-local (contains explicit memory terms), that is, the relaxation is "retarded." The fluctuation-dissipation relation can sometimes be described phenomenologically, as in the case of Brownian motion [1] (cf. [18] and [19] and references therein for a discussion of fluctuation-dissipation relations in systems driven by colored noise). In some cases the noise and dissipation can be calculated from a microscopic model of the dynamical system of interest, a heat bath, and their coupling. The latter approach was first applied by Bogoliubov [20] to the problem of a linear oscillator coupled to a phonon bath; the corresponding quantum analysis of the dynamics of a linear oscillator coupled to a heat bath was first given by Schwinger [21] (see also Senitzky [22]).

If the noise driving a system is of nonthermal origin, the statistical distribution of the system in the stationary state (if indeed one exists) is non-Gibbsian. However, for sufficiently weak noise and nonbifurcational parameter values, the distribution still has a maximum (maxima) at the attractor(s) and is Gaussian near the maximum (maxima), just as in the case of thermal equilibrium. The dependence of the parameters of the Gaussian distribution(s) on the shape of the power spectrum of the driving noise is discussed briefly in §2.2. The spectral densities of the fluctuations of systems driven by thermal noise and by nonthermal colored noise are also considered there.

It is not only important to understand the behavior of a fluctuating system near its most probable states; the tails of the distribution are also of great interest for various experimental measures. The tails describe the distribution for states of the system that are reached only rarely and come about mainly from occasional large "outbursts" of noise that "push" the system far away from the small-fluctuation region in phase space. In §2.3 we present an idea for a systematic approach to the problem of occasional large fluctuations in systems driven by Gaussian noise with a power spectrum of arbitrary shape. The approach is based on the concept of the optimal path, i.e., on the physical assumption (proved a posteriori) that the paths along which the system moves to a given point in the course of fluctuational "outbursts" are concentrated around a particular most probable path, called the optimal path.

In §2.4 the application of the method of the optimal path is illustrated via an example of a system driven by "strongly colored" (quasi-monochromatic) noise, i.e., noise with a power spectrum that exhibits a narrow peak at a finite frequency. The pattern of fluctuations caused by such a noise differs drastically from that of a white-noise-driven system.

In §2.5 we further explore the idea of optimal paths by investigating the statistical distribution of the paths along which a system arrives at a given point. The corresponding pre-history problem is formulated, and recent experimental data on the first direct observation of optimal paths is discussed.

A qualitative feature of the kinetics of bistable (and multistable) systems is the onset of noise-induced transitions (noise-induced switching) from one (meta)stable state to another. Such transitions occur as a result of large fluctuations, and they can be analyzed within the scope of the method of the optimal path. The analysis and its application to systems driven by noise with power spectra of various shapes, including exponentially correlated and quasi-monochromatic noise, are considered in §2.6. Some further perspectives for future research in the area of large fluctuations in systems driven by weak noise are outlined in §2.7.

2.2. Spectral density of fluctuations and statistical distribution near a stable state

A ubiquitous (but by no means all-inclusive) description of the dynamics of a system driven by nonthermal noise is a set of coupled stochastic differential equations (frequently called Langevin equations) of the form

$$(2.7)\quad \dot{x}_n = -g_n(\boldsymbol{x}) + f_n(t), \quad n = 1, 2, \ldots, L, \qquad \boldsymbol{x} = (x_1, x_2, \ldots, x_L).$$

Here the $g_n(\boldsymbol{x})$ are functions of the state $\boldsymbol{x}$ of the system at time t. The $f_n(t)$ are the components of a zero-mean ($\langle f_n(t)\rangle = 0$) Gaussian noise, with correlation functions and spectral densities

$$(2.8)\qquad \phi_{nm}(t) = \langle f_n(t)\, f_m(0)\rangle, \qquad \Phi_{nm}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t}\phi_{nm}(t).$$

Thus, the components $f_n(t)$ for different n may be interrelated, as embodied in those correlators $\phi_{nm}(t)$ that do not vanish. The $f_n(t)$ in (2.7) are assumed to be independent of the state $\boldsymbol{x}$ of the system, i.e., the noise is "additive." In general the noise is not of thermal origin, and in that case the dissipative contributions to $g_n(\boldsymbol{x})$ are not related to the noise $f_n(t)$. Systems driven by noise of thermal origin are discussed in §2.2.3.

If the noise is sufficiently weak, the system experiences mostly small fluctuations about its stable state(s) (attractor(s)). We shall only consider systems with attractors of the simplest form, namely, fixed points in phase space (foci or nodes). The generalization to limit cycle attractors (tori) is straightforward, but the case of more complicated attractors associated with dynamical chaos (see Thompson and Stewart [23] and references therein) and the interplay of noise-induced fluctuations and dynamical chaos lie outside the scope of the present chapter.

2.2.1. Spectral densities of fluctuations in nonthermal systems. Let us begin by considering the spectral densities of the fluctuations of the variables $x_1, x_2, \ldots, x_L$,

$$\widetilde{Q}_{nm}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} Q_{nm}(t), \tag{2.9}$$

with

$$Q_{nm}(t) = \left\langle \left(x_n(t) - x_n^{st}\right)\left(x_m(0) - x_m^{st}\right)\right\rangle . \tag{2.10}$$

These experimentally accessible spectral densities constitute a useful way to characterize the dynamical behavior of the fluctuating system in the stationary state. Here $\boldsymbol{x}^{st}$ is the stable state of the noiseless system, that is, the stationary solution of (2.7) in the absence of the noise:

$$g_n(\boldsymbol{x}^{st}) = 0. \tag{2.11}$$

Simple expressions for the $\widetilde{Q}_{nm}(\omega)$ can be obtained by noting that, according to linear response theory, the response of the system to weak forces, including random ones (i.e., noise of low intensity), is given in terms of linear susceptibilities $\chi_{nm}(t)$ [24]. Specifically, the noise-induced change $\delta x_n(t)$ in x_n is of the form

$$\delta x_n(t) = \sum_m \int_{-\infty}^{t} d\tau\, \chi_{nm}(t-\tau) f_m(\tau). \tag{2.12}$$

If one identifies $\delta x_n(t)$ with the difference $x_n(t) - x_n^{st}$ that appears in (2.10), one finds from (2.9), (2.10), and (2.12) that the spectral densities $\widetilde{Q}_{nm}(\omega)$ of the fluctuations can be expressed in terms of the power spectra $\Phi_{nm}(\omega)$ of the components of the noise as

$$\widetilde{Q}_{nm}(\omega) = \sum_{n'm'} G_{nm}^{n'm'}(\omega) \Phi_{n'm'}(\omega), \tag{2.13}$$

where

$$G_{nm}^{n'm'}(\omega) = \widetilde{\chi}_{nn'}(\omega)\widetilde{\chi}^*_{mm'}(\omega) \tag{2.14}$$

and $\widetilde{\chi}_{nm}(\omega)$ is the one-sided Fourier transform of the linear susceptibility,

$$\widetilde{\chi}_{nm}(\omega) = \int_0^\infty dt\, e^{i\omega t}\, \chi_{nm}(t). \tag{2.15}$$

Care should be taken when evaluating the spectral densities of the fluctuations from the formulas (2.9) and (2.10) since (2.7) is time-irreversible and therefore fluctuations and initial values of the x_n are eventually "forgotten" when time goes forward but not when time goes backward. In (2.9) and (2.10) it has been assumed that the system was "prepared" at $t = -\infty$.

The susceptibilities $\widetilde{\chi}_{nm}(\omega)$ can be calculated by diagonalizing the matrix $\boldsymbol{g}$ whose elements are $g_{nm} = (\partial g_n/\partial x_m)_{\boldsymbol{x}=\boldsymbol{x}^{st}}$. In terms of the unitary matrix $\boldsymbol{\Lambda}$ that performs the diagonalizing transformation,

$$\sum_{n'm'} \Lambda^{-1}_{nn'} g_{n'm'} \Lambda_{m'm} = \alpha_n \delta_{nm}, \qquad \det\boldsymbol{\Lambda} = 1, \tag{2.16}$$

we can write

$$\widetilde{\chi}_{nm}(\omega) = \sum_{n'} \Lambda_{nn'}\Lambda^{-1}_{n'm}(\alpha_{n'} - i\omega)^{-1}. \tag{2.17}$$

Because of the assumed stability of the state $\boldsymbol{x}^{st}$, the real parts of the eigenvalues α_n of the matrix $\boldsymbol{g}$ are positive.

Equations (2.13)–(2.17) describe in explicit form the dependence of the spectral densities of the fluctuations of a noise-driven system on the shape of the power spectrum of the noise: $\widetilde{Q}_{nm}(\omega)$ is simply the sum of the products of the spectral components $\Phi_{n'm'}(\omega)$ of the noise and the Green functions $G_{nm}^{n'm'}(\omega)$. The Green function $G_{nm}^{n'm'}(\omega)$ expresses the spectral density of fluctuations of the variables x_n, x_m if the driving noise is white with correlators $\langle f_k(t) f_l(t')\rangle = D\delta(t-t')\delta_{kn'}\delta_{lm'}$. For a one-dimensional system ($L = 1$ in (2.7)), the relations (2.13) and (2.14) reduce to

$$\widetilde{Q}(\omega) \equiv \widetilde{Q}_{11}(\omega) = G(\omega)\Phi(\omega), \tag{2.18}$$

where

$$\Phi(\omega) \equiv \Phi_{11}(\omega), \tag{2.19}$$

and

$$G(\omega) \equiv G_{11}^{11}(\omega) = (\alpha^2 + \omega^2)^{-1}, \qquad \alpha \equiv \alpha_1 = g'_1(x_1^{st}). \tag{2.20}$$

The simple and instructive expression (2.18) also holds for the spectral density of the fluctuations of a particle with coordinate q and momentum $p = \dot{q}$ driven by colored noise and described by the equation of motion

$$\ddot{q} + 2\Gamma\dot{q} + U'(q) = f(t). \tag{2.21}$$

This equation of motion can be written in the form (2.7) with $x_1 = q$, $g_1 = p$, $x_2 = p$, and $g_2 = -2\Gamma p - U'(q)$. Note that only x_2 is directly driven by the colored noise; the equation for x_1 contains no explicit noise term. The spectral density of the fluctuations of the coordinate,

$$\widetilde{Q}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \langle (q(t) - q^{st})\,(q(0) - q^{st}) \rangle, \tag{2.22}$$

is given by (2.18) with

$$G(\omega) = [(\omega^2 - \omega_o^2)^2 + 4\Gamma^2\omega^2]^{-1}, \qquad \omega_o = [U''(q^{st})]^{1/2}. \tag{2.23}$$

Here q_{st} is the position of the particle at the minimum of the potential $U(q)$, i.e., the solution of $U'(q^{st}) = 0$.

The relations (2.13) between the spectral densities of the fluctuations of the system and the noise, and, in particular, the special case (2.18), show that the system "filters" the noise, and that the spectrum of a system driven by colored noise reflects features of the spectrum of the noise as well as characteristics of the system itself. This makes it possible to use such systems in a judicious way to investigate the power spectrum of an unknown noise. A particularly useful system for this purpose is the underdamped oscillator described by (2.21) with a tunable frequency ω_o and with a damping Γ (the "bandwidth" of the function $G(\omega)$) that is much smaller than ω_o and than the characteristic bandwidth of the power spectrum $\Phi(\omega)$ under investigation. In this case the spectral density $\widetilde{Q}(\omega)$ of the fluctuations of the oscillator contains a narrow peak near ω_o of halfwidth Γ and height $\Phi(\omega_o)/4\Gamma^2\omega_o^2$. We note, however, that a delicate problem may arise if the noise is not very weak. As pointed out by Ivanov et al. [25] in the context of the problem of the absorption spectra of localized vibrations of impurities in crystals and analyzed in detail by Dykman and Krivoglaz [26] (see also [27]), an anharmonicity in the potential $U(q)$ of a noise-driven underdamped oscillator may result in a nonlinear response to noise as evidenced by a strong distortion of the peak of the spectral density of the fluctuations. The distortion is caused by the fact that the anharmonicity leads to an amplitude-dependence of the vibration frequency. As a consequence, noise-induced fluctuations of the amplitude cause fluctuational frequency straggling. If this straggling exceeds the small frequency "uncertainty" Γ associated with the damping, the shape of the spectral peak is substantially modified from the Lorentzian shape obtained from the linear approximation, and its height differs from $\Phi(\omega_o)/4\Gamma\omega_o^2$. The noise-induced distortion of the peak of $\widetilde{Q}(\omega)$ as described by Dykman and Krivoglaz [26] has been observed and investigated in detail quantitatively in analog electronic experiments [28]. The intensity of the noise that leads to a distortion of the spectrum is much smaller than that resulting in the distortion of the probability distribution caused by the nonlinearity of the potential.

One general consequence of (2.18)–(2.20) is that the time correlation function of a one-degree-of-freedom system driven by colored noise is a smooth function near $t = 0$ (cf. Sancho [29], who considers the particular case of exponentially correlated noise). As can be seen from (2.10) and (2.18)–(2.20), this function is given by the expression

$$Q(t) \equiv Q_{11}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\frac{\Phi(\omega)}{\alpha^2+\omega^2}. \tag{2.24}$$

If $\Phi(\omega)$ were constant (white noise), the derivative $Q'(t)$ would be singular as $t \to 0$ since the integral over ω of $\omega/(\alpha^2+\omega^2)$ diverges. If, however, $\Phi(\omega)$ decays to zero with increasing ω, as is the case for any physical noise, this derivative necessarily vanishes since the integral of $\omega\Phi(\omega)/(\alpha^2+\omega^2)$ converges and $\Phi(\omega)$ is an even function of ω (and also the stationary correlation function $Q(t)$ is an even function of t).

2.2.2. Statistical distribution near the maximum. The expressions obtained above in the case of weak noise lead in a straightforward way to an expression of the parameters of the statistical distribution $p(\boldsymbol{x})$ of the system near a maximum in terms of the spectral densities $\Phi_{nm}(\omega)$. It follows from (2.10) that the components β_{nm} of the matrix β formed from the means of the products of the displacement components are given by

$$\begin{aligned}\beta_{nm} &= Q_{nm}(0) \equiv \langle (x_n(0) - x_n^{st})\,(x_m(0) - x_m^{st})\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \widetilde{Q}_{nm}(\omega).\end{aligned} \tag{2.25}$$

By the ergodic hypothesis that permits the interchange of time averages and ensemble averages, one can interpret the matrix elements β_{nm} as second-order moments of the statistical distribution $p(\boldsymbol{x})$:

$$\beta_{nm} = \int d\boldsymbol{x}\,(x_n - x_n^{st})\,(x_m - x_m^{st})\,p(\boldsymbol{x}). \tag{2.26}$$

Since the driving noise is zero-mean Gaussian, the higher moments of $p(\boldsymbol{x})$ can also be expressed in terms of the β_{nm}. Indeed, according to (2.12),

$$\begin{aligned}&\langle (x_{n_1} - x_{n_1}^{st})\cdots(x_{n_k} - x_{n_k}^{st})\rangle \\ &\qquad = \sum_{\{m_j\}}\int_{-\infty}^{0} dt_1\cdots dt_k\,\chi_{n_1m_1}(-t_1)\cdots\,\chi_{n_km_k}(-t_k) \\ &\qquad\quad \times \langle f_{m_1}(t_1)\cdots f_{m_k}(t_k)\rangle,\end{aligned} \tag{2.27}$$

and therefore all the odd moments of $p(\boldsymbol{x})$ vanish while the relationships among the even moments are similar to those of the noise. For example,

$$\begin{aligned}&\langle (x_{n_1} - x_{n_1}^{st})\cdots(x_{n_4} - x_{n_4}^{st})\rangle \\ &\qquad = \beta_{n_1n_2}\beta_{n_3n_4} + \beta_{n_1n_3}\beta_{n_2n_4} + \beta_{n_1n_4}\beta_{n_2n_3}\end{aligned} \tag{2.28}$$

(cf. (2.3)). This implies that the statistical distribution $p(\boldsymbol{x})$ is Gaussian near the maximum, i.e., of the form [24]

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{L/2}(\det\beta)^{1/2}} \times \exp\left(-\frac{1}{2}\sum_{nm}\beta^{-1}_{nm}(x_n - x_n^{st})\,(x_m - x_m^{st})\right). \tag{2.29}$$

The expression for the distribution takes an even simpler form for a one-dimensional system. It follows from (2.29) that in this case

$$p(x) = (2\pi\beta)^{-1/2}e^{-(x-x^{st})^2/2\beta}, \qquad \beta \equiv \beta_{11} = Q(0), \tag{2.30}$$

with $x \equiv x_1$ (an alternative derivation of (2.30) based on the path-integral formulation is presented in §2.3).

An important feature of the statistical distribution arises from the fact that the spectral density of the noise $\Phi(\omega)$ and hence β scale as D. As a result, it is evident in (2.29) and (2.30) that the width of the Gaussian distribution is proportional to the noise intensity D, i.e., the distribution becomes increasingly narrower with decreasing noise. The distribution also becomes narrower as the maximum of the power spectrum of the noise moves away from the range of large susceptibility of the system, i.e., the range where the Green functions $|G^{n'm'}_{nm}(\omega)|$ are large.

2.2.3. Spectral density of fluctuations in thermal equilibrium. Thermal equilibrium fluctuations in a dynamical system arise through its coupling to a thermal bath which is itself a dynamical system of many degrees of freedom [24]. In some cases, the description of the entire coupled system can be reduced to a set of stochastic equations of motion for the dynamical variables of interest in which the effects of the bath appear as potential shifts (and/or mass renormalizations), dissipative contributions, and random forces (noise). However, if the random forces resulting from this reduction have finite correlation times (colored noise), the form of these equations differs from (2.7) even in the simplest cases. Correspondingly, the shape of the spectral density of the fluctuations of the system differs from that given in §2.2.1. We briefly illustrate these points through a simple specific "generic" model [5], [6], [20]–[22], [30], in which the coupling between the system and the bath is assumed to be linear in the coordinate of the system. The coupled system–bath Hamiltonian function H for this model is of the form

$$H = H_0 + H_b + H_i, \tag{2.31}$$

where

$$H_0 = \frac{1}{2}p^2 + U(q), \qquad H_i = -q\Xi. \tag{2.32}$$

Here q and p are the coordinate and momentum of the system, H_0 and H_b are, respectively, the Hamiltonian functions of the system and the bath in the absence of coupling, $U(q)$ is the potential energy of the isolated system, and Ξ is only a function of the dynamical variables of the bath, so that the dependence of the coupling energy on the system of interest is linear in the system coordinate q (an analysis similar to that outlined below can be carried out for the case of coupling proportional to the momentum p).

Not only the evolution of the system but also that of the bath is affected by the coupling between the two. If the coupling is sufficiently weak, the response of the bath to the perturbation H_i is given by linear response theory and hence only requires consideration of the evolution of the bath in the absence of the system. This linear response can be described with the help of a generalized susceptibility $K(t)$ as

$$\Xi(t) = f(t) + \int_{-\infty}^{t} d\tau\, K(t-\tau)q(\tau), \tag{2.33}$$

where it has been assumed that the coupling was switched on at $t = -\infty$. Here $f(t)$ is the instantaneous value of the bath-dependent quantity Ξ in the absence of the bath–system coupling. This instantaneous value fluctuates in time. If Ξ is itself a sum of many "elementary" uncorrelated contributions arising from different degrees of freedom of the bath (e.g., a bath of oscillators or particles each interacting individually with the system), then the discussion in the Introduction leads to the conclusion that $f(t)$ is a Gaussian random process which we take to have a zero-mean (a nonzero-mean can be removed by a proper redefinition of variables). Since the bath has been evolving since $t = -\infty$, the random process is stationary. The second term in (2.33) arises from the system–bath interaction and represents the way in which the bath dissipates excess energy of the system that is introduced by the interaction. It follows from the fluctuation-dissipation theorem [24] that the Fourier transform of the susceptibility $K(t)$ is related to the power spectrum $\Phi(\omega)$ of the noise $f(t)$ and the temperature T of the bath (in energy units) as

$$\mathrm{Re}\widetilde{K}(\omega) = \frac{1}{\pi T} P \int_0^{\infty} d\Omega\, \Phi(\Omega) \frac{\Omega^2}{\Omega^2 - \omega^2}, \qquad \mathrm{Im}\widetilde{K}(\omega) = \frac{\omega}{2T}\Phi(\omega), \tag{2.34}$$

where

$$\widetilde{K}(\omega) \equiv \int_0^{\infty} dt\, e^{i\omega t}\, K(t) \tag{2.35}$$

and P denotes the principal value integral. The expressions (2.32) and (2.33) lead to a stochastic equation for the dynamical variable q,

$$\ddot{q} + U'(q) - \int_{-\infty}^{t} d\tau\, K(t-\tau)q(\tau) = f(t). \tag{2.36}$$

The integral in (2.36) describes a dissipative delayed "self-action" of the system mediated by the bath. According to (2.34) and (2.35), if the noise $f(t)$ were white the susceptibility $K(t)$ would be proportional to the derivative of the δ-function (cf. Ford et al. [30]), $K(t) \propto d\delta(t)/dt$, and the integral in (2.36) would be proportional to $\dot{q}(t)$, leading to the usual dissipation linear in the instantaneous value of the momentum, as in Brownian motion (cf. (2.21)). We note, however, that this limit leads to a self-energy divergence problem similar to the well-known divergence encountered in quantum electrodynamics: the real part of the susceptibility $\widetilde{K}(\omega)$ diverges and as a result the renormalized (because of the coupling to the bath) frequency of the small-amplitude vibrations of the system is infinite. There is no physical significance underlying this divergence: it can be removed in a way that is standard in quantum electrodynamics; furthermore, it does not arise if the power spectrum of the noise decays to zero as $\omega \to \infty$, i.e., for colored noise.

An important case where the dynamics of the system can be described within a quasi-white-noise approximation and where divergences do not arise occurs when the coupling is weak compared not only to the characteristic frequencies of the bath but also to those of the system: if $|\widetilde{K}(\omega)|$ is small compared to the squared frequencies $\omega^2(E)$ of the eigenvibrations with energies $E \leq T$ and if the dependence of $\widetilde{K}(\omega)$ on ω is smooth for $\omega \sim \omega(E)$, the dynamics of the energy and of the slowly varying portion of the phase of the system on a time scale coarse-grained over $t \sim \omega^{-1}(E)$ are the same as those of white-noise-driven Brownian vibrations. The noise intensity for such motion is $\Phi[\omega(E)] \approx$ const. and the friction coefficient is given by $\mathrm{Im}\widetilde{K}[\omega(E)]/\omega(E) \approx$ const. (cf. Bogoliubov [20], where a corresponding microscopic derivation was given for the first time; see also [30]). The perturbative corrections to these results due to the color of the noise have been considered by Carmely and Nitzan [32] (see also references therein).

The explicit solution of (2.36) for colored noise $f(t)$ with an arbitrary spectrum $\Phi(\omega)$ can be obtained when the force $U'(q)$ is linear, or when the noise is sufficiently weak that the force can be linearized about the equilibrium position q^{st}, $U'(q) \approx \omega_0^2(q-q^{st})$. The spectral density of the fluctuations within this approximation can be seen from (2.36) to be given by

$$\widetilde{Q}(\omega) = \frac{\Phi(\omega)}{|\omega_0^2 - \omega^2 - \widetilde{K}(\omega)|^2}. \tag{2.37}$$

It is instructive to compare (2.37) with the undelayed relaxation result (2.18) with (2.23). In both cases the spectral density $\widetilde{Q}(\omega)$ is proportional to the power spectrum $\Phi(\omega)$ of the driving noise, but the coefficients are different. In particular, in (2.37) the form of the coefficient depends, through $\widetilde{K}(\omega)$, on the shape of $\Phi(\omega)$ itself. Therefore, if the coupling is sufficiently strong that $|\widetilde{K}(\omega)| \geq \omega_0^2$, the structure of $\Phi(\omega)$ near the maximum is not reflected directly in $\widetilde{Q}(\omega)$. However, for weak coupling and low-frequency noise, where both $|\widetilde{K}(\omega)|^{1/2}$ and the characteristic width of $\Phi(\omega)$ are small compared to ω_0, the features of $\Phi(\omega)$ are clearly reproduced in $\widetilde{Q}(\omega)$.

We note that (2.36) and (2.37) hold regardless of the inter-relation between the characteristic frequencies of the system and $|\widetilde{K}(\omega)|^{1/2}$ provided that the system–bath coupling is weak compared to the characteristic frequencies of the bath. At the same time, (2.36) is *exact* for a particular model of a harmonic oscillator and a bath composed of a set of harmonic oscillators with a coupling function Ξ in (2.32) that is linear in the bath oscillator coordinates [30] (cf. [33], where $\widetilde{Q}(\omega)$ was given for the corresponding quantum problem).

Thermal equilibrium fluctuations in systems described by (2.36) have been investigated for various forms of the potential $U(q)$ and of the spectrum $\Phi(\omega)$ (sometimes the "retarded" term in (2.36) is written in a form where $q(\tau)$ is replaced by $p(\tau)$ and the kernel is transformed accordingly; cf. [34] and references therein). Among the most recent results related to the color of the noise we mention the observation by means of analog electronic simulation [35] of the onset of an additional peak in the spectral density of the velocity fluctuations (i.e., of the fluctuations in $\dot{q}$) in a system with a periodic cosine potential. The simulated system was described by (2.36) with a cosine potential $U(q)$ and with exponentially correlated noise (cf. (2.6)), but the effects of the renormalization of the potential and the mass related to the real part of $\widetilde{K}(\omega)$ were omitted. A double-peaked spectrum was observed (see Fig. 2.1), and the peaks were attributed to intrawell vibrations and to the interplay of motion over the barrier and the color of the noise.

To understand this behavior, consider the special case of a harmonic oscillator described by (2.37). The spectral density of the velocity fluctuations is given by $\omega^2\widetilde{Q}(\omega)$. If the spectrum of the noise is of the form $\Phi(\omega) = AT/(1+\omega^2t_c^2)$ and we neglect $\mathrm{Re}\widetilde{K}(\omega)$ in (2.37), then $\omega^2\widetilde{Q}(\omega)$ exhibits a sharp peak at the oscillator frequency ω_o provided that $\omega_o \gg A/4(1+\omega_o^2t_c^2)$. On the other hand, for small ω_o such that $\omega_o \ll (A/t_c^2)^{1/3}$ and $At_c > 2$, the spectrum $\omega^2\widetilde{Q}(\omega)$ shows a peak at the frequency $\omega \approx (A/t_c^2)^{1/3}$. This peak only appears when the noise is colored and does not arise for white noise. The motion in a cosine potential is characterized by a broad spectrum of eigenfrequencies, all the way down to zero, and therefore both of these peaks may coexist. The position of the color-induced peak in Fig. 2.1 is satisfactorily reproduced by a calculation using a flat potential $U(q) = \text{const.}$ (equivalent to setting $\omega_o = 0$ in $\omega^2\widetilde{Q}(\omega)$) with the appropriate noise parameter values [35].

The results of this section indicate that the statistical distribution of the fluctuations of a noise-driven system near its maxima and the spectral density of the fluctuations can be found explicitly for sufficiently weak colored noise. Explicit solutions can be found for systems driven by nonthermal noise (with unretarded relaxation), and also for equilibrium systems driven by thermal noise (whose relaxation is retarded). The shape of the distribution near a maximum is Gaussian in all cases, while the shape of the spectral density of the fluctuations is given by relatively simple expressions and is in general proportional to the power spectrum of the driving noise, with the coefficient

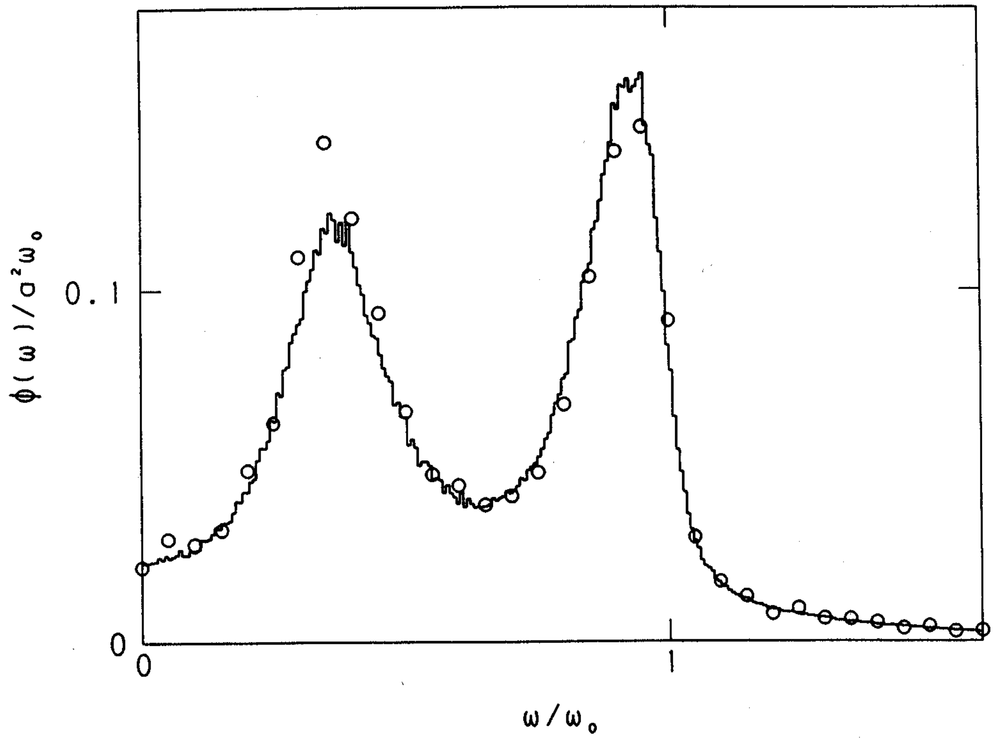

FIG. 2.1. *Spectral density of fluctuations of the velocity of a system in "thermal equilibrium" at "temperature" T fluctuating in a cosine potential* $U(x) = -A\cos x$ *as measured in an analog experiment (the circles show the results of a digital simulation)* [35]. *The low-frequency peak is due to the color of the noise. The data are for exponentially correlated noise, with* $t_c = 10/\sqrt{A}$, $D = 2A\sqrt{A}$, $T = A$; *the frequency is measured in units of the eigenfrequency of intrawell vibrations.*

depending on the frequency and, in the case of thermal fluctuations, on the shape of the power spectrum of the noise.

2.3. Large fluctuations: Method of the optimal path

The small fluctuations analyzed in §2.2 are the most probable fluctuations, and as such they determine the noise-induced "smearing" of the system about its stable positions. Another important problem for noise-driven systems is that of determining the probabilities of occasional large fluctuations. Large fluctuations result from "outbursts" of the driving noise which cause the system to move far from the stable states in phase space. These large fluctuations determine the shape of the tails of the statistical distribution of the system where the distribution is small. In this section we study the probabilities of large fluctuations for monostable systems driven by colored noise. To demonstrate the ideas and to focus sharply on those results that are specifically related to the color of the noise we concentrate on the simplest type of systems, namely, those described by the equation of motion (2.7) with a single dynamical variable,

$$\dot{x} + U'(x) = f(t). \tag{2.38}$$

We shall call x a "spatial" variable.

If the system is monostable, i.e., if the potential $U(x)$ has only one minimum at $x = x^{st}$ $[U'(x^{st}) = 0]$, its intrinsic motion is characterized by the relaxation time $t_r = 1/U''(x^{st})$. The spatial scale of the most probable fluctuations of the system is given by the root-mean-square displacement $\Delta x = \beta^{1/2}$ about the stable position x^{st}. According to (2.25), Δx is proportional to the square root of the noise intensity D (cf. the last paragraph in §2.2.2). On the other hand, the decay of correlations of the noise is characterized by the correlation time t_c given by the reciprocal width of the narrowest peak (or dip) of the power spectrum $\Phi(\omega)$ of the noise defined in (2.2). (In general, colored noise may be characterized by several times equal to the reciprocal positions (excluding the peak at $\omega = 0$ if present) and widths of all the peaks of $\Phi(\omega)$, but it is obviously the smallest of the reciprocal widths that gives the time over which a value of $f(t)$ is forgotten.)

"Large" fluctuations are those that cause the system to move away from x^{st} by a distance that substantially exceeds Δx. Apart from the spatial scale, there is also a time scale associated with large fluctuations. It is obvious that the sojourn in the vicinity of a remote point x and the return to x^{st} takes a time of order

$$t_o = \max(t_r,\ t_c). \tag{2.39}$$

Large noise "outbursts" that cause the system to stray far away from the stable state x^{st} are rare. If the farthest state x reached as a consequence of such an outburst is sufficiently far from x^{st}, then the time $T(x)$ between these large noise "outbursts" substantially exceeds t_o. It is easy to imagine the character of such large fluctuations. During a time $t \lesssim T(x)$ (but $t \gg t_o$) the system fluctuates mostly near the stable state. Then the system makes an "excursion" to x of characteristic duration t_o. The successive excursions to x are thus statistically independent of one another, since the previous excursion has been forgotten by the time the next excursion occurs. Of course, in addition to the excursions to a given x, other excursions to remote points may be taking place. The duration of each of them is of order t_o, since this is the only time that characterizes the correlation of fluctuations or the deterministic evolution of the system (whichever takes more time). The intervals between excursions to any extreme value that is sufficiently far from the stable state are also statistically independent of one another. It is evident from the "physical" notion of the probability $p(x)dx$ as the relative length of time spent in a small vicinity dx around the point x (ergodic hypothesis) that $p(x)/p(x^{st}) \sim t_o/T(x)$. It is also clear from the above picture that $T(x)$ might be called a mean first-passage time (MFPT) to the point x from the regime of small fluctuations about the stable state.

A typical trajectory $x(t)$ of a noise-driven system that illustrates the arguments presented above is sketched in Fig. 2.2. One can single out a path (a portion of this trajectory) that arrives at a given x having started at some point within $\Delta x \sim D^{1/2}$ of x^{st} (it is not useful to specify the starting point of the motion to an accuracy sharper than the fluctuational smearing of x^{st}). Each such path is noise-driven and therefore random. Furthermore, different paths that arrive at x are mutually independent. Therefore, these paths can themselves be described as random processes, and one can associate a probability density with the realization of a particular path $x(t)$. This probability density is a functional, $\wp[x(t)]$, since the random quantity is itself a function and not a variable [36] (see also [37] and references therein).

Since the point x lies far from the attractor, the probability density for the realization of any particular path that reaches x at time t is small. Furthermore, the probability densities for paths that on their way to x pass different points at a given time prior to reaching x differ considerably (exponentially for systems driven by Gaussian noise) if these points differ by an amount that substantially exceeds Δx. Therefore, one might expect that there is a group of paths that are close to one another (lying within a range Δx of one another) along which the system is most likely to move toward a given x at time t. One can further imagine that these paths surround an "optimal path" which represents the most probable path for arrival at x at

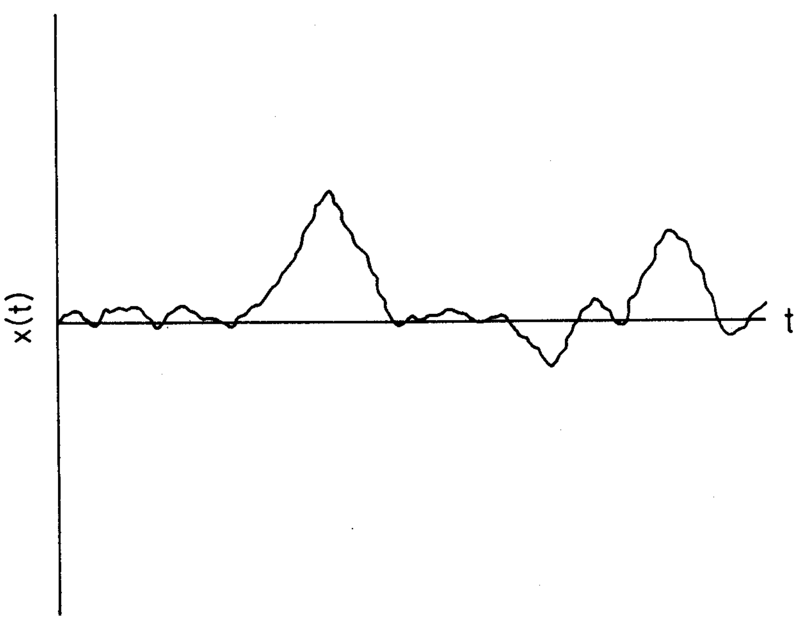

FIG. 2.2. *A sketch of the trajectory $x(t)$ of a noise-driven system exhibiting a large occasional fluctuation.*

time t. Recent experimental data on the visualization of an optimal path in a noise-driven system, and the related problem of the motion on the way to a given point (the pre-history problem) are considered in §2.5.

As shown below, the probability density that the system is found in a state x (the statistical distribution $p(x)$) in a system driven by weak Gaussian noise decreases exponentially sharply with increasing separation $|x - x^{st}|$. It is therefore important to be able to calculate $p(x)$ at least to logarithmic accuracy, i.e., to find the leading terms in $\ln p(x)$. To do this it suffices to calculate only the probability density for the realization of the optimal path. The ideas underlying this calculation are outlined in the next subsection.

2.3.1. Variational problem for the optimal path. A convenient approach to the analysis of large fluctuations in noise-driven systems is based [33], [38] on Feynman's idea [36] of the direct inter-relation between the probability densities of the paths of the system and those of the noise (see also [39]–[45]). This inter-relation arises from the fact that each path of the noise results in an associated path of the dynamical variables; in particular, each path $f(t)$ in (2.38) results in an associated path $x(t)$. As a consequence, the probability density for reaching a given point in the phase space of the system at a given instant is determined by the integral of the probability density functional for the noise over those noise trajectories that bring the system to that point at that instant. For a point that is remote from the stable state, the probability densities for all appropriate trajectories of the noise (and of the system) are very small and, as noted earlier (see also below), for Gaussian noise the probability densities differ exponentially for different trajectories. Therefore, the integral over the paths can be calculated by the method of steepest descent, and it is precisely the optimal path that corresponds to the extremal solution. The path-integral approach was applied recently to large fluctuations in colored-noise-driven systems in several papers in addition to those cited above [46], [47].

To find the stationary distribution $p(x)$ of a monostable system it is convenient [38] to express it in terms of the transition probability density $w(x, 0;\, x_a,\, t_a)$ for the transition from a point x_a occupied at some instant $t_a < 0$ to the point x at the instant $t = 0$. Since the initial state of the system and of the noise are forgotten over the time interval t_o given in (2.39), one has

$$p(x) = w(x, 0;\, x_a,\, t_a), \qquad -t_a \gg t_o. \tag{2.40}$$

The transition probability density w can in turn be written in terms of a path integral over the trajectories of the driving noise. For the system (2.38),

$$w(x, 0;\, x_a,\, t_a) = \frac{\int_{x(t_a)=x_a} \mathcal{D}f(t)\wp[f(t)]\delta[x(0) - x]}{\int \mathcal{D}f(t)\wp[f(t)]}. \tag{2.41}$$

Here $\wp[f(t)]$ is the probability density functional for the noise trajectories $f(t)$. Equation (2.41) expresses the fact that $w(x, 0;\, x_a,\, t_a)$ is the integral over all realizations of the noise $f(t)$ which move the coordinate $x(t)$ from the value x_a at $t = t_a$ to x at $t = 0$. The weighting factor $\wp[f(t)]$ gives the probability of a realization, and the denominator in (2.41) is simply a normalization factor.

For Gaussian noise the probability density functional $\wp[f(t)]$ is Gaussian [36], i.e., it is exponential in a bilinear functional of $f(t)$. For white noise with $\Phi(\omega) = D$ the form of this functional can be argued as follows. We begin by discretizing the noise $f(t)$ into a sequence of values $f(t_i)$ where $\Delta = t_{i+1} - t_i$ is very small ($\Delta \to 0$). The quantities $f(t_i)$ are random numbers rather than random functions. If we carry out this discretization process for white zero-mean Gaussian noise of intensity D such that $\langle f(t)\, f(t')\rangle = D\delta(t-t')$, we obtain $\langle f(t_i)\, f(t_j)\rangle = (D/\Delta)\delta_{ij}$. The distribution function of the $f(t_i)$ (which is *not* a functional) to within a normalization factor is of the form

$$P[\cdots f(t_i)\cdots] = \exp\left(-\frac{\Delta}{2D}\sum_i f^2(t_i)\right). \tag{2.42}$$

In the limit $\Delta \to 0$ the multivariable distribution function (2.42) goes over into the probability density functional

$$\wp[f(t)] = \lim_{\Delta\to 0} P[\cdots f(t_i)\cdots] = \exp\left(-\frac{1}{2D}\int dt\, f^2(t)\right). \tag{2.43}$$

The path-integral formalism was used for white noise by Wiener [48].

When the noise driving the system is colored, the form of the exponent in the expression for $\wp[f(t)]$ is more complicated. It is important to note, however, that it can be expressed entirely in terms of the power spectrum $\Phi(\omega)$ of the noise. To do this we observe that due to the stationarity of the noise, the Fourier components f_ω of the noise are δ-correlated in frequency,

$$\langle f_\omega\, f_{\omega'}\rangle = 2\pi\delta(\omega+\omega')\Phi(\omega), \qquad f_\omega = \int_{-\infty}^{\infty} dt\, e^{i\omega t} f(t). \tag{2.44}$$

The form of the probability density functional $\tilde{\wp}[f_\omega]$ for δ-correlated noise f_ω is similar to (2.43):

$$\tilde{\wp}[f_\omega] = \exp\left(-\frac{1}{4\pi}\int_{-\infty}^{\infty} d\omega\, f_{-\omega}[\Phi(\omega)]^{-1} f_\omega\right) \tag{2.45}$$

(cf. [36], [39], and [42]). In writing (2.45) we have taken into account that $\Phi(\omega)$ is an even function of ω, which in turn is a consequence of the fact that the time correlation function $\phi(t)$ of the noise is even in t. It follows from (2.44) and (2.45) that the probability density functional $\wp[f(t)]$ can be written as

$$\wp[f(t)] = \exp\left(-\frac{1}{2D}\int dt\, f(t)F(-id/dt)f(t)\right), \quad F(\omega) = \frac{D}{\Phi(\omega)}. \tag{2.46}$$

In going from (2.45) to (2.46) we assumed that $\Phi^{-1}(\omega)$ can be expanded as a series in ω that converges for finite ω. The operator $F(-id/dt)$ is then self-adjoint within the class of functions $f(t)$ that are sufficiently smooth and that vanish as $t \to \pm\infty$, and for which the integral $\int dt\, f(t)F(-id/dt)f(t)$ exists. The smoothness condition on $f(t)$ imposes no physical restriction in practice since noise generated by real physical sources does not have singularities. The results considered below refer mostly to the case where $\Phi^{-1}(\omega)$ is a finite series in ω; in particular, for exponentially correlated noise as in (2.6), $\Phi^{-1}(\omega) = (1 + \omega^2 t_c^2)/D$. This corresponds to $f(t)$ being a component of a Markov process with the total number of components equal to the degree of the polynomial $\Phi^{-1}(\omega)$ divided by two [45]. Since the power spectrum $\Phi(\omega)$ is positive for all ω, the argument of the exponential in (2.46) is negative for all $f(t)$.

The crucial point for the mathematical formulation of the method of the optimal path in systems driven by weak Gaussian noise is that if a system such as that described by (2.38) has been moved from a stable state in the phase space to another point by an external force, it must have been subjected to finite forcing applied over some time. Although different realizations of the force can result in the same final destination of the system at time t, the point is that the trajectories leading there are essentially deterministic, not random, and that they are independent of the noise intensity D. Therefore, if D is sufficiently small, the probabilities of all of these realizations are seen from (2.46) to be exponentially small. Moreover, it is also seen from (2.46) that the probabilities of different realizations differ exponentially from each other. Thus one would expect there to exist a realization $f(t)$ which is much more probable than the others, and it is just this realization that is associated with the optimal path $f_{opt}(t)$ of the noise. The optimal path $x_{opt}(t)$ of the system is obtained from the equation of motion (e.g., from (2.38)) with $f(t) = f_{opt}(t)$.

It follows from (2.40), (2.41), and (2.46) and from the above arguments that the statistical distribution $p(x)$ is exponentially small for points x that are removed from the stable state x^{st} by a distance that substantially exceeds the root-mean-square displacement $\Delta x \sim \langle (x - x^{st})^2 \rangle^{1/2} \propto D^{1/2}$. Since we have limited ourselves to seeking $p(x)$ to logarithmic accuracy in the noise intensity it is convenient to write it as

$$p(x) = C(x)p(x^{st})e^{-R(x)/D}. \tag{2.47}$$

Here, the argument $R(x)$ of the exponential is just the value of the integral $(1/2)\int dt\, f(t)F(-id/dt)f(t)$ in (2.46) for the optimal path $f_{opt}(t)$. The factor $C(x)p(x^{st})$ (the coefficient $p(x^{st})$ is written explicitly simply for convenience) is a smooth function of x on the scale $\Delta x \propto D^{1/2}$ that allows for the normalization (the denominator in (2.41)) and for the fact that not only the optimal path but a "tube" of paths surrounding it contribute to a steepest-descent evaluation of the integral in the numerator in (2.41). The width of this tube is determined by the form of the probability density functional $\wp$ of (2.46) and also by the form of the equation of motion of the system and the value of x. It is obvious from (2.46), however, that $\wp[f(t)]$ drops sharply for those $f(t)$ that differ from $f_{opt}(t)$ by more than $\sim D^{1/2}$, and therefore this width is proportional to $D^{1/2}$

as well. The analysis of the prefactor $C(x)$ in (2.47) for some limiting cases is briefly discussed below.

To calculate the argument $R(x)$ of the exponential in (2.47), we note that it is the motion of the system from the stable state x^{st} to a given point x that we are monitoring. Therefore, the integral in the argument of the exponential in (2.46) should be minimized subject to the inter-relationship (2.38) between the optimal path $f_{opt}(t)$ and the associated path $x_{opt}(t)$. Therefore, $R(x)$, $f_{opt}(t)$, and $x_{opt}(t)$ are given by the solution of the following variational problem [45]:

$$\begin{aligned} R(x) &= \min \Re[f(t),\, x(t);\, x], \\ \Re[f(t),\, x(t);\, x] &= \frac{1}{2}\int_{-\infty}^{\infty} dt\, f(t)F(-id/dt)f(t) \\ &\quad + \int_{-\infty}^{0} dt\, \lambda(t)[\dot{x} + U'(x) - f(t)], \end{aligned} \tag{2.48}$$

with the boundary conditions

$$\begin{aligned} &x(t \to -\infty) = x^{st}, \qquad x(t=0) = x, \\ &f(t \to \pm\infty) = 0, \\ &\lambda(t \to -\infty) = 0, \qquad \lambda(t > 0) = 0, \end{aligned} \tag{2.49}$$

and the relation (2.38). The first term in the functional $\Re$ in (2.48) is the argument of the exponential in (2.46) (without the coefficient $-1/D$). The second term allows for the inter-relationship between $f(t)$ and $x(t)$; $\lambda(t)$ is an undetermined Lagrange coefficient. With this term, the functional $\Re$ should be minimized independently with respect to both $f(t)$ and $x(t)$.

The boundary conditions (2.49) correspond to the physical picture of the motion described above. Prior to arrival at a remote point x at the instant $t = 0$ as a result of a large fluctuation, the system spends a long time that substantially exceeds t_o (of the order of the mean first-passage time $T(x)$) near the stable state x^{st}. Here the noise is weak, so that $|x - x^{st}| \propto |f(t)| \propto D^{1/2}$. The boundary conditions on $x(t)$ and $f(t)$ for $t \to -\infty$ express this fact to within an error proportional to $D^{1/2}$ (reflecting the uncertainty in the position of the system near x^{st} and of the noise near zero). This error can be neglected since all contributions to $R(x)$ that vanish when $D \to 0$ are ignored. The conditions (2.49) for $t > 0$ follow from the fact that the calculation of the statistical distribution $p(x)$ to logarithmic accuracy does not require us to follow the further evolution of the system once it has reached the given point x; therefore, the driving noise $f(t)$ can be allowed to decay back toward zero for $t > 0$ "on its own" independent of $x(t)$, and therefore one can set $\lambda(t) = 0$ (this boundary condition has also been taken into account in (2.48) by setting the upper limit in the second integral in $\Re$ equal to zero). We note that $\lambda(t)$ can be discontinuous for $t = 0$. At the same time, $f(t)$ itself and also several derivatives of $f(t)$ are continuous (except for the case of white noise; see below). Because of this continuity, there arises a "postaction": the decay of $f(t)$ for

$t > 0$ does influence the behavior of $f(t)$ for $t < 0$ even though it does not contribute explicitly to $R(x)$.

An alternative procedure that accounts for the inter-relationship between $f(t)$ and $x(t)$ in the functional integral (2.41) is to multiply the integrand in the numerator of (2.41) by the functional (whose effect must be accommodated in the prefactor $C(x)$ in (2.47))

$$\int \mathcal{D}x(t)\delta[\dot{x} + U'(x) - f(t)] = \int\int \mathcal{D}\left(\frac{z(t)}{2\pi}\right)\mathcal{D}x(t) \times \exp\left(i\int dt\, z(t)[\dot{x} + U'(x) - f(t)]\right). \tag{2.50}$$

Note that $i\mathcal{D}z(t)$ is closely related to $\lambda(t)$ in (2.48). The Gaussian integral over $\mathcal{D}f(t)$ can then easily be calculated using standard methods [36], and the statistical distribution $p(x)$ to logarithmic accuracy is obtained by minimizing the remaining functional of $x(t)$ and of the auxiliary variable $z(t)$. This approach was used by Luciani and Verga [46]. In contrast to the differential variational equation for the functional $\Re$ in (2.48) found by the first procedure and considered in the next subsection, the variational equations obtained from the approach of Luciani and Verga are integral equations. An advantage of the present formulation is that it deals with the functions $f(t)$ and $x(t)$ that correspond to physical observables, so that intuitive arguments can be used when seeking the solution for the optimal paths. The approach also allows one to formulate the boundary conditions needed to obtain the statistical distribution and the transition probabilities (see also §2.6) so that the substantial difference between the two problems becomes obvious. Another advantageous feature of the approach is that the solutions can be immediately tested experimentally.

Since $\Phi(\omega)$ is proportional to D, the operator $F(-id/dt) = D/\Phi(-id/dt)$ in (2.48) does not change with a rescaling of the noise intensity. and therefore the function $R(x)$ is independent of D. The dependence of $p(x)$ on D as given in (2.47) is thus of the activation type. and $R(x)$ can be called an "activation energy" for reaching a point x. The concept of the activation energy is meaningful and the approximation (2.47) holds provided that

$$R(x) \gg D. \tag{2.51}$$

In this case, the distribution $p(x)$ is exponentially small for a given x and. as noted earlier, the average interval between successive outbursts of the noise that bring the system to a given x (the MFPT). $T(x) \sim t_o \exp[R(x)/D]$. greatly exceeds both the relaxation time t_r of the system and the correlation time t_c of the noise.

2.3.2. Variational equations and their analysis in limiting cases. The (deterministic) set of variational equations describing the optimal paths $f_{opt}(t)$ and $x_{opt}(t)$ follows from (2.48):

$$F(-id/dt)f(t) - \lambda(t) = 0. \tag{2.52}$$

$$\dot{\lambda}(t) - U''(x)\lambda(t) = 0, \qquad \dot{x} + U'(x) = f(t), \tag{2.53}$$

with

$$f(t) \equiv f_{opt}(t), \qquad x(t) \equiv x_{opt}(t). \tag{2.54}$$

The set (2.52) and (2.53) with the boundary conditions (2.49) constitute a boundary-value problem. This problem can be solved numerically for an arbitrary system potential $U(x)$ and for an arbitrary shape of the power spectrum $\Phi(\omega)$ of the noise.

A simple procedure can be followed when $F(\omega) \propto \Phi^{-1}(\omega)$ is a polynomial in ω^2 of finite degree M. The procedure uses the fact that the function $U'(x)$ is linear in $x - x^{st}$ near the stable state x^{st}. As a consequence, (2.52) and (2.53) are linear for $t \to -\infty$ where x is close to x^{st}, and the solution for $f(t)$, $\lambda(t)$, and $x(t) - x^{st}$ for $t \to -\infty$ is a linear combination of the exponentials $\exp(\mu t)$ with $\mathrm{Re}\mu > 0$. The values of μ are obtained from the secular equations

$$F(-i\mu_n) = 0, \quad \mathrm{Re}\mu_n > 0, \quad n = 1, \ldots, M; \qquad \mu_o = U''(x^{st}). \tag{2.55}$$

The solution contains $M + 1$ coefficients. They can be found from the condition that $x(0) = x$ and from M relationships between $f(t)$, $df/dt, \ldots, d^{2M-1}f/dt^{2M-1}$ for $t = 0$. To arrive at these relationships we first note that the function $f(t)$ and its derivatives $df/dt, \ldots, d^{2M-1}f/dt^{2M-1}$ are continuous at $t = 0$. This follows from the fact that (2.52) is a $2M$th-order differential equation for $f(t)$, and the function $\lambda(t)$ on the right-hand side of this equation is seen from (2.53) to be continuous for $t < 0$ and for $t > 0$ (where $\lambda = 0$). Thus, a discontinuity of λ can only occur at $t = 0$. Hence, $f(t)$ and its derivatives should be continuous for all t. On the other hand, the solution of (2.52) for $f(t > 0)$ where $\lambda = 0$ is of the form

$$f(t) = \sum_{n=1}^{M} [A_n \exp(\mu_n t) + B_n \exp(-\mu_n^* t)] \quad \text{for } \lambda = 0 \quad (t > 0). \tag{2.56}$$

Because of the continuity of $f(t)$ and its derivatives, the A_n, B_n in (2.56) are functions of the coefficients of the solution for $t \to -\infty$, and it follows from the condition $f(t) \to 0$ for $t \to \infty$ that the M functions that would lead to divergence of $f(t)$ vanish, i.e., that $A_1 = 0, \ldots, A_M = 0$.

The solutions of the variational equations (2.52) and (2.53) can be obtained in explicit form for several limiting cases. The simplest case is that of fluctuations in a quadratic potential,

$$U(x) = \frac{1}{2}\alpha x^2. \tag{2.57}$$

A Fourier transform of (2.52) and (2.53) over time yields

$$f_\omega = \int_{-\infty}^{\infty} dt\, e^{i\omega t} f(t) = \frac{\lambda(t = 0)}{F(\omega)(\alpha + i\omega)}, \tag{2.58}$$

$$x = \frac{1}{2\pi}\lambda(t=0)\int_{-\infty}^{\infty}\frac{d\omega}{F(\omega)(\alpha^2+\omega^2)}, \tag{2.59}$$

$$R(x) = \frac{\pi x^2}{\int \frac{d\omega}{F(\omega)(\alpha^2+\omega^2)}}. \tag{2.60}$$

The statistical distribution $p(x)$ obtained from (2.47) and (2.58)–(2.60) coincides with the result (2.30), (2.24) which was obtained in an entirely different way.

2.3.3. Activation energy for noise of small correlation time. Comparison to other approaches. Another limiting case for which $R(x)$ can be obtained explicitly is that of "weakly colored" noise, i.e., noise whose characteristic correlation time t_c is small compared to the relaxation time of the system, $t_r = [U''(x^{st})]^{-1}$. This is the limit in which t_r^{-1} is much smaller than all characteristic frequencies of the power spectrum $\Phi(\omega)$ of the noise, as illustrated in Fig. 2.3(a). One expects the system to be influenced primarily by low-frequency noise fluctuations, $\omega \lesssim t_r^{-1}$, while high-frequency fluctuations are mostly filtered out because the system can not follow them (see, however, §2.4). Therefore, it is the low-frequency part of $\Phi(\omega)$ that determines the main features of the fluctuations of the system.

The limit where the finiteness of the correlation time of the noise can be neglected altogether corresponds to a white noise driver. In this limit, i.e., to zeroth order in t_c/t_r, the optimal path is described by the equations

$$f(t) = 2U'(x), \quad \lambda(t) = 2F(0)U'(x), \quad \dot{x} = U'(x), \quad \text{for } t \le 0, \tag{2.61}$$

$$f(t) = 0 \qquad \text{for } t \ge 0, \tag{2.62}$$

$$f(t) \equiv f_{opt}(t), \qquad x(t) \equiv x_{opt}(t), \qquad t_c/t_r \to 0.$$

We note that the optimal path $x_{opt}(t)$ as given by (2.62) is a "time-inverted" path of the system in free motion, i.e., in the absence of external noise, as described by (2.38) with $f(t) = 0$. We also note that for a white-noise-driven system the optimal path of the noise $f_{opt}(t)$ is discontinuous, which is reasonable: since the noise is temporally uncorrelated, it can be assumed to vanish (i.e., it can be forced to achieve a root-mean-square value equal to zero within the optimal path method) immediately upon having "brought" the system to a given point.

At first glance, one might expect the corrections to (2.62) and to the corresponding expression for $R(x)$ due to a finite correlation time of the noise to be of order t_c^2/t_r^2 because $F(-id/dt)$ in (2.52) is a series in powers of $t_c^2 d^2/dt^2$. However, because of the discontinuity of the optimal path $f_{opt}(t)$ at $t = 0$ there

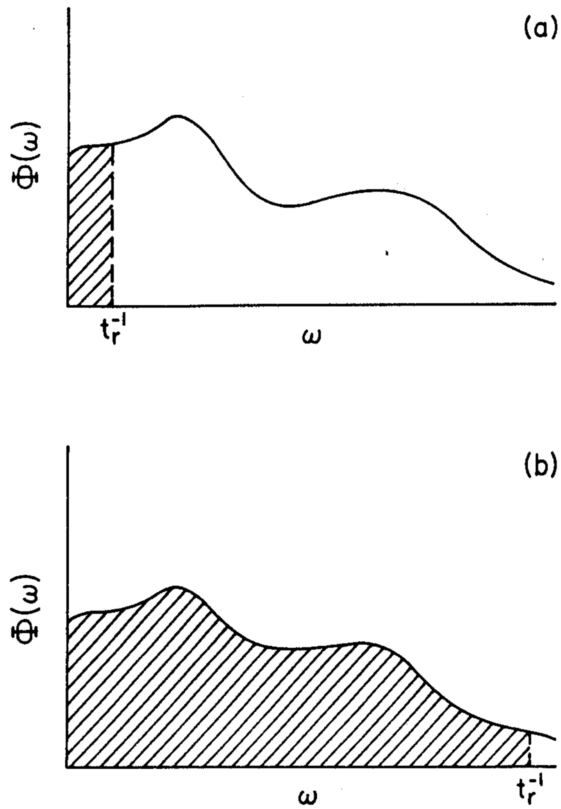

FIG. 2.3. *A sketch of the power spectrum $\Phi(\omega)$ of colored noise with all correlation times being of the same order of magnitude. The spectral components of the noise that influence a noise-driven system most substantially are those with frequencies smaller than or of the order of the characteristic reciprocal relaxation time t_r^{-1} of the system. The corresponding areas of $\Phi(\omega)$ are shaded. It is evident that if the correlation time of the noise is small compared to t_r* (a) *it is the shape of $\Phi(\omega)$ for small ω that determines the fluctuations in the system, while for the large correlation times of the noise* (b) *nearly all Fourier components of the noise are involved in determining the fluctuations of the system.*

appears a quickly varying contribution to $f_{opt}(t)$ for $|t| \lesssim t_c$ that gives rise to a correction of order t_c/t_r to the activation energy $R(x)$. One obtains from (2.52) and (2.53) [45]

$$R(x) \doteq 2F(0)[U(x) - U(x^{st})] + F(0)[U'(x)]^2\tilde{t}_c, \tag{2.63}$$

where

$$\tilde{t}_c = 2F(0)\int_0^\infty dt\, \frac{\phi(t)}{D}, \qquad |\tilde{t}_c| \ll t_r, \tag{2.64}$$

and $\phi(t)$ is the time correlation function of the noise as given in (2.2). Note that the ratio $\phi(t)/D$ is independent of the noise intensity. The first term in

$R(x)$ in (2.63) is the well-known result for systems driven by white noise and can be obtained directly from (2.62). It leads to a Boltzmann distribution $p(x)$ in (2.47) [24] with an "effective temperature" of $D/2F(0) = (1/2)\Phi(0)$. This is in agreement with the observation (cf. Fig. 2.3(a)) that the effective intensity of the "white" noise is proportional to $\Phi(0)$ for $|\tilde{t}_c| \ll t_r$. The second term in $R(x)$ is the correction due to the color of the noise. We stress that this correction, although small compared to the main term, can change the statistical distribution $p(x)$ exponentially strongly, since it can substantially exceed the characteristic noise intensity D. Another point to note is that this correction term may be positive or negative depending on the shape of the power spectrum of the noise. In other words, the color of the noise may either "squeeze" or "extend" the statistical distribution. The former situation occurs when $\tilde{t}_c > 0$ since $R(x)$ then exceeds its white-noise value. The latter occurs if $\tilde{t}_c < 0$. An example of a power spectrum $\Phi(\omega)$ that leads to a negative finite-correlation-time-induced correction to $R(x)$ is considered in §2.4.

If the noise is exponentially correlated (cf. (2.6)), the correlation time $\tilde{t}_c$ in (2.64) is precisely the time t_c that parametrizes the correlation function of the noise, and the correction to $R(x)$ is then positive. Equation (2.47) with (2.63) in this case coincides with the result of others (see [49] and [50] and also [17], [51]–[53], and references therein). These results have been obtained by several methods. In particular, the approaches based on the derivation of "effective Fokker–Planck equations" for the statistical distribution by various approximation procedures have been reviewed in detail by Lindenberg et al. [17].

The approach of [49] and [50] exploits the fact that the exponentially correlated noise $f(t)$ itself, and the composite system $[x(t), f(t)]$ consisting of the dynamical system of interest and the noise, can be viewed as Markov processes, with the equation for $x(t)$ being of the form (2.38) and that for $f(t)$ being of the form

$$\dot{f}(t) = -t_c^{-1} f(t) + \xi(t), \qquad \langle \xi(t)\, \xi(0)\rangle = D t_c^{-2} \delta(t), \tag{2.65}$$

where $\xi(t)$ is Gaussian white noise. The evolution of the joint probability density $\bar{p} \equiv \bar{p}(x, f; t)$ of the variables x, f is described by the Fokker–Planck equation

$$\frac{\partial}{\partial t}\bar{p} = \frac{\partial}{\partial x}[U'(x) - f]\bar{p} + t_c^{-1}\frac{\partial}{\partial f} f p + \frac{1}{2} D t_c^{-2} \frac{\partial^2}{\partial f^2}\bar{p}. \tag{2.66}$$

For small noise intensities the eikonal approximation can be used to solve this equation. In particular, one can seek a stationary solution of the form [54]

$$\frac{\bar{p}(x, f)}{\bar{p}(x^{st}, 0)} = \exp\left(-\frac{S(x, f)}{D}\right) \tag{2.67}$$

(see also [49] and [50] and references therein). To lowest order in D the equation for $S(x, f)$ that follows from (2.66) is a first-order nonlinear differential equation of the form of a Hamilton–Jacobi equation,

$$\begin{aligned} H\left(x, f; \frac{\partial S}{\partial x}, \frac{\partial S}{\partial f}\right) &\equiv [f - U'(x)]\frac{\partial S}{\partial x} - f t_c^{-1}\frac{\partial S}{\partial f} + \frac{1}{2}t_c^{-2}\left(\frac{\partial S}{\partial f}\right)^2 \\ &= 0, \end{aligned} \tag{2.68}$$

so that $S(x, f)$ can be associated with the mechanical action of an auxiliary dynamical system described by the variables x and f, the associated momenta $\partial S/\partial x$ and $\partial S/\partial f$, and the Hamiltonian function H. In the general case of finite t_c this equation cannot be solved analytically because of the lack of detailed balance [15] in the system. (We note that the eikonal approximation was applied in [55] and [56] when considering Markovian physical systems without detailed balance, in particular, the problem of the transitions between stable states of such systems [56]; in [38] a path-integral formulation was applied to this problem.) For small t_c, however, the solution (2.68) can be obtained easily. To lowest order in t_c it is of the form

$$S(x, f) = 2[U(x) - U(x^{st})] + f^2 t_c. \tag{2.69}$$

By substituting (2.69) into (2.67) and integrating over f one arrives at the "Boltzmann" distribution $p(x) \propto \exp[-2U(x)/D]$. The corrections to $S(x, f)$ that lead to the distribution (2.47) with $R(x)$ given by (2.63) can be obtained by perturbation theory in t_c [49], [50]. The method also makes it possible to find, again for exponentially correlated noise, the color-induced corrections not only to the exponent of the distribution (as in (2.63)) but also the terms of order t_c in the prefactor.

There is an immediate parallel between the path-integral formulation presented above as applied to the particular case of exponentially correlated noise, and the eikonal equation (2.68). To show this we first note that, for the extreme path, the integral $\int_{-\infty}^{\infty} dt\, f(t)F(-id/dt)f(t)$ in the functional $\Re$ in (2.48) can be replaced by $\int_{-\infty}^{0} dt\, f(t)F(-id/dt)f(t)$ since $\lambda(t) = F(-id/dt)f(t)$ vanishes for $t > 0$. Furthermore, the functional $\Re$ can be considered as a function of time t', $\widetilde{\Re}[f(t), x(t); t']$, if the upper limits in both integrals in $\Re$ are replaced by the running time t'. For exponentially correlated noise $F(-id/dt) = 1 - t_c^2 d^2/dt^2$ and the first term in (2.48) can be integrated by parts, so that the resulting functional can be written as

$$\widetilde{\Re}[f(t), x(t); t'] = \int_{-\infty}^{t'} dt\, L(f, \dot{f}, x, \dot{x}; \lambda), \tag{2.70}$$

$$L(f, \dot{f}, x, \dot{x}; \lambda) = \frac{1}{2}[f(t) + t_c \dot{f}(t)]^2 + \lambda(t)[\dot{x} + U'(x) - f(t)]. \tag{2.71}$$

(Here in the integration by parts we have taken into account that for the optimal path $\dot{f}(t') = -t_c^{-1} f(t')$ at the upper limit t' of the integral: cf. (2.52), (2.53), and (2.56).) The functional $\widetilde{\Re}$ in (2.70) can be viewed as a mechanical action of a system with dynamical variables x, f and with Lagrangian L. The generalized moments p_x, p_f are of the standard form [57]

$$p_x = \frac{\partial L}{\partial \dot{x}}, \qquad p_f = \frac{\partial L}{\partial \dot{f}}, \tag{2.72}$$

and the Hamiltonian function $H(x, f; p_x, p_f)$ is given by the Legendre transform

$$\begin{aligned} H(x, f; p_x, p_f) &= p_x\dot{x} + p_f\dot{f} - L \\ &= -p_x[U'(x) - f] - t_c^{-1} f p_f + \frac{1}{2} t_c^{-2} p_f^2. \end{aligned} \tag{2.73}$$

The expression (2.73) for the Hamiltonian function obviously coincides with (2.68) provided that account is taken of the fact that $p_x = \partial S/\partial x$ and $p_f = \partial S/\partial f$ [57]. The mechanical action S in (2.68) is just the minimum of the functional $\widetilde{\mathfrak{R}}$ in (2.70), and the equation $H = 0$ corresponds to the fact that this minimal value is independent of t' for given x. Thus, the path integral formulation and the eikonal approximation in the Fokker–Planck equation give identical results for the case of exponentially correlated noise.

We note that there is yet another alternative way of reducing to a mechanical problem the calculation of the tails of the statistical distribution for systems driven by noise that is itself a Markov process and/or a component of one. The method is based [38]–[43] on expressing the white noise that drives the noise that in turn drives the system only in terms of the dynamical variable of the system and thereby excluding the colored noise at this point, substituting this expression into the probability density functional (2.43) for the white noise [36], and minimizing the resulting functional of the dynamical variable in the argument of the exponent. This program is very similar to that presented in §2.3.1. The advantage of the formulation in §2.3.1 (see also [45]) is that it is based directly on the power spectrum of the noise that drives the system and is not limited to Markov processes. A further advantage is that it is straightforward to write the boundary conditions (see (2.49)). Yet another advantage, as mentioned earlier, is that the formulation presented here appeals to physical intuition and that therefore many peculiar features inherent to systems driven by colored noise can be understood in physical terms (in this context, see §2.4 below). Finally, we note that the analysis of the tails of the stationary distribution for certain types of Markov systems without detailed balance was performed in a different way in the mathematical paper of Ventzel and Freidlin [58] (see also [59]).

2.3.4. Statistical distribution for noise with large correlation time. The variational equations (2.52) and (2.53) can also be solved analytically when all correlation times of the noise greatly exceed the relaxation time of the system, i.e., $t_c \gg t_r$. Thus t_r^{-1} exceeds all characteristic frequencies of the power spectrum $\Phi(\omega)$ of the noise, as shown in Fig. 2.3(b), so that all Fourier components of the noise are "incorporated" in and affect the dynamics of the system. In this case one can physically picture the optimal fluctuation path as follows [60] (see also [61]): the noise $f(t)$ varies with a slow characteristic increment/decrement time t_c, and the system follows this variation adiabatically, i.e., the value of the coordinate $x(t)$ is given by the expression

$$x(t) \approx x^{ad}(t), \qquad U'[x^{ad}(t)] = f(t), \qquad t_r \ll t_c. \tag{2.74}$$

In other words, when $f(t)$ varies slowly the system occupies the time-dependent minimum of the "adiabatic" potential $U(x) - xf(t)$.

To calculate the activation energy $R(x)$ to reach a given point x in the approximation (2.74) it is convenient to change from the differential equation (2.52) for $f(t)$ to an integral equation that relates $f(t)$ to $\lambda(t)$. Allowing for the inter-relations (2.2) and (2.46) between $F(\omega)$ and the time correlation function $\phi(t)$ of the noise, and also for the boundary conditions (2.49), one obtains from (2.52)

$$f(t) = D^{-1} \int_{-\infty}^{\infty} dt'\, \phi(t-t')\lambda(t'). \tag{2.75}$$

The activation energy $R(x)$ can in turn be written as

$$R(x) = \frac{1}{2} \int_{-\infty}^{\infty} dt\, f(t)\lambda(t). \tag{2.76}$$

The subsequent analysis depends on whether the function $U''(x)$ is positive in the interval (x^{st}, x), i.e., whether $|U'(x)|$ increases monotonically as the coordinate moves away from the stable-state value x^{st} to a given x, or whether in this interval the potential $U(x)$ has an inflection point x^{infl} where $U''(x)$ changes sign. The adiabatic approximation (2.74) only holds in the former case, since $U''(x)$ is a measure of the local reciprocal relaxation time around the minimum of the adiabatic potential $U(x) - xf(t)$, and the criterion for the slowness of the noise in (2.74) can therefore in general be expressed as

$$t_c U''(x^{ad}) \gg 1. \tag{2.77}$$

The evolution of the adiabatic potential with increasing $f(t)$ for these two situations, namely, one in which $U''(x)$ is positive throughout and one in which $U''(x)$ changes sign, is shown, respectively, in Figs. 2.4(a) and (b).

If (2.77) is in fact satisfied so that the adiabatic approximation holds, then the function $\lambda(t)$ is seen from (2.53) to vary (increase in absolute value) much more rapidly than $f(t)$ for $t < 0$,

$$\lambda(t) = \lambda(0) \exp\left(\int_0^t dt'\, U''[x(t')]\right). \tag{2.78}$$

Bearing in mind that $x(t=0) = x$ one obtains from (2.74), (2.75), and (2.76)

$$U'(x) = f(0) = D^{-1}\phi(0)\lambda(0)/U''(x) \tag{2.79}$$

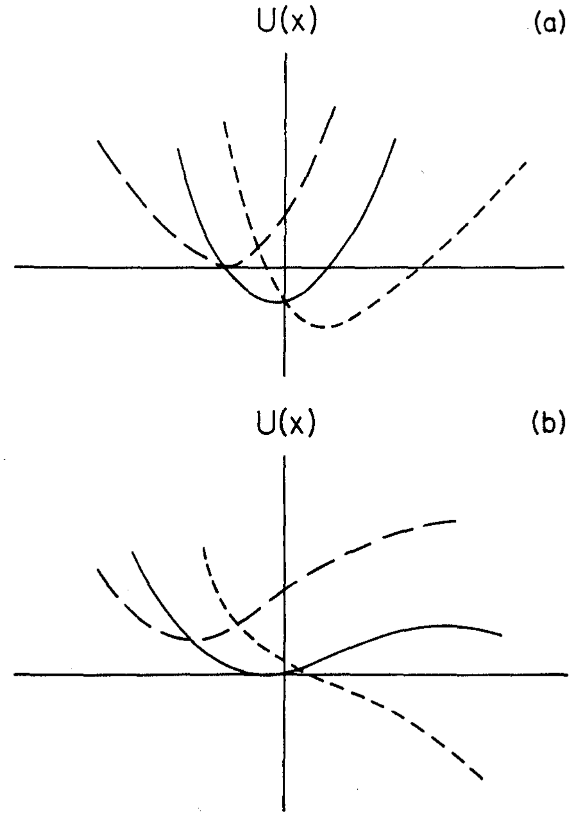

FIG. 2.4. *Evolution of the adiabatic potential* $U(x) - xf$ *with increasing force* f *for "bare" potentials* $U(x)$ *that* (a) *do not have and* (b) *do have an inflection point* $x^{infl}, U''(x^{infl}) = 0$. *In the case* (b) *the initial minimum of the adiabatic potential becomes more shallow with increasing* $f/U'(x^{infl})$ *and eventually disappears when* $f/U'(x^{infl}) > 1$.

and

$$R(x) = \frac{f(0)\lambda(0)}{2U''(x)}, \tag{2.80}$$

so that finally [45]

$$R(x) = \frac{D[U'(x)]^2}{2\phi(0)}. \tag{2.81}$$

For exponentially correlated noise $D/\phi(0) = 2t_c$, and (2.81) then yields precisely the result obtained in [43], [46], and [60].

When the correlation time of the noise is large, it is possible not only to determine the argument of the exponential in the expression for the statistical distribution (as done above) but also the prefactor of the distribution. Because of the adiabatic character of the response of the system, the probability $p(x)dx$ for the dynamical variable of the system to lie in the interval dx around a given x is equal to the probability $\tilde{p}(f)df$ for the noise to lie in the interval df about $f = U'(x)$. Therefore,

$$p(x) = \frac{1}{[2\pi\phi(0)]^{1/2}} U''(x) \exp\left(-\frac{[U'(x)]^2}{2\phi(0)}\right). \tag{2.82}$$

For values of x relatively close to x^{st}, where $U'(x) \approx U''(x^{st})(x - x^{st})$, equation (2.82) agrees with equation (2.30) obtained for this range of x with a noise power spectrum of arbitrary shape, if one allows in (2.24) for the fact that $U''(x^{st})$ substantially exceeds all characteristic frequencies of the noise.

If the potential $U(x)$ has an inflection point x^{infl} so that $U''(x^{infl}) = 0$, then for points x lying close to x^{infl}, and for points x that lie on the opposite side of x^{infl} than does x^{st}, the adiabatic approximation (2.74) does not hold since the local relaxation time of the system becomes very large where $U''(x)$ is small. It is obvious from Fig. 2.4(b), however, that the probability of reaching the point x^{infl} and also the region beyond x^{infl} is simply determined by the probability of the force $f(t)$ reaching the "critical" value $U'(x^{infl})$. Having been brought to the inflection point by a large outburst of noise, the system does not need strong additional forcing to move further; in effect, it moves further "on its own." The dominant term in $R(x)$ can therefore be written as

$$R(x) = \frac{D|U'_m|^2}{2\phi(0)}, \qquad |U'_m| = \max_{\tilde{x}\in(x^{st},x)} |U'(\tilde{x})| \tag{2.83}$$

(see [43], [46], [60], and [61] for exponentially correlated noise and [45] for the general case).

It is obvious from (2.83) that the distribution beyond x^{infl} is flat, i.e., that $R(x)$ in this region is independent of x. This flatness is apparent in the results of the numerical calculations of $R(x)$ for exponentially correlated noise carried out by Bray et al. [43] for the case of a quartic bistable potential of the form

$$U(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4. \tag{2.84}$$

The function $R(x)$ is plotted in Fig. 2.5 (we have used only the data for the region $x < 0$). The numerical data clearly demonstrate the evolution of the shape of $R(x)$ with varying noise correlation time, from $R(x) \propto U(x)$ in the white-noise limit $t_c \to 0$ to a function with a nearly flat section between $x^{infl} = -1/\sqrt{3}$ and $x = 0$ for large t_c. The data were obtained by solving equations of the type (2.52) and (2.53). More precisely, instead of a system of equations consisting of a second-order equation for $f(t)$ and first-order equations for $\lambda(t)$ and $x(t)$, a fourth-order equation for $x(t)$ was constructed (the force $f(t)$ was excluded at the initial stage of the path integral formulation, as mentioned above) [43]. A self-similar solution of the form of $y(x) = \dot{x}(t)$ was

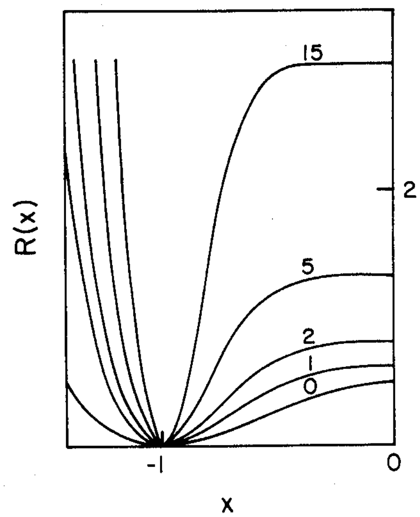

FIG. 2.5. *Change of the activation energy $R(x)$ to reach point x with changing correlation time t_c of the exponentially correlated driving noise* [43]. *The system is described by* (2.38) *with the quartic bistable potential* (2.84). *The data for the stable state $x = -1$ are plotted. The curves are labeled by the value of t_c. The $t_c = 0$ (white noise) curve corresponds to the Boltzmann distribution.* $R(x) = 2[U(x) - U(x = -1)]$.

sought, so that the problem was reduced to a second-order equation for $y(x)$. We note that this approach, although very effective in the problem considered in [43], is not applicable in the general case of colored noise (in particular in the case of quasi-monochromatic noise considered in the next section), since $\dot{x}(t)$ takes on different values for given x for different t along the optimal path.

The slowing-down of the motion of the system gives rise to corrections to the activation energy that are nonanalytic in t_c [43], [45]. The order of magnitude of the corrections can be estimated by noting that in the vicinity of the point of inflection the equation of motion of the system is of the form

$$\delta\dot{x} = \delta f(t) - \frac{1}{2}U'''(x^{infl})(\delta x)^2, \tag{2.85}$$

where

$$\delta x \equiv x - x^{infl}, \qquad \delta f(t) \equiv f(t) - U'(x^{infl}). \tag{2.86}$$

If the motion in this region lasts for a time $\delta t \ll t_c$, then during this time $f(t)$ varies by an amount $\delta f(t) \sim U'(x^{infl})\delta t/t_c$, and this variation is of order $\delta x/\delta t$ and also of order $(\delta x)^2$. Hence $\delta x \propto t_c^{-1/3}$ and $\delta t \propto t_c^{1/3}$. It follows from the latter estimate and from (2.80) that the nonanalytic correction to $R(x)$ is of order $t_c^{1/3}$ as well. The problem was analyzed in detail by Bray et al. [43]. The explicit expression for the correction to the activation energy of the transition obtained in [43] is given in §2.6 below.

To summarize: it follows from the results of the present section that the dependence of the tails of the statistical distribution of a system driven by Gaussian noise on the noise intensity is of the activation type. The problem of calculating the activation energy for reaching a given point can be reduced, via the method of the optimal path, to a variational problem which in turn in many cases translates to a boundary-value problem involving a set of ordinary differential equations. The form of the equations is determined by the shape of the power spectrum of the noise. They can be solved analytically in several limiting cases, including those of large and small characteristic correlation times of the noise. In general it is straightforward to investigate the equations numerically.

2.4. Fluctuations induced by quasi-monochromatic noise

The general formulation presented above clearly demonstrates that small fluctuations about stable states of noise-driven systems as well as the statistical distribution of large fluctuations depend crucially on the shape of the power spectrum of the noise. In many cases the noise turns out to be "truly colored": its power spectrum is peaked at a certain frequency ω_o, and the halfwidth Γ of the peak is much smaller than ω_o,

$$\Gamma \ll \omega_o. \tag{2.87}$$

An example of such noise is "normal" (incoherent) nearly monochromatic light. Such light possesses a specific "color," and it would therefore be reasonable to use the terminology "colored" for any noise with a similar peaked power spectrum. However, to avoid confusion, we have used and will continue to use the standard terminology; to distinguish this sort of noise from other forms of colored noise we will therefore call it quasi-monochromatic noise (QMN) [62].

QMN is generated by a variety of noisy systems capable of singling out a frequency and can be viewed as the result of filtering broad-band noise through a highly selective system. Examples of such systems include various electromagnetic or acoustic high-Q cavities: their eigenvibrations excited at random by an external noise produce QMN [5], [6]. Another well-known example is that of local and quasi-local (resonant) vibrations of impurities in crystals [63] (see [64] for recent work); such vibrations are mainly characterized by a single frequency and are coupled to a broad band of other modes of the crystal. Their thermal fluctuations are a typical example of QMN. The spectral densities of fluctuations of local and resonant vibrations have been thoroughly investigated both theoretically and experimentally (see [33], [63], and [65] for reviews). Yet other examples of QMN are provided by random vibrations of fragments of macromolecules (in particular, enzymes [66]), and of engineering structures.

The simplest type of QMN (which will be the specific noise that we refer to as QMN below) is that produced by a harmonic oscillator of frequency ω_o

and friction coefficient Γ driven by white noise (cf. (2.21)):

$$\ddot{f} + 2\Gamma\dot{f} + \omega_o^2 f = \xi(t), \qquad \langle \xi(t)\,\xi(0)\rangle = 4\Gamma\tilde{D}\delta(t). \tag{2.88}$$

The power spectrum of QMN (i.e., the spectral density of the fluctuations of the oscillator) is of the form

$$\Phi(\omega) = \frac{4\Gamma\tilde{D}}{(\omega^2 - \omega_o^2)^2 + 4\Gamma^2\omega^2} \tag{2.89}$$

(cf. (2.18) and (2.23)). The characteristic noise intensity $D = \Phi_{max}(\omega)$ as introduced in (2.5) is proportional to the characteristic intensity $\tilde{D}$ of the noise $\xi(t)$ and is equal to

$$\begin{aligned} D &= \frac{4\Gamma\tilde{D}}{\omega_o^4} && \text{for } \omega_o^2 \le 2\Gamma^2, \\ &= \frac{\tilde{D}}{\Gamma(\omega_o^2 - \Gamma^2)} && \text{for } \omega_o^2 \ge 2\Gamma^2. \end{aligned} \tag{2.90}$$

Noise with the power spectrum (2.89) with parameters Γ and ω_o that are unrelated beyond the constraint (2.87) has sometimes been called "harmonic noise" [66]. Such noise is a simple example of colored noise with two correlation times given by Γ^{-1} and ω_o^{-1}. Although strong effects related to the presence of two times are expected to arise (and indeed do arise) when $\omega_o \gg \Gamma$, some features already become apparent for $\omega_o^2 > 2\Gamma^2$ when the peak of the power spectrum $\Phi(\omega)$ in (2.89) is positioned at a finite frequency, in contrast to that of the exponentially correlated noise. Changes in the structure of $\Phi(\omega)$ related to the shift of its peak to finite frequency are expected to affect the fluctuations of the system noticeably when the position of the peak is of the order of the reciprocal relaxation time t_r^{-1} of the system (cf. Fig. 2.3). However, some effects arise even for large t_r. In particular, it can be seen from (2.64) that the parameter $\tilde{t}_c$ in the expression for the correction to the activation energy $R(x)$ when the correlation time of the noise is finite is of the form [45]

$$\tilde{t}_c = -\frac{\omega_o^2 - 4\Gamma^2}{2\Gamma\omega_o^2}, \quad \omega_o t_r \gg 1, \quad \Gamma t_r \gg 1. \tag{2.91}$$

It follows from (2.63) and (2.91) that, depending on the ratio ω_o^2/Γ^2, the correction to $R(x)$ is either positive (if $\omega_o^2 < 4\Gamma^2$) or negative (if $\omega_o^2 > 4\Gamma^2$) and therefore the distribution $p(x)$ becomes either squeezed or extended relative to the distribution associated with white noise.

The most interesting and nontrivial effects associated with a noise that is characterized by two correlation times rather than a single one might be expected to occur when these times are substantially different, as is the case for quasi-monochromatic noise with $\Gamma \ll \omega_o$, and the value of the reciprocal relaxation time lies between them, i.e.,

$$\Gamma \ll t_r^{-1} \ll \omega_o. \tag{2.92}$$

This is the case that is analyzed further in the present section [45], [62]. The analysis is presented in the context of the simplest model of a noise-driven system, as described in (2.38).

2.4.1. Double-adiabatic approximation for a QMN-driven system. To gain insight into the characteristic features of the fluctuations of the dynamical variable $x(t)$ in the parameter range (2.92) we note that, when (2.87) is fulfilled, the QMN $f(t)$ consists mostly of nearly periodic random vibrations of frequency ω_o. In fact, according to (2.88), $f(t)$ is of the form

$$f(t) = \sum_{\nu=\pm} f_\nu(t) e^{i\nu\omega_o t} + \delta f(t), \qquad \langle(\delta f)^2\rangle \ll \langle|f_\pm|^2\rangle. \tag{2.93}$$

The complex amplitudes $f_\pm(t)$ of the random vibrations vary smoothly in time; their correlation time is equal to Γ^{-1},

$$\frac{\langle|\dot{f}_\pm|^2\rangle}{\langle|f_\pm|^2\rangle} \sim \Gamma^2 \ll \omega_o, \tag{2.94}$$

and the nonresonant addition $\delta f(t)$ is small on the average (it has been omitted when estimating $\langle|f_\pm|^2\rangle$).

The rapidly oscillating random force $f(t)$ gives rise to fast oscillations of the noise-driven system, i.e., to rapidly oscillating terms $x_\pm(t)\exp(\pm i\omega_o t)$ in the coordinate $x(t)$ in (2.38) in addition to a smooth contribution $x_c(t)$. Because of the inequality $\omega_o t_r \ll 1$, the complex amplitudes $x_\pm(t)$ of these vibrations as well as the contribution $x_c(t)$ vary smoothly over the period $2\pi/\omega_o$. This makes the concept of nearly periodic vibrations meaningful, and this is the "first" adiabaticity.

The "second" adiabaticity is related to the fact that the position $x_c(t)$ of the center of the forced vibrations depends on their amplitude. In the parameter range $t_r \ll \Gamma^{-1}$ not only do the amplitudes $x_\pm(t)$ of the vibrations of the system follow the amplitudes $f_\pm(t)$ of the vibrations of the force adiabatically, but so does $x_c(t)$ (however, the adiabaticity of $x_c(t)$ breaks down in the vicinity of certain singular points of the distribution; see below).

The above arguments have been constructed having in mind a solution $x(t)$ of the dynamical equation (2.38) in the form of a superposition of fast-oscillating and smooth terms,

$$x(t) = \sum_{\nu=\pm} x_\nu(t) e^{i\nu\omega_o t} + x_c(t). \tag{2.95}$$

Equation (2.38) is the only equation for the variables $x_\pm(t)$ and $x_c(t)$, and therefore two of these three must be specified separately in some fashion. To do this we note that if the amplitudes $x_\pm(t)$ and $x_c(t)$ were time-independent, then the force $U'(x)$ would be a series in $\exp(\pm i\omega_o t)$. The coefficients of this series would be expressed in terms of $x_\pm, x_c$ and they would obviously be of order $t_r^{-1} \approx U''(x^{st})$. The same expansion is physically meaningful if

$x_\pm, x_c$ vary smoothly with t compared to $\exp(\pm i\omega_o t)$. When substituting the corresponding expansion into (2.38) it is convenient to set the x-dependent coefficients of the rapidly oscillating terms $\exp(\pm i\omega_o t)$ on the left-hand side of the equation equal to the corresponding coefficients in $f(t)$ on the right-hand side. Then to lowest order in $(\omega_o t_r)^{-1}$ and Γ/ω_o, i.e., upon neglect of $U'(x)$ and $\dot{x}_\pm$ compared with $\omega_o x_\pm$, one finds

$$x_\nu(t) = (i\nu\omega_o)^{-1} f_\nu(t), \qquad \nu = \pm. \tag{2.96}$$

To write the equation for $x_c(t)$ it is convenient to introduce an auxiliary three-variable-dependent potential $V(x_c, x_+, x_-)$,

$$V(x_c, x_+, x_-) = \frac{1}{2\pi}\int_0^{2\pi} d\psi\, U(x_c + x_+ e^{i\psi} + x_- e^{-i\psi}), \tag{2.97}$$

which is thus the value of $U(x)$ averaged over the period $2\pi/\omega_o$ for constant $x_c, x_\pm$. Then

$$\dot{x}_c = -V_c' + \delta f(t), \qquad V_c' \equiv \frac{\partial}{\partial x_c} V(x_c, x_+, x_-), \tag{2.98}$$

i.e., if one neglects $\delta f(t)$ the function $x_c(t)$ is smooth. It is easy to see that the rapidly oscillating corrections to x_c, e.g., those oscillating as $\exp(\pm 2i\omega_o t)$ and arising from the nonlinearity of $U(x)$ (which has *not* been assumed small), are of order $(\omega_o t_r)^{-1}$. Since the amplitudes $f_\pm(t)$ and thus $x_\pm(t)$ practically do not vary over the time $\sim t_r \ll \Gamma^{-1}$, the solution of (2.98) upon neglecting $\delta f(t)$ is given by the adiabatic approximation:

$$x_c \approx x_c^{ad}, \qquad V_c'(x_c^{ad}, x_+, x_-) = 0, \qquad x_c^{ad} \equiv x_c^{ad}(x_+, x_-). \tag{2.99}$$

Equations (2.96) and (2.99) show that, with varying amplitudes $f_\pm$ of the rapidly oscillating noise components, the amplitude of the forced oscillations of the system varies as well; the latter variation causes the change of the effective potential $V(x_c, x_+, x_-)$ that determines the motion of the center x_c of the vibrations and, because the relaxation time of the system is short compared to the decrement/increment time of $f_\pm$, the center x_c occupies the minimum of the potential V for given x_+, x_-.

A naive picture of the optimal fluctuation that brings the system to a given point x constructed with the help of (2.95), (2.96), and (2.99) is as follows. The amplitude $|f_+(t)|$ of the noise increases from a root-mean-square value of order $\tilde{D}^{1/2}/\omega_o$ with an increment of order Γ, and the amplitude $|x_+(t)|$ increases accordingly. At the same time, the center x_c of the vibrations shifts (initially $x_c = x^{st}$ within an accuracy of $\tilde{D}^{1/2}/\omega_o^2$). Finally the vibrating coordinate $x(t)$ reaches, for the first time, a given value x. This obviously happens at the turning point of the vibrations, i.e., when the deviation of $x(t)$ from the instantaneous position $x_c(t)$ of the center of the vibrations is maximal. If the time is set so that this event occurs at the time origin $t = 0$, then

$$x_c(0) + x_+(0) + x_-(0) = x, \qquad x_+(0) = x_-(0). \tag{2.100}$$

Although the subsequent behavior of the system once it has reached the given x is not of interest in the present context, we note that, with an overwhelming probability, the amplitude $|f_+(t)|$ decays with a decrement Γ for $t > 0$, and the coordinate $x(t)$ follows this decay adiabatically and returns to the stable position.

This picture makes it straightforward to obtain the expression for the activation energy $R(x)$ for reaching a given point x. Indeed, (2.99) and (2.100) give the values of $x_c(0)$ and $x_\pm(0)$ in terms of x, and the value of $|x_+(0)|$ gives the minimal (optimal) value of the amplitude $|f_+(0)|$ of the noise necessary to bring the system to a given x. The statistical distribution of the squared amplitudes $|f_+|^2$ is exponential: it is of the form of the energy distribution of an underdamped oscillator in thermal equilibrium at temperature $\widetilde{D}$,

$$p(|f_+|^2) = (\widetilde{D}/2\omega_o^2)\exp\left(-2\omega_o^2|f_+|^2/\widetilde{D}\right) \tag{2.101}$$

(the energy of the oscillator is $(1/2)\dot{f}^2 + (1/2)\omega_o^2 f^2 = 2\omega_o^2|f_+|^2$). We note that the functions f_+ and f_- here are related to f and $\dot{f}$ via (2.93) without δf together with the relation $\dot{f} = i\omega_o \sum_\nu \nu f_\nu \exp(i\nu\omega_o t)$. The term δf contains the small contributions that would otherwise be included in f_+ and f_- (and would give rise to small rapidly oscillating terms in $f_\pm$). Omission of δf simply places these contributions back in $f_\pm$. Allowing for the inter-relation (2.90) between the maximum value D of the power spectrum of the noise and the effective temperature $\widetilde{D}$, and for (2.96), one obtains

$$R(x) = (2\omega_o^2/\Gamma)|x_+(0)|^2, \tag{2.102}$$

where $x_+(0)$ is related to x via (2.99) and (2.100). A consistent derivation of (2.102) based on the set of variational equations (2.52), (2.53) is given by Dykman [45].

One remarkable feature of the statistical distribution of a QMN-driven system is immediately seen from (2.99), (2.100), and (2.102) for systems with a symmetric potential $U(x)$ (no matter how nonlinear). If $U(x) = U(-x)$ and $x = 0$ is the stable equilibrium position, it is seen from (2.99) that $x_c = 0$ for all $x_\pm$ and thus $x_+(0) = x_-(0) = x/2$, so that

$$R(x) = (\omega_o^2/2\Gamma)x^2, \qquad U(x) = U(-x). \tag{2.103}$$

Thus, according to (2.103) the distribution is independent of the shape of the potential provided the latter is symmetrical. This invariance is a consequence of the fact that it is the amplitude of the rapidly oscillating noise that is fluctuating initially. The system follows these fluctuations adiabatically, and for high ω_o the amplitude of the forced oscillations is nearly independent of the shape of $U(x)$.

The independence of the activation energy $R(x)$ of the curvature $U''(0)$ of the potential $U(x)$ for $x = 0$ and also of its nonlinearity was tested in analog electronic experiments [62]. The data obtained for symmetric potentials are shown in Fig. 2.6. The results were compared with (2.103) and also with the exact expression (2.30) for the Gaussian statistical distribution of a system with the parabolic potential $U(x) = \alpha x^2/2$: the "steepness" parameter β in (2.30) when the power spectrum of the noise is of the form (2.89) takes the form

$$\beta = \frac{\Gamma D(\omega_o^2 - \Gamma^2)}{\alpha\omega_o^2} \frac{\alpha(\alpha^2 + \omega_o^2) - 4\Gamma^2\alpha + 2\Gamma\omega_o^2}{(\alpha^2 + \omega_o^2)^2 - 4\Gamma^2\alpha^2} \tag{2.104}$$

(to lowest order in Γ/α and in α/ω_o the value of β^{-1} as given in (2.104) coincides with $2(dR(x)/dx^2)/D$ as given in (2.103)). The agreement of the data measured for the harmonic potential with the expression (2.104) was excellent within the experimental uncertainty of $\pm 2\%$. Furthermore, the data are nearly independent of the curvature $U''(0) = \alpha$ for a broad range of values of α and of the anharmonicity of the potential $U(x)$.

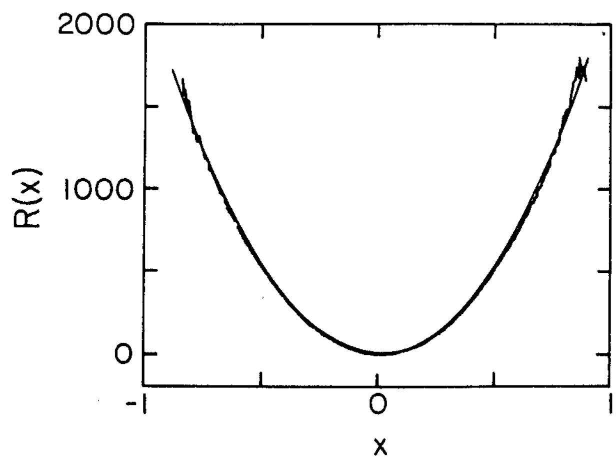

FIG. 2.6. *The activation energy $R(x)$ for reaching a point x by a system* (2.38) *driven by quasi-monochromatic noise in the case of a symmetric parabolic potential $U(x) = (1/2)x^2$, as obtained from an analog experiment* [62]. *The parameters of the QMN are: $\omega_o = 9.81$, $\Gamma = 0.021$, and the characteristic intensity $D = 160$ (note that the effective noise intensity $\Gamma D/\omega_o^2 \approx 0.035 \ll 1$). The experimental data for $U(x) = (1/2)x^2 + (1/4)x^4$, and for $U(x) = (1/6)x^2$ were found to be coincident with those for $U(x) = (1/2)x^2$ within the experimental error. The smooth curve represents the exact theory* (2.30) *with* (2.104).

2.4.2. Quasi-singularity of the activation energy: Breakdown of the adiabatic approximation. For asymmetric potentials $U(x)$ the activation energy $R(x)$ as given in (2.99), (2.100), and (2.102) is asymmetric. The specific feature that follows from these expressions is that $R(x)$ may be singular for some $x = \bar{x}$ if the values of the coordinate that lie beyond $\bar{x}$ (with respect to x^{st}) are adiabatically inaccessible. The onset of such "forbidden" regions can be understood on the basis of our previous description of the motion: with increasing amplitude $|f_+|$ of the noise, the amplitude $|x_+|$ of the forced vibrations increases as well, and as a result the limiting point x of (2.100) reached in the course of the vibrations also shifts. The dependence of the latter shift on the variation of x_+, allowing for the fact that the position of the center of the forced vibrations x_c itself depends on x_+, is of the form

$$\frac{dx}{dx_+} = 2(V''_{cc} - V''_{c+})V''_{cc}, \quad x_+ = x_-, \quad x_c = x_c^{ad}(x_+, x_-). \tag{2.105}$$

Here V'' stands for a second derivative of $V(x_c, x_+, x_-)$ and the subscripts $c, +, -$ stand for differentiation with respect to x_c, x_+, x_-, respectively. We have taken into account that the dependence of x_c on $x_\pm$ in the adiabatic approximation is given by (2.99) and that $V'_+ = V'_-$ when $x_+ = x_-$. It is seen from (2.105) that for $x_\pm = \bar{x}_\pm$ such that

$$V''_{cc}(x_c^{ad}, \bar{x}_+, \bar{x}_-) = V''_{c+}(x_c^{ad}, \bar{x}_+, \bar{x}_-), \tag{2.106}$$

a further increase of the amplitude of the vibrations does not drive the system into the region beyond the point

$$\bar{x} = x_c^{ad}(\bar{x}_+, \bar{x}_-) + \bar{x}_+ + \bar{x}_-. \tag{2.107}$$

This obviously happens because the dependence of the adiabatic position of the center of vibrations, $x_c^{ad}(x_+, x_-)$, on the amplitude $|x_+|$ for $x_\pm = \bar{x}_\pm$ becomes too fast and with increasing amplitude the system as a whole actually moves away from $\bar{x}$. It follows from (2.102) and (2.105)–(2.107) that the activation energy $R(x)$ has a square-root singularity for $x = \bar{x}$ in the adiabatic approximation [45], $R(x) = R(\bar{x}) - [C(x - \bar{x})]^{1/2}$, where the value of C can easily be expressed in terms of the derivatives of the potential V for $x_\pm = \bar{x}^\pm$.

Penetration into the "forbidden" region occurs via fluctuations for which the adiabatic approximation (2.99) does not hold. For high-frequency noise the nonadiabaticity comes primarily from contributions to the variables x_c and $x_\pm$ which, although still slow compared to the rapidly oscillating factors $\exp(\pm i\omega_o t)$, vary over time scales that are not slow compared to the relaxation time of the system, i.e., $\dot{x}_c \sim \dot{x}_\pm \sim t_r^{-1}$. The onset of these terms is related to the onset of a large (compared to its value of $O((\Gamma/\omega_o)|f_+|)$ in the adiabatically accessible range) nonresonant term $\delta f(t)$ in the driving noise (2.93) which varies over the same characteristic time, $|\delta\dot{f}/\delta f| \sim t_r^{-1}$.

To estimate the steepness of the activation energy $R(x)$ in the adiabatically inaccessible range we note that, according to (2.46) and (2.89), the magnitude of the logarithm of the probability of a "smooth" (compared to $\exp(\pm i\omega_o t)$) fluctuation $\delta f(t)$ of duration $\sim t_r$ is of order $\omega_o^2(\delta f)^2 t_r/\Gamma^2 D$. Since in the range of x in question the only time scale for slow variables is t_r, the deviations $x - \bar{x}, x_c - x_c^{ad}(\bar{x}_+, \bar{x}_-), x_\pm - \bar{x}_\pm$ caused by the force $\delta f(t)$ are all of order $\delta f t_r$, and the activation energy $R(x)$ must be of the form $R(x) \sim \omega_o^2(x - \bar{x})^2/t_r\Gamma^2$, i.e., the value of $R(x)$ changes by an order of magnitude compared with the value (2.102) in an extremely narrow interval $|x - \bar{x}| \sim (\Gamma t_r)^{1/2}$.

A consistent theory that describes the smearing out of the square-root singularity of $R(x)$ arising from the adiabatic approximation has been given by Dykman et al. [62]. It is based on the solution of the boundary-value problem (2.49), (2.52), and (2.53) for a QMN-driven system. The result that makes it possible to follow the change of $R(x)$ from the comparatively smooth function far from the threshold point $\bar{x}$ to a steep function beyond $\bar{x}$ is of the form

$$
\begin{aligned}
R(x) &= \frac{2\omega_0^2}{\Gamma}\left(x_+^2(0) + \frac{3}{64}\frac{\delta f^2(0)}{\Gamma V''_{cc}}\right),\\
x &= 2x_+(0) + x_c^{ad}(x_+(0), x_-(0)) + \frac{3}{4}\frac{\delta f(0)}{V''_{cc}},\\
x_+(0) = x_-(0) &= \frac{\delta f(0)}{8\Gamma}\left(1 - \frac{V''_{c+}}{V''_{cc}}\right)\\
&\quad - \frac{\delta f^2(0)}{64\Gamma V''^2_{cc}}\left(7V'''_{cc+} - 5V'''_{ccc} - V'''_{c++} - V'''_{c+-}\right).
\end{aligned}
\tag{2.108}
$$

Here all the derivatives of the potential $V(x_c, x_+, x_-)$ are calculated for $x_\pm = \bar{x}_\pm(0), x_c = x_c^{ad}[\bar{x}_+(0), \bar{x}_-(0)]$; the function $\delta f(0)$ is the value of the smooth component of the noise at the instant $t = 0$ when the system reaches the given x.

Far from the threshold $x = \bar{x}$ in the adiabatically accessible range of x where the force $\delta f(0)$ is small $[O(\Gamma)]$, the expression (2.108) for $R(x)$ coincides with (2.102), and the function $R(x)$ is smooth, $|d\ln R(x)/dx| \sim 1$. In the vicinity of $\bar{x}$ the terms containing the force $\delta f(0)$ become substantial and $R(x)$ becomes steep. Far in the adiabatically inaccessible range $|d\ln R(x)/dx| \sim (\Gamma t_r)^{-1/2} \gg 1$, in complete agreement with the qualitative estimate given above.

We stress that it is the argument of the exponential of the statistical distribution that becomes very steep. Therefore, the distribution $p(x)$ itself is expected to vanish extremely sharply in the adiabatically inaccessible range. In contrast to the change of the character of the distribution at the inflection point of the potential when the correlation time of the noise is large (cf. §2.3.4), the threshold point $\bar{x}$ given by (2.106) and (2.107) is not immediately associated with a singular point of the potential of the system (although it is determined entirely by the potential and does not depend on the parameters of the noise). In this sense the appearance of a singularity in the distribution is "hidden," i.e., there is no a priori reason to expect any unusual behavior at $x = \bar{x}$.

The onset of an extremely sharp behavior of the logarithm of the statistical distribution in QMN-driven systems was observed in an analog electronic experiment [62]. The potential of the system was of the quartic bistable form (2.84). The singular points $\bar{x}$ as given by (2.106) and (2.107) are

$\bar{x} = \pm(5/3)^{1/2}$. If, for example, the system is placed initially in the vicinity of the stable state $x = -1$ and the quasi-stationary distribution over the range of fluctuations about this state is investigated, then only the point $\bar{x} = -(5/3)^{1/2}$ plays a role. The results for $R(x)$ obtained in [62] for this case are shown in Fig. 2.7. It is seen from Fig. 2.7 that $R(x)$ is indeed extremely steep beyond the point $x = -(5/3)^{1/2}$, although it is of course not vertical, as would be predicted within the adiabatic approximation. The nonadiabatic theory (2.108) is in good agreement with the experiment (the data were obtained for $\omega_0 = 9.81, \Gamma = 0.021$ and the theory does not contain any adjustable parameters). Another most peculiar feature of the quasi-stationary distribution plotted in Fig. 2.7 is that it spreads beyond the point $x = 0$, i.e., over the range of attraction of another stable state. We shall address this feature in §2.6 in the context of the transition probabilities between stable states, in particular those for QMN-driven systems.

It follows from the results of the present section that for a narrow-band high-frequency driving noise the shape of the statistical distribution of the system is not determined by the fluctuations at frequencies lower than the reciprocal relaxation time of the system (cf. Fig. 2.3). Instead, the distribution is determined by the high-frequency fluctuations. The shape of the tails of the distribution is qualitatively different from that for white-noise-driven systems, and the logarithm of the distribution (and not only the distribution itself) can be extremely steep.

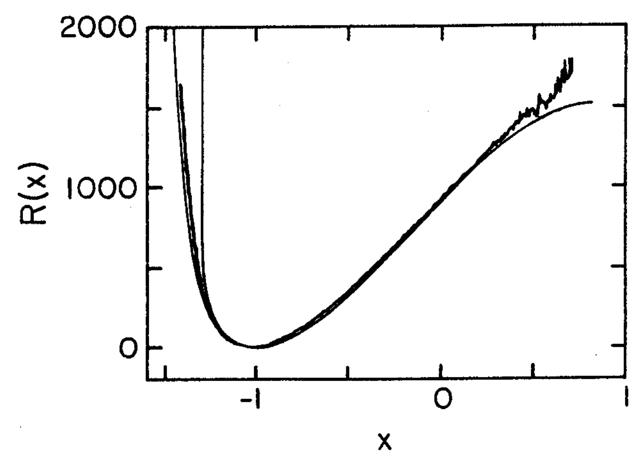

FIG. 2.7. *The activation energy $R(x)$ for reaching a point x starting from an equilibrium position $x^{st} = -1$ in the QMN-driven system with potential $U(x) = -(1/2)x^2 + (1/4)x^4$ as obtained from an analog experiment* [62]. *The eigenfrequency and bandwidth of the QMN are the same as in Fig.* 2.6, *and the noise intensity is $D = 189$. The experimental data (jagged line) are compared with the double-adiabatic theory (full curve, singular at $x = -\sqrt{5/3}$) and the expression* (2.108)*(the nonsingular full curve).*

2.5. Pre-history problem

The above analysis of large fluctuations in a system driven by Gaussian noise was based on the concept of the optimal path. It was assumed on physical grounds, and then demonstrated by making use of the path-integral method, that among the various paths along which the system can arrive at a given point x there is an optimal path that corresponds to the most probable fluctuation among very infrequent ones. This path starts in the vicinity of the stable equilibrium point x^{st}, and the duration of the motion far from the range of small fluctuations about x^{st} is given by the time t_{o} of (2.39), namely, by the larger of the relaxation time t_r of the system and the noise correlation time t_c. Although the concept is physically and mathematically clear, the optimal paths of both the system and the noise have so far only been described as the solutions of the variational problem (2.48).

In the present section (see also [67]) we introduce physically observable characteristics related to the optimal path of a system that further elucidate the approach. Such characteristics can be expressed in terms of the *"pre-history" probability density* $p_h(x,t;x_f,t_f)$ that the system fluctuating about a given stable state x^{st} was at a point x at time $t < t_f$, given that at time t_f it is at x_f. The calculation of this probability density constitutes a *pre-history problem.* In the general case of a multistable system $p_h(x,t;x_f,t_f)$ is not a standard two-time conditional probability; rather, it is the ratio of the three-time transition probability density $w(x_f,t_f;x,t;x_i,t_i)$ (the probability density of the transitions $x_i \to x \to x_f$) to the standard two-time transition probability density $w(x_f,t_f;x_i,t_i)$, with the initial value of the coordinate x_i lying in the vicinity of x^{st} and the initial instant t_i having been assumed to be such that $t-t_i$ and t_f-t_i substantially exceed the relaxation time of the system and all correlation times of the noise but, at the same time, are small compared to the reciprocal probability W^{-1} of the escape from the given stable state (see §2.6). As a consequence, both t_i and x_i have ostensibly dropped out from $p_h(x,t,;x_f,t_f)$,

$$p_h(x,t;x_f,t_f) = \frac{w(x_f,t_f;x,t;x_i,t_i)}{w(x_f,t_f;x_i,t_i)}, \qquad W^{-1} \gg t-t_i,\ t_f-t_i \gg t_c, t_r. \tag{2.109}$$

From the definition of the pre-history probability density it obviously follows that

$$\int dx\, p_h(x,t;x_f,t_f) = 1. \tag{2.110}$$

Under quasi-stationary conditions only the instant t_f when the system has been observed at x_f is singled out, and therefore

$$p_h(x,t;x_f,t_f) = p_h(x,t-t_f;x_f,0). \tag{2.111}$$

The concept of the pre-history probability density appeals to the fact that the visits of the system to a given point x_f are infrequent and that the corresponding noise outbursts leading to these visits are therefore mutually independent and uncorrelated. The probability density of crossing a given point at a given instant before arrival at a final destination x_f is then precisely the quantity that characterizes the distribution of the paths arriving at this final point. Since (by definition) the optimal path $x_{opt}(t; x_f)$ for reaching x_f at the instant $t = 0$ is the most probable of these paths, the function $p_h(x, t; x_f, t_f)$ for a given $t - t_f$ should have a sharp maximum if x lies on this path, i.e., if $x = x_{opt}(t - t_f; x_f)$.

By investigating the pre-history probability density experimentally one can therefore not only visualize the optimal paths themselves, but one can also directly test the general concepts of the optimal path and the optimal fluctuation presented in §2.3. Such an investigation should establish the range of parameters within which the concepts are applicable to a given system and for a given type of noise.

2.5.1. General expression for the pre-history probability density. The pre-history probability density p_h depends on the stable state that had been occupied initially and from which the system arrives at x and x_f. Persisting with our goal of simplicity, in what follows we give the theoretical formulation for the case of a monostable system; the generalization to the case of multistable systems is straightforward (the experimental data discussed below refer to a bistable system). We note that for a monostable system one can set $t_i = -\infty$ in (2.109), and $p_h(x, t; x_f, t_f)$ can then be expressed in terms of a two-time probability density: it is given by the ratio $p_2(x, t, ; x_f, t_f)/p(x_f, t_f)$. The other simplification made below is that the pre-history problem will be formulated in the context of the simplest type of equation of motion of a noise-driven nonlinear system, namely, (2.38). As always, we will assume the driving noise to be Gaussian. It is convenient to express the two- and three-time transition probability densities of (2.109) in the form of a path integral similar to that used in (2.41) and based on the fact [36] that each path of the noise $f(t)$ generates a certain trajectory $x(t)$ of the dynamical system. The weighting factor in the integral over the paths of the noise is the probability density functional $\mathcal{D}[f(t)]$ that determines the probability density of a given realization $f(t)$. The appropriate expression follows from (2.109) and is

$$(2.112)\quad p_h(x, t; x_f, t_f) = \frac{\int_{x(-\infty)=x_i} \mathcal{D}f(t')\, \wp[f(t')]\delta[x(t) - x]\delta[x(t_f) - x_f]}{\int_{x(-\infty)=x_i} \mathcal{D}f(t')\, \wp[f(t')]\delta[x(t_f) - x_f]}.$$

Here, just as in (2.41), the δ-functions and the lower limits of the integrals show that the paths $f(t')$ that contribute to the transition probability densities are those that generate system paths that start at a point x_i a long time before arrival at x_f, pass through x_f at the instant t_f, and, in the case of the three-time transition probability density $w(x_f, t_f; x, t; x_i, -\infty)$, also pass through the point x at the instant t. The explicit form of the probability density functional $\wp[f(t)]$ in the case of colored Gaussian noise with the power spectrum $\Phi(\omega)$ is given in (2.46).

The realizations of the force $f(t)$ in (2.112) that bring the system to given point(s) at given instant(s) are "deterministic," i.e., although of fluctuational origin, they do not depend on the noise intensity D. However, their probabilities do, and therefore for sufficiently small D both numerator and denominator in (2.112) are exponentially small (the denominator was already considered in §2.3). It is thus reasonable to calculate them to logarithmic accuracy and to write the pre-history probability density as

$$\begin{aligned} p_h(x,t;x_f,t_f) &\equiv p_h(x,t-t_f;x_f,0) \\ &= C(x,x_f;t-t_f)\exp\left(-\rho(x,x_f;t-t_f)/D\right), \end{aligned} \tag{2.113}$$

where $C(x,x_f;t-t_f)$ is a smooth function and where

$$\begin{aligned} &\rho(x,x_f;t) = \widetilde{\rho}(x,x_f;t) - \widetilde{\rho}(x_f,x_f;0), \\ &\widetilde{\rho}(x,x_f;t) \gg D, \qquad \widetilde{\rho}(x_f,x_f;0) \gg D. \end{aligned} \tag{2.114}$$

The two terms in $\rho(x,x_f;t)$ come from the two transition probability densities in (2.109). They are determined by the values of the argument of the exponential in the probability density functional (2.46) for the optimal paths of the noise that result in the appropriate trajectories of the system. The inequalities in (2.114) express how weak the noise intensity D must be for the concept of the optimal path to be applicable. The further analysis is completely analogous to that in §2.3 and shows that the variational problem for the optimal path that arrives at the point x_f at the (running) time $t'=0$ via the point x at the instant $t'=t$ is of the form

$$\begin{aligned} &\widetilde{\rho}(x,x_f;t) \equiv \min \Re[f(t'),x(t');x,x_f;t], \\ &\Re[f(t'),x(t');x,x_f;t] = \frac{1}{2}\int_{-\infty}^{\infty} dt'\, f(t')F(-id/dt')f(t') \\ &\qquad\qquad + \int_{-\infty}^{0} dt'\, \lambda(t')[\dot{x} + U'(x) - f(t')], \end{aligned} \tag{2.115}$$

with the boundary conditions

$$\begin{aligned} &x(-\infty) = x^{st}, \quad x(t) = x, \quad x(0) = x_f, \\ &f(\pm\infty) = 0, \\ &\lambda(-\infty) = 0, \qquad \lambda(t'>0) = 0. \end{aligned} \tag{2.116}$$

In a sense, the functional $\Re$ in (2.115) coincides exactly with that in (2.48) and, as in (2.48), the minimum in (2.115) should be taken with respect to $f(t')$ and $x(t')$ independently since their inter-relation as given by the equation of motion (2.38) has been taken into account by introducing the undetermined coefficient $\lambda(t')$. The difference between the present variational problem and that considered in §2.3 for the optimal path arriving at x_f directly lies in the boundary conditions: in the present case the system is to pass a given point x at time t before arriving at x_f. Obviously, the value of $\widetilde{\rho}(x_f, x_f; 0)$ as given by (2.115) and (2.116) coincides exactly with the activation energy (2.48) for reaching the point x_f,

$$\widetilde{\rho}(x_f, x_f; 0) = R(x_f), \tag{2.117}$$

which is not surprising since $R(x_f)$ in (2.48) determines the logarithm of the transition probability density $w(x_f, 0; x^{st}, -\infty)$.

Since $\widetilde{\rho}(x_f, x_f; 0)$ is determined by the optimal path $x_{opt}(t'; x_f)$ that arrives at x_f without the additional constraint of passing a given point at a given previous time, one has the inequality $\widetilde{\rho}(x, x_f; t) \geq \widetilde{\rho}(x_f, x_f; 0)$. It is obvious from (2.115) and (2.116) that the equality holds only when x coincides with $x_{opt}(t; x_f)$ (for a given t). Thus the maximum of the probability density $p_h(x, t; x_f, 0)$ is indeed achieved on the optimal path,

$$\rho[x_{opt}(t; x_f), x_f; t] = 0. \tag{2.118}$$

According to (2.114) and (2.115), p_h decreases exponentially away from the optimal path, and for weak noise its shape is Gaussian near the maximum, i.e.,

$$\rho(x, x_f; t) \approx \frac{[x - x_{opt}(t; x_f)]^2}{2\sigma(t; x_f)} \tag{2.119}$$

when $|x - x_{opt}(t; x_f)|$ is sufficiently small (in particular, as compared to $|x^{st} - x_f|$). The dispersion of $p_h(x, t; x_f, 0)$ has been denoted by $D\sigma(t; x_f)$. In view of the normalization (2.110), equations (2.113), (2.114), and (2.119) give not only the argument of the exponential but also the prefactor in the pre-history probability density,

$$p_h(x, t; x_f, 0) = [2\pi D\sigma(t; x_f)]^{-1/2} e^{-[x - x_{opt}(t; x_f)]^2 / 2D\sigma(t; x_f)}. \tag{2.120}$$

(We have taken into account that the prefactor C in (2.113) and (2.114) is smooth on the scale $\Delta x \approx (D\sigma)^{1/2}$ and have therefore set it equal to its value at $x = x_{opt}(t; x_f)$.)

The analysis of the pre-history transition probability has thus been reduced to the calculation of two functions: the optimal path $x_{opt}(t; x_f)$, and the dispersion parameter $\sigma(t; x_f)$. The calculation of these functions for various types of power spectra of the noise can be done numerically. (The variational equations are obviously of the form (2.52) and (2.53), but the boundary conditions for the problem of the optimal path itself and the pre-history problem as a whole are different.) In some cases these functions can be calculated analytically.

2.5.2. Pre-history probability density for systems driven by white noise. Simple analytical results can be obtained if the driving noise is white, whence the operator F in (2.115) is equal to 1. The optimal path in this case is given by (2.62), while the expression for $\sigma(t;x_f)$ can be shown to be of the form [67]

$$\sigma(t;x_f) \equiv \tilde{\sigma}[x_{opt}(t;x_f);x_f], \tag{2.121}$$

with

$$\tilde{\sigma}(x;x_f) = [U'(x)]^2 \int_x^{x_f} dy\, [U'(y)]^{-3}. \tag{2.122}$$

Experimental data [67] on the pre-history probability density are shown in Figs. 2.8 and 2.9. They were obtained via analog electronic simulation of Brownian motion as described by (2.38) with the potential (2.84). The system was initially placed in a state within the range of attraction of the stable equilibrium position $x^{st} = -1$. The paths that arrived at a given point x_f were stored, and it was their distribution that gave, by definition, the pre-history probability density. It is seen clearly in Fig. 2.8 that the function $p_h(x,t;x_f,t_f)$ obtained in this way exhibits a very sharp ridge. This ridge provides the visualization of the optimal path. The shape of the optimal path $x_{opt}(t;x_f)$, and the characteristic width $\tilde{\sigma}(x;x_f)$ as defined by assuming the function p_h to be Gaussian near the maximum, are shown in Fig. 2.9. It is obvious that the function $\tilde{\sigma}(x;x_f)$ is strongly nonmonotonic: the pre-history probability density is very sharp for x near the final point x_f and its peak is

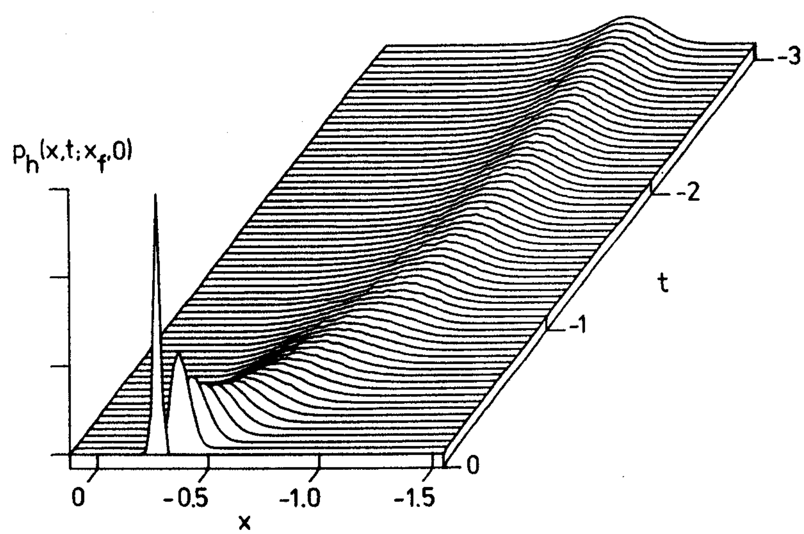

FIG. 2.8. *The pre-history probability density $p_h(x,t;x_f,0)$ for the white-noise-driven system with the bistable quartic potential* (2.84) [67]. *The final position is $x_f = -0.3$, and the characteristic noise intensity is $D = 0.07$.*

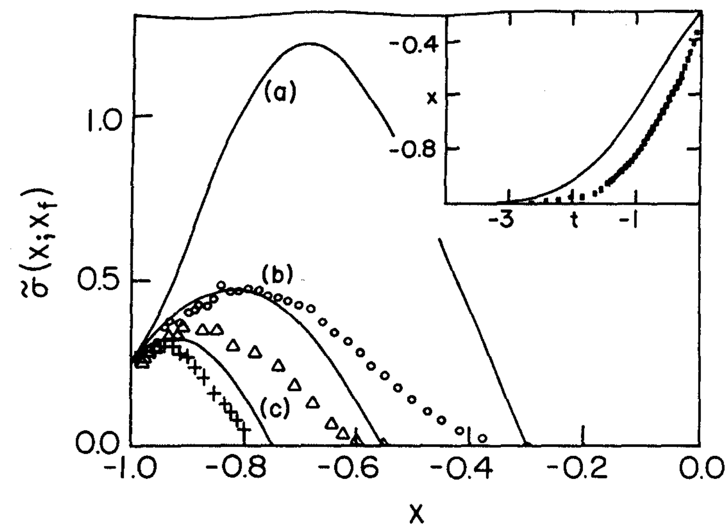

FIG. 2.9. *The dispersion parameter* $\tilde{\sigma}(x; x_f)$ *of the pre-history probability density (cf. Fig.* 2.8) *as a function of* x *measured in the analog experiment and compared with the theory* (2.122) (*curves*) [67] *for* (a) $x_f = -0.30$, $D = 0.07$ (*circles*); (b) $x_f = -0.55$, $D = 0.0265$ (*triangles*); (c) $x_f = -0.75$, $D = 0.0085$ (*pluses*). *Inset: the optimal path* (*curve*) *as given by* (2.62) *is compared with a path along the ridge* (*data points*) *of the experimental distribution* $p_h(x, t; x_f, 0)$ *for* $x_f = -0.30$, $D = 0.07$.

comparatively narrow for x close to the equilibrium position $x^{st} = -1$, while for intermediate x it is much flatter and the width $\tilde{\sigma}(x; x_f)$ is correspondingly larger.

The surprising behavior of the width $\tilde{\sigma}(x; x_f)$ of the ridge of the pre-history probability density observed in [67] can be explained on the basis of (2.121) and (2.122). In particular, it immediately follows from these equations that for x close to x_f, $\tilde{\sigma}(x; x_f) = (x_f - x)/U'(x_f)$ is small, i.e., the bunch of paths that have arrived at x_f is sharply squeezed in the immediate vicinity of x_f. This is also evident from qualitative arguments: for x_f remote from stationary points the velocity $|U'(x)|$ of the motion along the optimal path is of order unity, and the motion from x to x_f for x close to x_f takes a short time. Therefore, the diffusional smearing (which gives rise to smearing of the bunch of paths) should be small. It is also evident that for x close to the stable state x^{st} the width of the peak of p_h should coincide with that of the stationary statistical distribution of the system and that in fact $p_h(x, t; x_f, t_f) \to p(x)$ for $t \to -\infty$, where $p(x)$ is the stationary distribution considered in §§2.2 and 2.3 (the distribution over x does not depend on whether in the far future $(t_f - t \to \infty)$ there might be an outburst of noise that drives the system to x_f). Both of these features are seen very clearly in Fig. 2.8. The dome-like shape of $\tilde{\sigma}(x; x_f)$ is related to the particular shape of the potential (2.84); we note, however, that such a shape will always arise for bistable potentials provided

x_f is sufficiently close to the saddle point x_s [$(U'(x_s) = 0$ and $U''(x_s) < 0)$] because it follows from (2.121) and (2.122) [67] that $\widetilde{\sigma}(x; x_f)$ is proportional to $(x_f - x_s)^{-2}$ for small $|x_f - x_s|$ and both $|x - x_s|$ and $|x - x^{st}| \gg |x_f - x_s|$, i.e., for x between x^{st} and x_f the dispersion $\widetilde{\sigma}(x, x_f)$ does become large.

It is evident from Fig. 2.9 that the expression (2.121) for $\widetilde{\sigma}(x; x_f)$ explains the experimental data, at least qualitatively. The discrepancy between theory and experiment is most likely related to the fact that the noise intensities investigated experimentally were not sufficiently small for the theory to be applicable; in particular, the width of the peak of p_h is seen from Fig. 2.8 to be comparable to the distance between the singular points of the potential $U(x)$ of the system, while in the theory the width of the peak has been assumed to be much smaller. Another important point to be addressed in the future is the interpretation of the data when x_f approaches a saddle point where $\widetilde{\sigma}(x; x_f)$ diverges and the present theory becomes inapplicable.

It follows from the results of the present section that the formulation of the pre-history problem has made it possible to visualize optimal paths in noise-driven systems and to investigate their statistical distribution. Through this approach it has been possible to provide direct experimental verification of the fundamental concept of the optimal path.

2.6. Probabilities of fluctuational transitions between coexisting stable states of noise-driven systems

It is a feature of many physical systems that they have two or more coexisting stable states. Among the many examples of such systems we mention interstitial atoms or molecules in solids that can occupy any elementary cell with equal probability [3], [4], active (lasers) and passive optically bistable and multistable devices (see [68]–[70] and references therein), a relativistic electron in a Penning trap that displays bistability when excited by cyclotron resonant radiation [71], and biased Josephson junctions with coexisting oscillatory and steady states [7], [12]. A feature common to systems with coexisting stable states is the possible occurrence of fluctuational transitions (switchings) between these states. Because of its broad importance and interest, the problem of the transition probabilities between stable states has been considered in a large number of papers (see the reviews [12]–[17], [33], and [59] and references therein); it was probably Kramers' paper [72] that most influenced the modern developments in this field. (We note that in spite of the apparent simplicity of the formulation, the complete solution of the Kramers problem of the escape of a white-noise-driven particle with one degree of freedom from a potential well has only been obtained recently (cf. [73] and references therein).) The effects of color of the driving noise are still a matter of vivid discussion, although some results and some concepts have already been well established.

The physical concept of the probability W of a transition between stable states or, equivalently, of the probability of escape from a stable state, is based on the very fact that this probability is much smaller than both the relaxation rate t_r^{-1} of the system and the inverse correlation time t_c^{-1} of the noise,

$$Wt_o \ll 1, \qquad t_o = \max(t_r, t_c). \tag{2.123}$$

If the condition (2.123) is fulfilled, there are two clearly distinct time scales. Within a time $\sim t_o$, the system placed initially somewhere in the range of attraction of a given stable state (attractor) in the phase space approaches this attractor with an overwhelming probability for sufficiently weak noise, and forgets the initial position. Simultaneously, the noise correlations decay, i.e., the initial state of both the system and the noise are forgotten. For a while the system fluctuates about the attractor. The escape from the attractor occurs as a result of a large occasional outburst of the noise which drives the system away from the region in phase space associated with the initially occupied attractor (drives the system to another attractor around which the system now fluctuates). The average frequency of such outbursts is given by W, and therefore the probability that such an outburst occurs over the time t_o is very small according to (2.123), thus leading to a self-consistent picture. Only if this description is valid can one appeal in a meaningful way to the notion of the probability of the transition (escape) from a given stable state; otherwise the transition probability would depend on the initial position of the system and/or the initial state of the noise, and one would arrive at a continuum of "transition probabilities" when considering a distribution of initial states. Such a continuum does not have much in common with an intuitively clear rate of the transition under consideration.

Since a statistical distribution in the vicinity of a stable state is generated over a time of order t_o regardless of the initial state of the system, it is obvious from the above picture that the results for the distribution of noise-driven systems considered in §§2.2–2.5 hold not only for monostable, but also for bistable and multistable systems. However, in these latter cases they yield not a stationary but a quasi-stationary statistical distribution for the population around an attractor; its integral over the corresponding region of the phase space (the total population around the attractor) slowly evolves in time because of transitions away from the attractor.

The criterion (2.123) places a restriction on the intensity of the noise: only for sufficiently weak noise is the concept of a transition probability sensible. For example, if a system is fluctuating in a double-well potential (cf. Fig. 2.10) and the noise is so strong that motion over the barrier is strongly excited, the concept of transitions between potential wells is obviously meaningless because the system is in fact not located in any single well. It was demonstrated earlier that, for Gaussian noise, the probabilities of the large outbursts, including those necessary for a transition to occur when the noise is weak, are exponentially small. As before, it is therefore most interesting to calculate these probabilities to logarithmic accuracy in the noise intensity. The results are presented below.

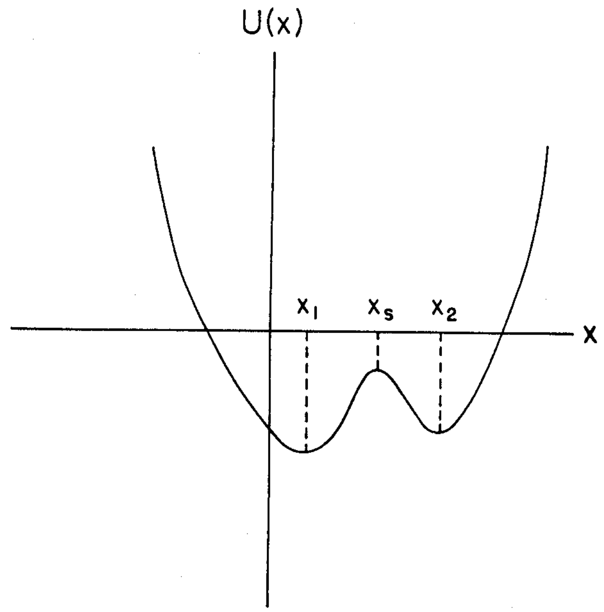

FIG. 2.10. *Sketch of a bistable potential $U(x)$; $x_{1,2}$ are the stable equilibrium positions, x_s is the saddle point.*

2.6.1. Method of optimal path in the problem of fluctuational transitions. As in the case of the rare fluctuations that determine the tails of the statistical distribution, the probabilities of different fluctuations (realizations of the paths of noise and of the corresponding paths of the system) that result in transitions between stable states differ exponentially strongly from each other when the noise is weak. Therefore, to logarithmic accuracy, the value of the probability W_{ij} of a transition from the ith to the jth stable state is given by the probability of realization of the most probable (optimal) appropriate path of the noise and, correspondingly, of the optimal path of the system. We shall consider W_{ij} for the simplest case when a system is described by one dynamical variable $x(t)$ and the equation of motion is of the form (2.38). As always, $f(t)$ is zero-mean Gaussian noise with the power spectrum (2.2) and the probability density functional (2.46). We assume the potential $U(x)$ of the system to be a double well (see Fig. 2.10); the stable states of the system are positioned at the minima of the potential, x_1 and x_2, and the local maximum of $U(x), x_s$, is the saddle point.

The difference between the problem of the tails of the statistical distribution and the problem of the transition probabilities lies in the following. In the former problem the further destiny of the system after its arrival at a given point in the phase space as a result of the large fluctuation was not of interest. The force $f(t)$ did not vanish at the moment $\tilde{t}$ of arrival ($\tilde{t} = 0$ in the variational functional (2.48)), and in the course of its decay for $t > \tilde{t}$ this force drove the system back toward the stable state occupied initially. In the case of the optimal path resulting in a transition, on the other hand, the system that was initially in one stable state (the state i) is to be in another stable state (the state j) when the optimal fluctuation of the noise has died out and $f(t)$ and its derivatives have become zero (to within their root-mean-square values proportional to $D^{1/2}$). Therefore, to logarithmic accuracy we write

$$W = C' t_o^{-1} e^{-R_i/D}, \tag{2.124}$$

where, for the system described by (2.38), R_i is the solution of the variational problem

$$\begin{aligned} R_i &= \min \Re[f(t), x(t)], \\ \Re[f(t), x(t)] &= \frac{1}{2}\int_{-\infty}^{\infty} dt\, f(t) F(-id/dt) f(t) \\ &\quad + \int_{-\infty}^{\infty} dt\, \lambda(t)[\dot{x} + U'(x) - f(t)]. \end{aligned} \tag{2.125}$$

Equations (2.124) and (2.125) are similar to (2.47) and (2.48), and just as in (2.48) the minimum here should be taken with respect to $f(t)$ and $x(t)$ independently; $\lambda(t)$ is an undetermined Lagrange coefficient. However, in contrast to (2.48), the upper limit in the second integral is not the instant of arrival at a given point but $+\infty$: as explained above, it is necessary to know in the present problem where the system is when $f(t)$ has decayed to zero, i.e., for $t \to +\infty$. Obviously, by that time the system can be not only in the stable state j but at any point in the range of attraction of the state j, including the boundary point x_s (the probability of a transition from x_s to x_j caused by the small fluctuations is $\sim 1/2$).

The point x_s is precisely the point where the optimal path of the system should end in the problem of the transition probability. Indeed, this is a stationary point, $U'(x_s) = 0$, and thus there occurs a slowing down of the motion of the system for $f(t) = 0$, i.e., the conditions of approaching x_s and of vanishing of $f(t)$ and its derivatives are fulfilled self-consistently for $t \to +\infty, x \to x_s$. Self-consistency requires that $\lambda(t)$ also vanish for $t \to \infty, x \to x_s$; this follows from the variational equation $\dot{\lambda}(t) = U''(x)\lambda(t)$ (cf. (2.52) and (2.53)) and from the fact that $U''(x_s) < 0$. Therefore, the boundary conditions for the variational problem (2.125) for the transition from the ith stable state are of the form

$$x(-\infty) = x_i, \quad x(\infty) = x_s, \quad f(\pm\infty) = 0, \quad \lambda(\pm\infty) = 0. \tag{2.126}$$

The boundary conditions are obviously of great importance in the problem of the transition probability. It is an advantage of the present path-integral formulation based on physically clear concepts that it facilitates the formulation of boundary conditions. We note that in the important case of noise $f(t)$ that is a component of an N-component Markov process (this occurs when $F(\omega) \propto \Phi^{-1}(\omega)$ is a polynomial of degree N in ω^2 and includes the case of exponentially correlated noise), the conditions (2.126) can be obtained in a different way [45]. In this case one can consider fluctuational transitions between stable states of an $(N+1)$-component Markov system (the $(N+1)$st component is the dynamical variable $x(t)$ itself) driven by white noise. To cause a transition, this noise must bring the system to a hypersurface that separates the ranges of attraction of two different stable states; the system will then go on to the other stable state from that originally occupied with a probability $\sim 1/2$. If we are interested in calculating the transition probability to logarithmic accuracy we must optimize not only over the paths of the noise but also over the final point of the system on the separating hypersurface [38]. It is precisely the point $f = \dot{f} = \cdots = f^{(N-1)} = 0, x = x_s$ (the saddle point of the multidimensional process) that gives the maximum probability, in complete agreement with (2.126).

The variational equations for the optimal paths $f_{opt}(t)$ of the noise and $x_{opt}(t)$ of the system that follow from (2.125) are of the form (2.52) and (2.53). It is a general feature of systems driven by colored noise, however, that the solutions for the problem of the tails of the statistical distribution and for that of the transition probability are quite different: it does not follow from the fact that the system has reached a saddle point that it will then go to a different stable state with probability $\sim 1/2$. In fact, in the general case (an example is given in the next subsection) it will come back to the initially occupied state with overwhelming probability, i.e.,

$$R_i \geq R(x_s; x_i). \tag{2.127}$$

Here $R(x_s; x_i)$ is the activation energy (2.48) for reaching the point x_s if the stable state i was occupied initially. The inequality (2.127) shows that the mean first-passage time to the point x_s does *not* give the reciprocal transition probability—the latter is in general exponentially larger than the former.

2.6.2. Transition probabilities for particular types of noise. The activation energy R_i for the transition from the ith stable state can be evaluated in explicit form in some limiting cases. The simplest case is that of a short correlation time t_c of the noise, such that the bandwidth of the noise substantially exceeds the reciprocal relaxation time of the system (see Fig. 2.3(a)). To zeroth order in t_c, i.e., in the white-noise limit, the solution of the variational equations for $f(t), x(t), \lambda(t)$ is of the form (2.62). The color-induced correction can be obtained from (2.125) by noting that the operator F is a series in $t_c^2 d^2/dt^2$; to find the lowest-order correction in t_c it suffices to allow for the linear term in this series in (2.125), while keeping for $f(t)$ the corresponding zero-t_c approximation (2.62). (This is a standard trick in the perturbation theory for variational problems [57] which does not work, however, for corrections to the statistical distribution that are nonanalytic in t_c^2; see §2.3.) The result is of the form [45]

$$R_i = 2F(0)[U(x_s) - U(x_i)] + F''(0) \int_{x_i}^{x_s} dx\, U'(x)[U''(x)]^2, \tag{2.128}$$

where

$$F''(\omega) \equiv \frac{d^2F(\omega)}{d\omega^2}, \qquad \left|\frac{F''(0)}{F(0)}\right| \ll t_r^2. \tag{2.129}$$

The sign of the color-induced correction coincides with that of $F''(0)$. This means that if the power spectrum $\Phi(\omega)$ of the noise has a maximum at $\omega = 0$, then the correction is positive, i.e., the color causes the transition probability to decrease, while in the opposite case (as, for example, when noise is produced by a harmonic oscillator described by (2.88) with $\omega_0^2 > 2\Gamma^2$ that filters white noise) the transition probability exceeds its white-noise-limit value. We note that the change of the transition probability due to the noise color is exponentially strong when the correction to R_i, although small compared to the main term, is nevertheless large compared to D. In the particular case of exponentially correlated noise (2.6), $F''(0)/F(0) = 2t_c^2$, and (2.128) goes over into the result of Klosek-Dygas et al. [49] obtained by seeking the solution of the appropriate Fokker–Planck equation (2.66) (or its adjoint equation for the mean first-passage time) in the eikonal approximation; not only the argument of the exponential but also a prefactor in the expression for the transition probability were obtained in [49]. A systematic analysis of the color-induced corrections to transition probabilities for exponentially correlated noise was given by Bray et al. [39], [43] using a path-integral formulation somewhat different from the present one (see §2.3); by making use of an instanton technique it was also possible to obtain a prefactor within this formulation for $t_c \ll t_r$ [74].

In the case of noise with a long correlation time such that the width of the power spectrum $\Phi(\omega)$ of the noise is small compared to t_r^{-1} (see Fig. 2.3(b)), as explained in §2.3 the system follows the noise adiabatically and occupies a minimum of the adiabatic potential $U(x) - xf(t)$ for $f(t)/U'(x^{infl}) < 1$ (x^{infl} is the inflection point of $U(x)$); see Fig. 2.4. When $f(t) = U'(x^{infl})$ this minimum transforms into the inflection point. For even larger $f(t)/U'(x^{infl})$ the special character of this point disappears, and the system "rolls down" to another stable state. Therefore, the transition probability to logarithmic accuracy is equal to the probability for $f(t)$ to take on the value $U'(x^{infl})$ (these arguments were given by Tsironis and Grigolini [60] specifically for exponentially correlated noise, but they certainly hold for other types of noise as well), i.e.,

$$R_i = \frac{D}{2\phi(0)}[U'(x^{infl})]^2, \qquad t_c \gg t_r \tag{2.130}$$

(see Dykman [45]; the corresponding expression for exponentially correlated noise was first obtained by Luciani and Verga [46]). As explained in §2.3, the correction to (2.130) is nonanalytic in t_r/t_c; it was obtained for exponentially correlated noise by Bray et al. [43] and is of the form

$$\delta R_i = \Lambda R_i t_c^{-2/3}[U'(x^{infl})|U'''(x^{infl})|]^{-1/3}, \qquad \Lambda \approx 2.05. \tag{2.131}$$

We note that the onset of substantial corrections to (2.130) related to the slowing down of the motion of the system near the minimum of the adiabatic potential when $f(t)$ approaches $U'(x^{infl})$ was noticed in [61].

The numerical results obtained by Bray et al. [43] for the activation energy of a transition of a system with a symmetric double-well potential of the form (2.84) driven by exponentially correlated noise are shown in Fig. 2.11. As might be expected upon inspection of (2.128)–(2.131), although the activation energy as a function of the correlation time of the noise, $R_i(t_c)$, increases monotonically with increasing t_c, the first derivative $R_i'(t_c)$ is nonmonotonic. To make the features of $R_i(t_c)$ more evident, the renormalized quantity [43] proportional to $[R_i(t_c) - R_i(0)]/t_c$ has been plotted in Fig. 2.11. It has a maximum at $\log(t_c/t_r) \approx 1.1$. These results and the nonmonotonicity of $[R_i(t_c) - R_i(0)]/t_c$ in particular have been confirmed quantitatively in detailed Monte Carlo simulations in [75].

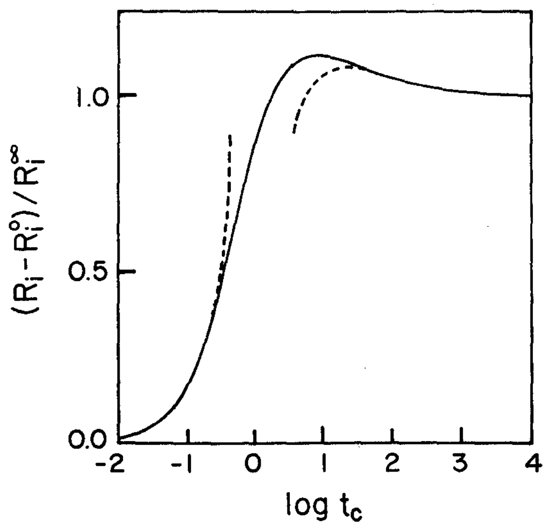

FIG. 2.11. *The activation energy $R = R_1 = R_2$ for the transitions between stable states of a symmetrical system with the potential $U(x) = -(1/2)x^2 + (1/4)x^4$ as a function of the correlation time t_c of the exponentially correlated driving noise* [43]. *The asymptotes given by the expressions* (2.128) (*the small t_c limit*), *and* (2.130), (2.131) (*the large t_c limit*) *are shown dashed. The value of R^o is that of R for $t_c = 0$, while R^∞ is given by* (2.130), *$R^\infty = 4t_c/27$, for the particular type of noise and the potential $U(x)$ considered in* [43].

2.6.3. Quasi-monochromatic noise. The features of the escape from a stable state related to the color of the driving noise are even more distinct when the noise has "true color," i.e., when its power spectrum $\Phi(\omega)$ contains a narrow peak at a finite frequency. We illustrate these features by considering as an example the QMN considered in §2.6. The shape of the power spectrum of this noise is given by (2.89), and we assume that the position ω_0 of the maximum of the spectral peak, the halfwidth Γ of the peak, and the relaxation time t_r of the system satisfy the inequality (2.92), $\Gamma \ll t_r^{-1} \ll \omega_0$. As explained in §2.4, in this case the motion of the system is a superposition of fast random oscillations and a smooth shift of their center. The amplitude of the oscillations is proportional to the amplitude of the oscillations of the noise, while the phase is shifted with respect to that of the noise oscillations by $\pi/2$. The amplitude varies over the characteristic time Γ^{-1}, which is the correlation time of the noise. The center of the vibrations follows this variation adiabatically.

This picture has been clearly confirmed in analog electronic experiments [62]. Two samples of the trajectories of a QMN-driven system described by (2.38) with the potential of the form (2.84) as obtained in [62] are shown in Fig. 2.12. The qualitative feature of the QMN-induced fluctuations that is obvious from Fig. 2.12 is that the trajectory $x(t)$ of the systems crosses the saddle point $x_s = 0$ (the potential barrier top) several times forwards and backwards and then goes back to the initially occupied state without completing a transition to the other stable state. Paths resulting in transitions are very much less frequent; they were observed to happen for larger amplitudes of fluctuational vibrations and, correspondingly, larger shifts of their centers.

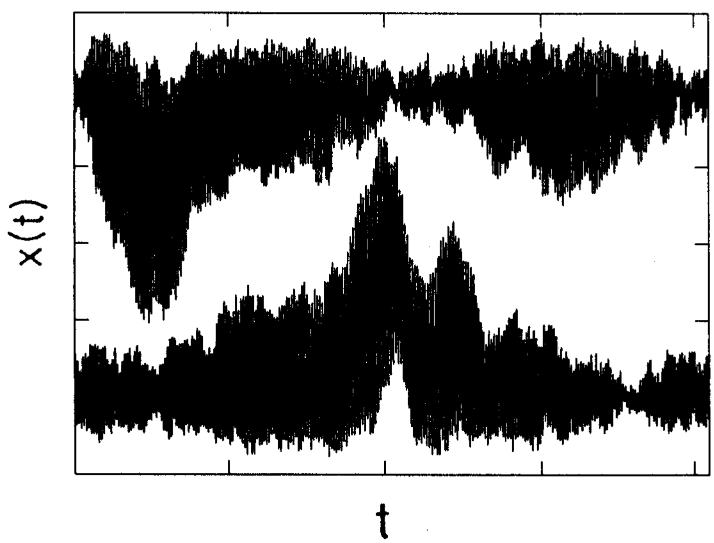

FIG. 2.12. *Two samples of the trajectory $x(t)$ of a symmetrical bistable system driven by quasi-monochromatic noise exhibiting an example of occasional large fluctuations from each of the attractors that result in passage across the saddle point (the potential is of the form* (2.84)*, so that $x_s = 0$), as observed in an analog experiment* [62]. *The eigenfrequency and halfwidth of the QMN are the same as in Fig.* 2.6, *and the noise intensity is $D = 192$.*

This observation is in complete agreement with the inequality (2.127): the activation energy $R(0)$ for the mean first-passage time to the point $x_s = 0$ is less than the activation energy R_i of fluctuational transitions (see below), and therefore the transition probability is exponentially smaller than the reciprocal mean first-passage time to the saddle point. The value of $R(0)$ is given immediately by (2.99), (2.100), and (2.102) [45],

$$R(0) = \frac{1}{5}\frac{\omega_o^2}{\Gamma}. \tag{2.132}$$

The double-adiabatic approximation explained in §2.4 does not hold for the entire optimal path resulting in a transition. One of the scenarios for escape is that, for some value $x_+ = x_+^{(o)}$, the "relaxation time" of the center of vibrations x_c diverges,

$$V_{cc}''(x_c^{ad}, x_+^{(o)}, x_-^{(o)}) = 0, \qquad x_+^{(o)} = x_-^{(o)}. \tag{2.133}$$

Having reached the corresponding value of x_c^{ad}, the center of vibrations goes over to the other branch of the solution of the equation $V_c'(x_c, x_+, x_-) = 0$ and approaches the saddle point as the amplitude $|x_+|$ of the forced vibrations (and thus the noise amplitude) falls to zero (the details are given in [45]). It follows from this picture that it is not the vibrating coordinate itself but the center of vibrations that should reach the saddle point for the transition to occur, and also that the transition probability is given by the probability of the realization of the noise amplitude resulting in the amplitude of the forced vibrations given by (2.133), so that, as explained when (2.102) was derived, the activation energy of the transition is of the form

$$R_i = (2\omega_o^2/\Gamma)|x_+^{(o)}|^2. \tag{2.134}$$

In the particular case that the potential is of the form (2.84), one obtains (see [45] and [76])

$$R_i = \frac{1}{3}\frac{\omega_o^2}{\Gamma}, \tag{2.135}$$

which indeed exceeds the activation energy (2.132) of the mean first-passage time. The value R_i given in (2.135) fits the activation energy of the transition obtained experimentally [62] very well. (The data for $R(x)$ have been discussed in §2.4; we note that the range where the distribution about state i is quasi-stationary extends to the values of x for which $R(x, x_i) = R_i$, i.e., far beyond the range of attraction to the state i; in the particular case of the potential (2.84) and $x_i = -1$ the boundary of quasi-stationarity is $x = \sqrt{2/3}$; cf. also experimental data in Fig. 2.7.)

In conclusion, the color of noise not only drastically changes the value of the activation energy of the transition probability, but it also changes the entire pattern of the transition. The transition probability differs exponentially strongly from the mean first-passage time to the saddle point (the top of the potential barrier)—the system can recross the saddle point many times without completing a transition. The method of the optimal path makes it possible to reduce the problem of the calculation of the activation energy of a transition to a variational problem with intuitively clear boundary conditions.

2.7. Conclusion

It follows from the above discussion that the present status of the investigation of colored-noise-driven dynamics is very promising. Several new theoretical and experimental results have been obtained within the last few years, including the prediction and observation of the features of large fluctuations in systems driven by quasi-monochromatic noise, and the visualization of optimal paths. Although some features of the dynamics are now well established qualitatively and the appropriate mathematical techniques have been developed, such as those used here to find optimal paths for large fluctuations and fluctuational transitions, there are still both qualitative and quantitative problems to be addressed. For example, if a bistable system is driven by noise, what is the shape of the far tails of the *stationary* distribution? This distribution is shaped by large noise outbursts that bring the system to a given point in the phase space from one rather than the other attractor with an overwhelming probability. A simple-minded picture, then, is that the full phase space is separated into the "strips" attributed to fluctuational arrivals from either attractor; this picture is similar to that of deterministic motion in the absence of noise where the phase space is separated into the ranges *from* which a system goes to one or the other attractor and, as in this latter picture, one can investigate the question of the structure of the separating manifold (fluctuational separatrix). However, the topology of the phase space with respect to fluctuational arrivals may be more complicated, and this simple-minded picture may not be generally applicable.

The outstanding problem of greatest importance is that of large fluctuations in systems driven by non-Gaussian noise. Since for Gaussian noise the probability of large noise outbursts decreases extremely rapidly with increasing magnitude of the outburst, even small deviations of the probability density functional from a Gaussian form can result in drastic changes in the transition probabilities and other quantities associated with large fluctuations in noise-driven systems. In many cases of physical interest the method of the optimal path will still be applicable, but more detailed inspection is necessary in each case.

There also exist a number of less "global" problems that are nevertheless interesting and important. These include, for example, a systematic algorithm for the numerical solution of the variational problem for optimal paths. the calculation of the prefactors in the expressions for the tails of the statistical distribution and for the transition probability, the shape of the spectral density of fluctuations in nonlinear (especially underdamped) noise-driven systems.

and the pre-history problem for bistable systems with the final point close to the saddle point. We plan to address some of these problems in the near future, and hope that this review will stimulate continued interest in phenomena related to the color of noise.

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