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Noise-Induced Spectral Narrowing in Nonlinear Oscillators.

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Abstract. – The spectral densities of the fluctuations of noise-driven underdamped nonlinear oscillators are discussed with particular reference to the large class of systems whose eigenfrequencies vary nonmonotonically with energy. It is shown by analogue electronic experiments and theoretically that, astonishingly, the widths of their spectral peaks can sometimes *decrease* with increasing noise intensity.

The study of vibrations and other oscillatory phenomena provides an important path towards a scientific understanding of the physical world. Almost without exception, the oscillators in question turn out to be nonlinear even though, in an earlier era, they were often studied in (frequently misleading) linear approximations. It is largely the discoveries of chaos and strange attractors in apparently deterministic nonlinear oscillators [1], and of noise-induced transitions and other exotic effects in nonlinear oscillators subject to fluctuations [2-5], that have contributed to the intensive investigations and accelerated progress in nonlinear science of the last two decades; it has also become apparent that such studies provide part of the essential basis for an understanding of the complexity that exists universally in the real world [6]. Nonlinear oscillators in both their underdamped and overdamped forms have been studied in a very wide range of situations, for example where they are driven by periodic forcings [1, 6, 7], or by noise [2-6], or by combined periodic and noisy forcings [8-10], or where they undergo bifurcations to self-oscillation as a control parameter is varied either without [11] or with [12] external fluctuations (these references being illustrative only, and in no way purporting to provide an overview of what is a huge literature).

In what follows, we pursue and develop what might appear, qualitatively at least, to be one of the simpler and more straightforward aspects of the subject: the spectral response of an underdamped nonlinear oscillator subject to a random force. In general, there will be one

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or more peaks in the spectral density $Q(\Omega)$ of the fluctuations, corresponding to the eigenfrequencies of the system. For small enough noise intensity, the peaks in question will be narrow and symmetrical, with spectral widths that are determined mainly by dissipation. For larger noise intensities, however, where nonlinearity becomes important, the variation of the eigenfrequency $\omega(E)$ with the energy E of the oscillator may be expected to give rise to additional contributions to the width: that is, to «frequency straggling» caused by the time variation of E. It is conventionally assumed, therefore, that spectral peaks broaden with increasing noise intensity.

The purpose of this letter is to point out that there exists a large class of underdamped nonlinear oscillators for which this conventional scenario does not necessarily hold true but where, instead, the spectral peaks can often become *narrower* with increasing noise intensity. As we shall demonstrate, this seemingly counterintuitive phenomenon can in principle occur in any system for which the dependence of $\omega(E)$ on E is significantly nonlinear and where, in addition, the slope $|\mathrm{d}\omega(E)/\mathrm{d}E|$ decreases with increasing E; the effect is at its most pronounced in systems for which $\omega(E)$ possesses an extremum at some energy E_e. A divergence is to be expected [13] in $Q(\Omega)$ at $\omega(E_e)$ in the limit of zero damping. Although it remains unclear whether it will be resolvable in practice in any real system, the residual effect of this singularity nonetheless plays an important role in the formation of the spectral peak when the noise intensity $T \sim E_e$ in a system with finite damping.

As a specific example of such a system, we consider the oscillator defined by

$$\ddot{q} + 2\Gamma\dot{q} + \mathrm{d}U(q)/\mathrm{d}q = f(t), \qquad U(q) = Aq + \frac{1}{2}q^2 + \frac{1}{4}q^4, \tag{1}$$

where $f(t)$ is a white Gaussian force with a correlator $\langle f(t)f(t')\rangle = 4\Gamma T\delta(t-t')$. Here, we assume that the linear dissipation constant $\Gamma \ll 1$. If $f(t)$ results from thermal fluctuations in a bath, and it is the bath that gives rise to the dissipation, then the noise intensity parameter T represents the temperature of the bath. With $A = 0$, we have the noise-driven single-well Duffing oscillator [3]. With $A \neq 0$, however, we have a system for which, as we shall show, $\omega(E)$ varies with E in precisely the manner discussed above. Despite its simplicity, (1) is a system of considerable interest: not only may we expect it to display the spectral narrowing phenomenon, but it is also directly related to local and resonant vibrations in certain doped crystals [14]. The linear term Aq in the potential $U(q)$ arises for inversely symmetrical defects when an electric field or external pressure are applied to the crystal, readily enabling the parameter A to be varied.

The dependence of the frequency $\omega(E)$ of the oscillations on the energy E of the system, as measured from the bottom of the potential well, is readily calculated [15] for different values of A, as shown in fig. 1*a*). Note that the bottom of the potential well does not occur at $q = 0$ for finite A, but at $q = q_{\mathrm{eq}}$ given by $q_{\mathrm{eq}}^3 + q_{\mathrm{eq}} + A = 0$. The «eigenfrequency of the oscillator» ω_0, defined as the frequency at the bottom of the potential well, is $\omega_0 \equiv \omega(0) = (1 + 3q_{\mathrm{eq}}^2)^{1/2}$, which increases monotonically with increasing $|A|$; but the *slope* of $\omega(E)$ at $E \to 0$ changes nonmonotonically. Using the small E asymptotics of ref. [16], we find that

$$\omega_0' \equiv (\mathrm{d}\omega(E)/\mathrm{d}E)_{E=0} = \frac{3}{4}(1 - 7q_{\mathrm{eq}}^2)/(1 + 3q_{\mathrm{eq}}^2)^{5/2}, \tag{2}$$

which is plotted as a function of A in fig. 1*b*). For $|A| > 8/7^{3/2} \simeq 0.43$, ω_0' is negative and the dependence of $\omega(E)$ on E becomes nonmonotonic. It is the corresponding extremum in $\omega(E)$ that would give rise to a singularity [13] in $Q(\Omega)$ in the absence of dissipation and which may be expected to lead to spectral narrowing in the dissipative system under consideration.

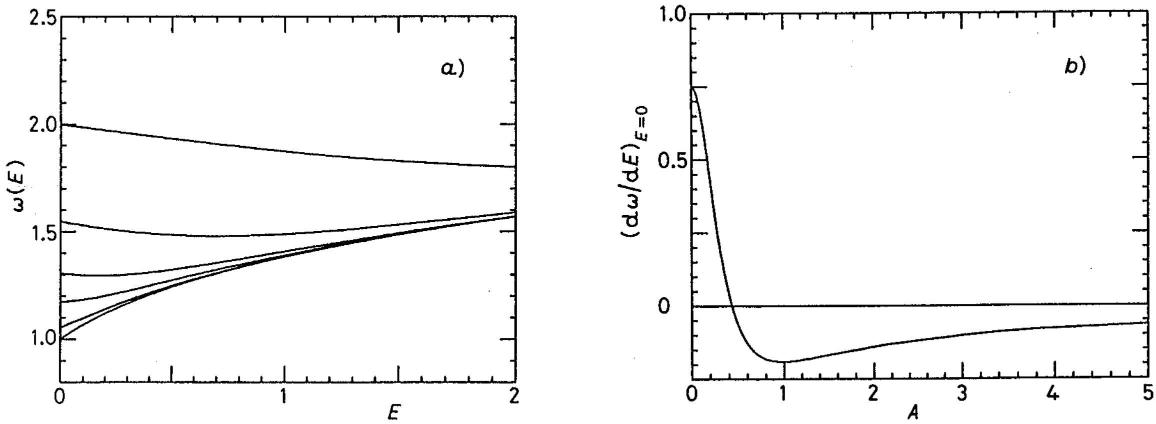

Fig. 1. – *a*) The calculated dependence of the oscillator eigenfrequency $\omega(E)$ on the energy E measured from the bottom of the potential well, for several values of the field strength parameter A in (1). From the bottom, the curves correspond to: $A = 0$; 0.2; 0.4; 0.6; 1.0; 2.0. *b*) The calculated dependence on A of the initial gradient $\omega_0' = [d\omega(E)/dE]_{E=0}$ of the curves shown in *a*).

We have tested this prediction with an analogue electronic circuit model of (1) driven by external quasi-white noise from a noise generator. Full details of the design and operation of the circuit and analysis system will be given elsewhere [15]; but they were very much in accordance with the general principles discussed previously [17]. Spectral distributions were measured for a wide range of the parameters A and T and their widths, characterized by the half-width at the half-maximum (HWHM), were measured. It was found that there was indeed a parameter range for which the width *decreased* with increasing T. As an example, the central parts of the principal peaks of two fluctuation spectra are shown with an expanded abscissa scale in fig. 2*a*); a vertical scaling factor of 0.6 has been applied to the $T = 0.814$ data to equalize the peak heights for more convenient comparison of their shapes. The only parameter changed between the two measurements was T. It is immediately evident that the peak for $T = 0.814$ is about 40% narrower than that for $T = 0.614$. The measured dependence of the width on the noise intensity for a series of such spectra is shown by the data points of fig. 2*b*).

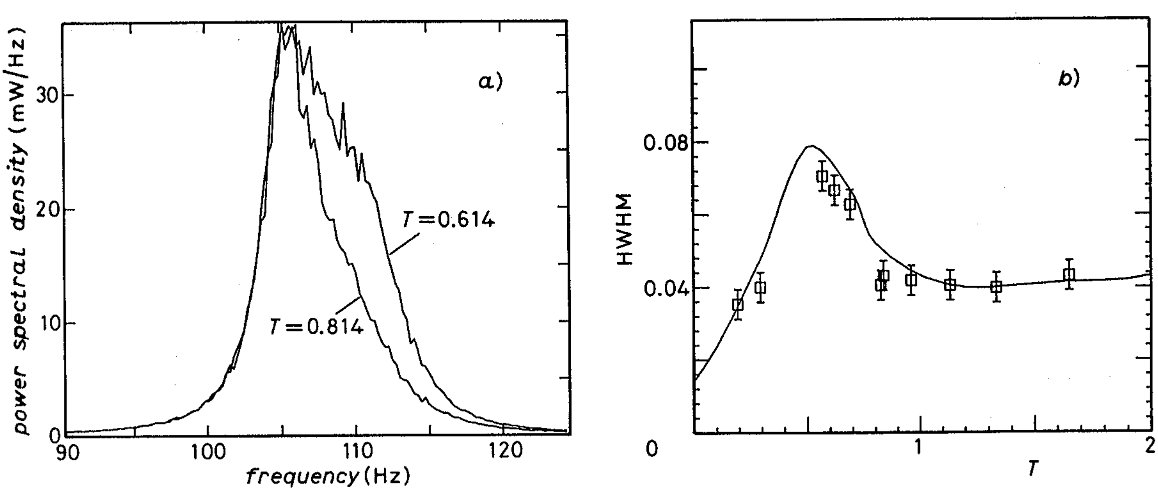

Fig. 2. – *a*) The experimental power spectral density measured for an electronic model of (1) with $A = 2$, using two different noise intensities T. *b*) The half-width at the half-maximum (HWHM) of the spectral peaks of the system (1) as measured for the analogue electronic circuit model (data points) and calculated (full curve) from (9) with $A = 2$.

An intuitive understanding of the spectral narrowing phenomenon can be gained by consideration of the effect of the cut-off frequency ω_e corresponding to the extremum in $\omega(E)$. At sufficiently large T, the peak starts to be «pressed» against ω_e: vibrations with larger and larger amplitudes are being excited as T increases, and their frequencies approach ω_e; but they cannot fall below ω_e. As a result, the peak becomes higher and higher, but without broadening any further on the ω_e side, and the HWHM correspondingly decreases.

These ideas are readily placed on a more quantitative basis [15], enabling us to calculate explicitly the spectral density of the fluctuations of the coordinate q, defined by

$Q(t)$, $Q(t) =$

$$Q(\Omega) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty \mathrm{d}t \, \exp[i\Omega t] \langle [q(t) - \langle q \rangle][q(0) - \langle q \rangle] \rangle . \tag{3}$$

To do so, we start from the Fokker-Planck equation associated with (1) and then follow the same procedure as that used earlier by Dykman and Krivoglaz [18]. Thus, the probability density $w(q, p, t; q_0, p_0, 0)$ for transitions from the phase space point (q_0, p_0) occupied at time zero to the point (q, p) occupied at time t may be written as

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial q}(pw) + \frac{\partial}{\partial p}[U'(q) w] + 2\Gamma \left[\frac{\partial}{\partial p}(pw) + T \frac{\partial^2 w}{\partial p^2} \right] . \tag{4}$$

If we define

$$\widetilde{W}(q, p, t) = \int \mathrm{d}q_0 \, \mathrm{d}p_0 [q_0 - \langle q_0 \rangle] \, w(q, p, t; q_0, p_0, 0) \, w_{\mathrm{st}}(q_0, p_0) , \tag{5}$$

where $w_{\mathrm{st}}(q_0, p_0)$ is the stationary distribution, we can then write

$$Q(t) = \int \mathrm{d}q \, \mathrm{d}p (q - \langle q \rangle) \, \widetilde{W}(q, p, t) . \tag{6}$$

The function $\widetilde{W}(q, p, t)$ satisfies the Fokker-Planck equation (4). We next make a canonical transformation to energy-angle (E, ϕ) variables (a procedure that is particularly useful because of our assumption that Γ is very small), and we introduce a set of functions $W_n(E, \Omega)$ enabling us to expand $W(E, \phi, \Omega)$ as $\sum W_n(E, \Omega) \exp[in\phi]$. The next step is to find a matrix representation of (4) on this expansion set and to average over ϕ, which is a fast-oscillating variable when compared to E. Finally, $W(E, \phi, \Omega)$ is approximated by the first term $W_{\pm 1}(E, \phi) \exp[\pm i\phi]$; this procedure is reliable because we are mainly interested in the spectral density in the immediate vicinity of ω_0. The resultant equation that one has to solve is

$$-i(\Omega - \omega(E)) \, W_1(E, \Omega) =$$

$$= 2\Gamma \left(1 + \overline{p^2} \frac{\mathrm{d}}{\mathrm{d}E} \right) \left(1 + T \frac{\mathrm{d}}{\mathrm{d}E} \right) W_1(E, \Omega) - 2\Gamma T \omega^2(E) \overline{(q_E)^2} \, W_1(E, \Omega) + q_1(E) \, w_{\mathrm{st}}(E) , \tag{7}$$

where

$$\begin{cases} \overline{p^2} = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \mathrm{d}\phi \, p^2(E, \phi) , \qquad \overline{(q_E)^2} = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \mathrm{d}\phi (\partial q(E, \phi)/\partial E)^2 , \\ \\ q_1(E) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \mathrm{d}\phi \, \exp[-i\phi] \, q(E, \phi) , \qquad w_{\mathrm{st}}(E) = Z^{-1} \exp[-E/T] , \end{cases} \tag{8}$$

$$W(E, \varphi, \Omega) = \frac{1}{\pi} \int_0^\infty dt \, e^{i\Omega t} \, \widetilde{W}(q(E, \varphi), p(E, \varphi), t)$$

$Z^{-1} = 2\pi \int_0^\infty \mathrm{d}E\omega^{-1}(E)\exp[-E/T]$ and we assume that $|\Omega - \omega(E)| \ll \omega_0$. The final expression obtained for $Q(\Omega)$ is then

$$Q(\Omega) \simeq Q_1(\Omega) = 2\,\mathrm{Re}\int_0^\infty \mathrm{d}E\,[\omega(E)]^{-1} q_1^*(E)\, W_1(E,\Omega)\,. \tag{9}$$

The conservative motion of the oscillator may be described by elliptic functions, so that the coefficients (8) may readily be evaluated, thus enabling a numerical solution of (7) to be obtained [15]. A comparison of the experimental spectral widths (points) with those calculated from (9) (full curve) is shown in fig. 2*b*). Despite the scatter of the data the agreement can be regarded as excellent.

In conclusion, we would emphasize that the remarkable phenomenon of noise-induced spectral narrowing reported in this letter arises purely from nonlinearity (cf. motional narrowing [19] of NMR lines, which occurs through a quite different mechanism). It is to be anticipated, not only in the particular model (1) investigated here, but in all underdamped systems whose eigenfrequencies vary nonmonotonically with energy. A fuller set of experimental data, and a more detailed discussion of the underlying physics, will be presented elsewhere [15].

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