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FLUCTUATIONS IN NONLINEAR SYSTEMS NEAR BIFURCATIONS CORRESPONDING TO THE APPEARANCE OF NEW STABLE STATES

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Fluctuations in nonlinear multidimensional dynamical systems caused by outer δ-correlated random forces are considered. The slowness of one of the motions at parameter values within the bifurcation region allows us to use the adiabatic approximation and hence to reduce the multidimensional problem to a one-dimensional problem. The exponent and the pre-exponential factor in the expression for the probability of the transition from the metastable equilibrium state of the system are calculated. The kinetics of fluctuations in the systems that are near the bifurcation point where two stable states coincide is considered. The results are illustrated with an example of the Duffing nonlinear oscillator in an external resonance field.

1. Introduction

For some problems of radiophysics, laser physics and nonlinear mechanics it is of interest to investigate the fluctuations in essentially nonlinear subsystems that interact with a medium and may be subjected to the action of regular external forces. These fluctuations have specific features near bifurcation points where, e.g., new stable states or limit cycles appear or disappear. The problem of fluctuations near bifurcation points in nonlinear systems was first considered for the example of the systems with a limit cycle. A series of papers (see e.g. [1–4]) was devoted to a detailed analysis of the case of the Van der Pol oscillator. The solution of such problem is essentially facilitated by the fact that in the Einstein–Fokker–Planck (EFP) equation for the relevant Markov process the variables (amplitude and phase) may be separated and that the equation is reduced to a one-dimensional equation. Under these circumstances it becomes possible, in particular, to investigate in detail the fluctuations in the neighbourhood of the bifurcation corresponding to a soft excitation of the limit cycle (see especially [2]). For some concrete systems the fluctuations near the bifurcation points of another kind were also considered (see e.g. [5,6]).

Along with the bifurcation points, where the roots $\lambda_{1,2}$ of the characteristic equation cross the imaginary axis and a limit cycle appears, those bifurcation points where $\lambda_1 = 0$ ($\lambda_i \neq 0$ for $i \neq 1$) are also of general type. In the latter points there appear or disappear two singular points in the phase space, e.g. a node and a saddle coincide with each other (see [7]).

In the range of parameter values where $\lambda_1 \ll \lambda_2, \lambda_3, \ldots$, one of the motions in the system becomes slow (a soft mode appears[8]). This results in the increase of fluctuations. The smallness of λ_1 permits the use of the adiabatic approximation for the description of the fluctuations and, in the case of weak random forces, the reduction (in the bifurcation region) of the, generally speaking, multi-dimensional problem to one-dimensional one.

Using this approach fluctuations are considered in section 2 in Markov systems for the case where one of the singular points appearing at the bifurcation is stable (a node or a focus) and corresponds to the equilibrium metastable state, while the second point is the saddle. In particular, the probability, W, of the escape from a metastable state over the saddle point to some other stable state of the system is calculated. Such problem is naturally reduced to the well known problem of the first moment of reaching the boundary by a Brownian particle[9]. It should be noted that not only the exponent in the expression for W (which depends strongly on the distance to the bifurcation point) may be determined in this way but also the pre-exponential factor. In this respect the analysis of the transition probability in multidimensional systems under study (which are within the bifurcation region) turns out to be more complete than that carried out in refs. 10 and 11 for a general case of multidimensional systems where W was determined only with logarithmic accuracy.

For systems with two parameters the bifurcations of a general type are described by the curves in the plane of the parameters a_1, a_2. These curves may have singular points – the spinode points. The shape of the curve near the spinode point K ($a_1 = a_{1K}$, $a_2 = a_{2K}$) is shown in fig. 1. In the parameter range bounded by solid lines in fig. 1 the system has two stable states and one

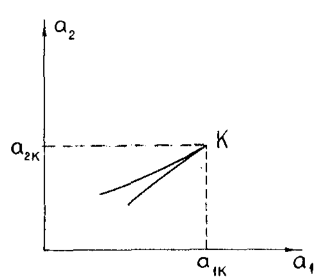

Fig. 1. The bifurcation curves in the two-dimensional parameter space near the spinode point K.

unstable equilibrium state (the saddle). With the approach to the point K (in the parameter space) these states come closer to each other (in the space of dynamical variables). Hence the probabilities of transitions between them grow rapidly, the system rigidity becomes weaker, and the fluctuations near the point K increase strongly. In a certain sense the point K is analogous to the critical point on the curve of the gas–liquid phase transition.

The theory of fluctuations near such a spinode point is given in section 3. The long-time asymptotics is found for the correlation function damping. Far from the point K where the stationary probability distribution function has two distinct maxima, this asymptotics is determined by the probability of the transition between them. In the immediate vicinity of K the asymptotics of the critical fluctuation damping is investigated numerically as a function of two parameters.

In section 4 the results obtained in sections 2 and 3 are used for the analysis of a concrete problem – the fluctuations near bifurcation points of a Duffing nonlinear oscillator interacting with a medium and subjected to a resonant force. The procedure of the reduction of the multidimensional problem to the one-dimensional one is given in the appendix.

Fluctuations in dynamical systems in the critical region have been considered recently by M. Mangel[12]). However, the methods and the problems in ref. 12 differ from those considered in the present paper.

2. Fluctuations near the metastable equilibrium state

Let us consider a multidimensional dynamical system subjected to the action of small random forces, $f_i(t)$, that are δ-correlated in time and distributed according to the stationary normal law. The equations of motion for dimensionless dynamical variables of the system, x_i, are of the form

$$\begin{aligned} \dot{x}_i &= P_i(\boldsymbol{a}, \boldsymbol{x}) + f_i(\boldsymbol{x}, t), \\ \langle f_i(\boldsymbol{x}, t) f_j(\boldsymbol{x}', t')\rangle &= 2\alpha_{ij}(\boldsymbol{x}, \boldsymbol{x}')\delta(t - t'). \end{aligned} \tag{1}$$

Here $\boldsymbol{a} = \{a_l\}$ denotes the set of parameters of the system. Near the bifurcation points, $\boldsymbol{a} = \boldsymbol{a}_{\mathrm{B}}$, corresponding to the appearance of a stable stationary state and the saddle point, the transformation of variables in eq. (1) may be performed in such a way that the functions P_i will be written as

$$\begin{aligned} &P_1(\boldsymbol{a}, \boldsymbol{x}) = \epsilon(\boldsymbol{a}) - \sum_{ij} b_{ij}x_ix_j + \cdots; \\ &P_i(\boldsymbol{a}, \boldsymbol{x}) = -A_ix_i - \sum_{jl} b_{ijl}x_jx_l + \cdots; \quad (i \geq 2); \\ &\qquad A_i' = \operatorname{Re} A_i, \quad A_i' > 0; \quad |\epsilon| \ll A_i'; \quad |\alpha_{ij}| \ll A_i' \end{aligned} \tag{2}$$

(cf. [7]), where similar equations are given in the absence of random forces). Here the parameter ϵ equals to zero in bifurcation points, and the variable x_1 describes the slow motion of the system in the bifurcation region (the $|x_i|$ are assumed to be small). With the accuracy up to small corrections $\sim \epsilon, \alpha$ the values of the parameters A_i, b_{ij}, α_{ij} may be determined from eq. (1) at $\epsilon = 0$, $x_i = 0$, $x_i' = 0$. In what follows the time scale is chosen in such a way that $\min A_i' \sim 1$.

Eqs. (1) and (2) describe multidimensional Langevin random process. The EFP equation corresponding to this process is analysed in the appendix. The separation of variables x_i into the fast and slow ones and the use of adiabatic approximation allow to reduce the EFP equation to the one-dimensional eq. (A.9) in the region of long times $t \gg t_0$ $(t_0 = \max(A_i')^{-1})$. The adiabatic approximation is valid with the accuracy to corrections $\sim \alpha^{1/3}$ in the most favourable case, $\epsilon = 0$. Eq. (A.9) corresponds to the Langevin process

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = -\frac{\mathrm{d}U}{\mathrm{d}y} + f(\tau); \quad U(y) = \frac{b}{3}y^3 - gy, \tag{3}$$

where

$$\begin{aligned} &y = \alpha^{-1/3}x_1; \quad \tau = \alpha^{1/3}t; \quad g = \alpha^{-2/3}\epsilon; \quad f(\tau) = \alpha^{-2/3}f_1(t); \\ &b = b_{11} > 0, \quad \alpha = \alpha_{11}; \quad \langle f(\tau)f(\tau')\rangle = 2\delta(\tau - \tau'). \end{aligned} \tag{4}$$

In the absence of a random force at $g > 0$ (if $b > 0$) eq. (3) has a stable stationary point $y = y_0$ and a saddle point $y = y_s$,

$$y_0 = \sqrt{g/b}, \quad y_s = -\sqrt{g/b}, \tag{5}$$

that are associated with the local minimum and maximum of the potential $U(y)$ (see fig. 2). The state y_0 is apparently metastable if ϵ is small enough.

The one-dimensional Markov process (3) may be investigated in a standard way. In particular, it is of interest to find the probability of the escape from the metastable state at $g \gg 1$. This probability may be characterized by the average time $\bar{\tau}(y, y_b)$ needed for the system (that was initially in some point y in the attraction range of the metastable state) to reach (for the first time) the boundary point y_b. This point is arbitrary to some extent but it must be far enough to the left from y_s (see fig. 2) in order that $\bar{\tau}(y, y_b)$ weakly (nonexponentially) depends on y_b.

The function $\bar{\tau}(y, y_b)$ satisfies the equation (see, e.g. [3,9]))

$$\frac{\partial^2\bar{\tau}(y, y_b)}{\partial y^2} - \frac{\mathrm{d}U}{\mathrm{d}y}\frac{\partial\bar{\tau}(y, y_b)}{\partial y} + 1 = 0. \tag{6}$$

This equation follows from the equation[3,9]) for the probability $W(y_b; y, \tau)$ of

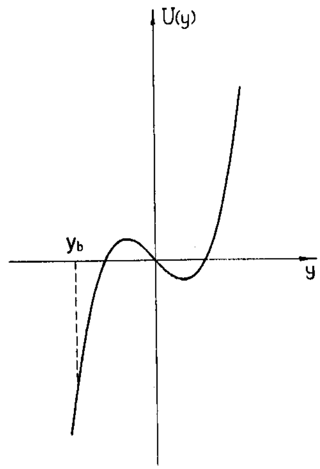

Fig. 2. The potential $U(y)$ near the metastable equilibrium state.

the reaching of the boundary y_b over the time τ,

$$\frac{\partial W(y_b;\, y, \tau)}{\partial \tau} = -\frac{\mathrm{d}U}{\mathrm{d}y}\frac{\partial W(y_b;\, y, \tau)}{\partial y} + \frac{\partial^2 W(y_b;\, y, \tau)}{\partial y^2},$$

that coincides with the first Kolmogorov's equation for the transition probability density $w(y_b, 0;\, y, -\tau)$. Eq. (6) may be easily obtained from this equation if the definition

$$\bar{\tau}(y, y_b) = \int_0^\infty \tau \frac{\partial W(y_b;\, y, \tau)}{\partial \tau}\, \mathrm{d}\tau$$

is taken into account.

Obviously $W(y_b;\, y_b, \tau) = 1$ and therefore $\bar{\tau}(y_b, y_b) = 0$. On physical grounds (taking into account the strong increase of the potential $U(y)$ (3) at large y) it is evident that in the problem under study $\partial\bar{\tau}/\partial y$ must not grow at $y \to \infty$. Solving the first order linear equation (6) for $\partial\bar{\tau}/\partial y$ and integrating the solution over y with allowance for the boundary conditions mentioned above one obtains

$$\bar{\tau}(y, y_b) = \int_{y_b}^{y} \mathrm{d}y' \exp[U(y')] \int_{y'}^{\infty} \mathrm{d}y'' \exp[-U(y'')]. \tag{7}$$

For the particular potential $U(y)$ (3) the integral (7) may be asymptotically calculated at $g^{3/2} \gg b^{1/2}$. The main exponential term is of the form:

$$\bar{\tau}(y, y_b) = W^{-1} = \frac{\pi}{\sqrt{bg}} e^Q; \quad Q = \frac{4}{3} \frac{g^{3/2}}{b^{1/2}} = \frac{4}{3} \frac{\epsilon^{3/2}}{\alpha b^{1/2}}. \tag{8}$$

In this approximation $\bar{\tau}$ does not depend on y, y_b and is connected with the probability W of the escape from the metastable state by the relation $W = \bar{\tau}^{-1}$.

The exponent Q in the expression for W depends on the distance to the bifurcation point (along the straight line ϵ) as $\epsilon^{3/2}$ and on the fluctuation intensity as $1/\alpha$.

The parameters ϵ, b in eq. (2) may be expressed simply through the set of parameters $\boldsymbol{a} = \{a_l\}$ in the functions $P_i(\boldsymbol{a}, \boldsymbol{x})$ of the initial eq. (1) (not yet in the standard form). Bifurcation values, $a_l = a_{lB}$, are found from the condition $\det|P_{ij}| = 0$, where $P_{ij} = (\partial P_i/\partial x_j)_{x=x_0}$, and $\boldsymbol{x}_0$ are the coordinates of a stationary point. The quantities $-A_i$ in eq. (2) are the eigenvalues of the matrix $\|P_{ij}\|$, $A_1(\boldsymbol{a}_B)$ being equal to zero (some A_i may be complex). Let S be the matrix that transforms $\|P_{ij}\|$ into a diagonal form at $\boldsymbol{a} = \boldsymbol{a}_B$:

$$(SPS^{-1})_{ij} = -A_i \delta_{ij}.$$

Expanding the functions P_i in eq. (1) in powers of $\boldsymbol{a} - \boldsymbol{a}_B$, $\boldsymbol{x}_0 - \boldsymbol{x}_{0B}$ ($\boldsymbol{x}_{0B}$ is the value of $\boldsymbol{x}_0$ at $\boldsymbol{a} = \boldsymbol{a}_B$) and going from the variables $\boldsymbol{x}$ to new variables $S(\boldsymbol{x} - \boldsymbol{x}_{0B})$ (which with an accuracy to ϵ coincide with the variables in eq. (2)) one obtains

$$\begin{aligned} \epsilon &= (\boldsymbol{a} - \boldsymbol{a}_B) \sum_j S_{1j} (\partial P_j(\boldsymbol{a}, \boldsymbol{x})/\partial \boldsymbol{a})_{\boldsymbol{x}=\boldsymbol{x}_{0B}, \boldsymbol{a}=\boldsymbol{a}_B}, \\ b &= -\sum_{jmn} S_{1j} (S^{-1})_{n1} (S^{-1})_{m1} (\partial^2 P_j/\partial x_n \partial x_m)_{\boldsymbol{x}=\boldsymbol{x}_{0B}, \boldsymbol{a}=\boldsymbol{a}_B}. \end{aligned} \tag{9}$$

Formulae (8) and (9) contain a simple expression for the probability of the escape from the metastable state in the general case of a multidimensional and multiparametrical system (which is near the bifurcation curve however). An example of how to apply these formulae to a particular case (the Duffing nonlinear oscillator) is given below in section 4.

3. Fluctuations near the point of the coincidence of two stable states

Near the spinode point on the bifurcation curve where stable states coincide (the point K in fig. 1) the coefficient b_{11} in eq. (2) tends to zero. Accordingly, the right-hand sides of the equations of motion (1) at $\boldsymbol{a} = \boldsymbol{a}_K$ (the values of the components of $\boldsymbol{a}_K$ correspond to the point K) and at the stationary values of variables $\boldsymbol{x}$ (they are assumed to be zero in the point K) after certain transformation of variables satisfy the conditions

$$(P_i)_K = 0; \quad (\partial P_i/\partial x_j)_K = -A_i\delta_{ij}; \quad A_1 = 0, \quad \mathrm{Re}\, A_{i>1} > 0; \quad (\partial^2 P_1/\partial x_1^2)_K = 0. \tag{10}$$

The probability distribution w of the slow motion x_1 is shown in the appendix to be described by a one-dimensional EFP equation. According to (A.10) this equation has the form

$$\begin{aligned} &\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial y}\left(\frac{\mathrm{d}U}{\mathrm{d}y} w\right) + \frac{\partial^2 w}{\partial y^2}; \quad U = \tfrac{1}{4}cy^4 - \tfrac{1}{2}g_1 y^2 - g_2 y; \\ &y = \alpha^{-1/4}x_1, \quad \tau = \alpha^{1/2}t. \end{aligned} \tag{11}$$

The error caused by the use of adiabatic approximation in the point K is of the order $\alpha^{1/4}$.

For long enough time the probability distribution w tends to a stationary solution of eq. (11) $w_{\mathrm{st}}(y)$:

$$w_{\mathrm{st}}(y) = Z^{-1}\exp[-U(y)]; \quad Z = \int_{-\infty}^{\infty} \mathrm{d}y \exp[-U(y)]. \tag{12}$$

In the range of $\boldsymbol{a}$ encompassed by the curves in fig. 1 where

$$g_1 > 0, \quad |\Delta| \leq 1, \quad \Delta = \frac{3\sqrt{3c}\,g_2}{2g_1^{3/2}}, \tag{13}$$

the function $w_{\mathrm{st}}(y)$ has two maxima, that correspond to two stable solutions of eq. (1) in the absence of random forces while outside this range there is only one maximum.

The character of fluctuations in the system depends essentially on the distance to the point K. At $g_1 \gg c^{1/2}$ the function $w_{\mathrm{st}}(y)$ has sharp maxima. In this case (provided the conditions (13) are satisfied) the system remains for a rather long time near one of the stationary states, fluctuating around it and only occasionally goes into another state. The ratio of the state populations here is (with a logarithmic accuracy) equal to

$$\begin{aligned} &w^{(1)}/w^{(2)} = \exp(\delta U), \quad \delta U = U^{(2)} - U^{(1)}; \\ &U^{(1,2)} \equiv U(y^{(1,2)}); \quad (\partial U/\partial y)_{y=y^{(1,2)}} = 0. \end{aligned} \tag{14}$$

The quantity $q = \delta U c g_1^{-2}$ is a function of the single parameter Δ and

$$q = \frac{4}{3\sqrt{3}}\Delta, \quad \text{at } |\Delta| \ll 1; \quad q = \frac{3}{4}\frac{\Delta}{|\Delta|}[1 - \tfrac{8}{9}(1 - |\Delta|)], \quad \text{at } 1 - |\Delta| \ll 1. \tag{15}$$

Here we have put $y^{(1)} > y^{(2)}$. It is seen from (A.11) that Δ does not depend on α, and $\delta U \sim \alpha^{-1}$.

The probability $W_{1\to2}$ of a transition between the states 1 and 2 may be estimated in the same way as in section 2 calculating the integral (7) with the corresponding potential $U(y)$ (11) asymptotically. The boundary point y_b must be placed between the saddle point $y^{(s)}$ and the point $y^{(2)}$. Then with the logarithmic accuracy:

$$W_{1\to2} = \text{const}\exp(-Q_1); \quad Q_1 = U^{(s)} - U^{(1)} = \frac{g_1^2}{c} q_1(\Delta),$$
$$U^{(s)} \equiv U(y^{(s)}). \tag{16}$$

Here the position of the saddle point $y^{(s)}$ is found as a mean root of the equation $\partial U/\partial y = 0$. The function $q_1(\Delta)$ in the limiting cases takes the form

$$q_1(\Delta) = \tfrac{1}{4} + \frac{2}{3\sqrt{3}}\Delta, \quad \text{at } |\Delta| \ll 1; \quad q_1(\Delta) = \tfrac{3}{4}[1 - \tfrac{8}{9}(1-\Delta)],$$
$$\text{at } 1-\Delta \ll 1; \tag{17}$$
$$q_1(\Delta) = \frac{16}{27\sqrt{6}}(1+\Delta)^{3/2}, \quad \text{at } 1+\Delta \ll 1.$$

It is evident that $W_{1\to2}/W_{2\to1} = w_{st}^{(2)}/w_{st}^{(1)}$. Because of the fact that the problem has been reduced to a one-dimensional one, expression (16) for $W_{1\to2}$ (and also eq. (8)) has a form similar to that of the expression obtained in Kramers' well-known work[13]) for the probability of the escape of a Brownian particle from the potential well.

In the range of parameter values $g_1 \leq c^{1/2}$, the maxima of the function $w_{st}(y)$ are smeared and the concept of the probability of a transition between stable states loses its sense. In this case the dynamics of the system in the long-time range $t \gg t_0$ is characterized by the decay of the time correlation function $\langle y(t)y(0)\rangle$ for the slow variable $y = \alpha^{-1/4}x_1$. Such correlators may be determined conveniently using eq. (11). After a standard separation of variables in eq. (11) (see, e.g. [2,14,15]))

$$w(y,\tau) = \sum_n C_n\, e^{-\lambda_n\tau} \exp[-U(y)/2]\psi_n(y) \tag{18}$$

we get the eigenvalue problem:

$$\frac{d^2\psi_n}{dy^2} + \left[\frac{1}{2}\frac{d^2U}{dy^2} - \frac{1}{4}\left(\frac{dU}{dy}\right)^2\right]\psi_n + \lambda_n\psi_n = 0. \tag{19}$$

This equation corresponds to the Schrödinger equation with a potential of the form of a polynomial of the sixth degree in y according to eq. (11) (cf. [15])).

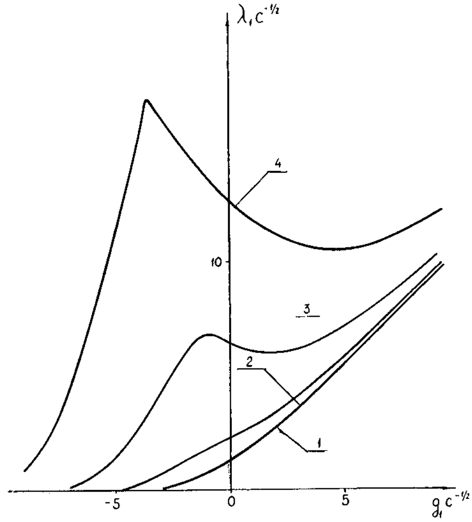

Fig. 3. The dependence of the lowest eigenvalue $\lambda_1 c^{-1/2}$ on the parameter $g_1 c^{-1/2}$. The curves 1–4 correspond to the parameter values $g_2 c^{-1/4} = 0$; 2; 5; 10.

The lowest eigenvalue in (19) is $\lambda_0 = 0$. The corresponding eigenfunction $\psi_0 \sim \exp[-U(y)/2]$ has no nodes (the stationary distribution $w_{st} \sim \psi_0^2(y)$). The behaviour of the time correlation function at large τ is determined by the lowest nonzero eigenvalue λ_1

$$\langle y(t)y(0)\rangle - \langle y(0)\rangle^2 \sim \exp(-\lambda_1\tau) = \exp(-\lambda_1\alpha^{1/2}t), \quad \tau \gg 1.$$

The dependence of $\lambda_1 c^{-1/2}$ on the parameters $g_1 c^{-1/2}$ and $g_2 c^{-1/4}$ was found by a numerical evaluation of eq. (19). The results are shown in fig. 3. In particular in the spinode point itself ($g_1 = g_2 = 0$)

$$\lambda_1 \approx 1.37c^{1/2}; \quad \langle y(t)y(0)\rangle \approx \exp[-1.37(c\alpha)^{1/2}t], \quad \sqrt{c\alpha}\,t \gg 1. \tag{20}$$

Note that at large $g_2 c^{-1/4}$ the dependence of $\lambda_1 c^{-1/2}$ on $g_1 c^{-1/2}$ has a rather sharp maximum.

4. Fluctuations near bifurcation points of the Duffing nonlinear oscillator in a resonance field

It is interesting to consider as an example the dynamics of such a well known nonlinear system as the Duffing oscillator interacting with a medium (and in the general case subjected to arbitrary white noise). The equation of the

Brownian motion for the oscillator is of the form

$$\frac{d^2q}{dt^2} + 2\Gamma\frac{dq}{dt} + \omega_0^2 q + \gamma q^3 = h\cos\omega t + \tilde{f}(t), \quad t > 0, \tag{21}$$

where $q(t)$ is the oscillator coordinate, Γ is the damping coefficient, γ is the nonlinearity parameter, h is the amplitude of the external field whose frequency ω is supposed to be close to ω_0, $\tilde{f}(t)$ is a random force.

It is convenient to go from $q(t)$ and $dq(t)/dt$ to smooth (in the time scale ω_0^{-1}) complex dimensionless functions $u(\tilde{\tau})$, $u^*(\tilde{\tau})$ (cf. [11]):

$$\begin{aligned} q(t) &= \left(\frac{2\omega\delta\omega}{3|\gamma|}\right)^{1/2}[u(\tilde{\tau})\exp(\mathrm{i}\omega t) + \text{c.c.}],\\ \frac{dq(t)}{dt} &= \mathrm{i}\omega\left(\frac{2\omega\delta\omega}{3|\gamma|}\right)^{1/2}[u(\tilde{\tau})\exp(\mathrm{i}\omega t) - \text{c.c.}],\\ \delta\omega &= |\omega - \omega_0|, \quad \tilde{\tau} = t\delta\omega. \end{aligned} \tag{22}$$

It follows from eq. (22) that

$$\frac{d^2q}{dt^2} + \omega^2 q = 2\mathrm{i}\omega[2\omega(\delta\omega)^3/3|\gamma|]^{1/2}\frac{du(\tilde{\tau})}{d\tilde{\tau}}\,\mathrm{e}^{\mathrm{i}\omega t}.$$

The substitution of this equation and eq. (22) into (21) allows to go from the second order differential equation for $q(t)$ to the set of two first order equations for the dynamical variables $u'(\tilde{\tau}), u''(\tilde{\tau})$ $(u = u' + \mathrm{i}u'')$. After averaging over the time $\sim\omega_0^{-1}$ they are reduced to a complex equation,

$$\begin{aligned} \frac{du}{d\tilde{\tau}} &= v + f; \quad v = v(u', u'')\\ &= -\frac{u}{\Omega} + \mathrm{i}u[|u|^2\,\mathrm{sign}\,\gamma - \mathrm{sign}(\omega - \omega_0)] - \mathrm{i}\sqrt{\beta}\,\mathrm{sign}\,h;\\ \Omega &= \frac{\delta\omega}{\Gamma}, \quad \beta = \frac{3|\gamma|h^2}{32\omega^3(\delta\omega)^3}; \quad f(\tilde{\tau}) = -\mathrm{i}\frac{\sqrt{3|\gamma|}}{(2\omega\delta\omega)^{3/2}}\tilde{f}(t)\,\mathrm{e}^{-\mathrm{i}\omega t};\\ &\langle f(\tilde{\tau})f(\tilde{\tau}')\rangle = 0; \quad \langle f(\tilde{\tau})f^*(\tilde{\tau}')\rangle = 4\alpha\delta(\tilde{\tau} - \tilde{\tau}'); \quad \alpha = \frac{\mathcal{D}}{\Omega^2}. \end{aligned} \tag{23}$$

Eq. (21) is valid when the characteristic frequency of the medium $\omega_{\text{eff}} \gg \omega_0$. But it was shown in refs. 16 and 11 that eq. (23) is valid also at $\omega_{\text{eff}} \sim \omega_0$ if the interaction with a medium is small enough (and in some other cases).

Along with the appearance of a friction force and a random δ-correlated force in the equations of motion the interaction with a medium results here in the renormalization of the oscillator frequency. In the case of an oscillator interacting with the bath we have in eq. (23) $\mathcal{D} = 3|\gamma|kT/8\omega^3\Gamma$ while in the case of the Duffing oscillator in the white noise field $\mathscr{D}$ is determined by the noise intensity.

The function v in (23) depends on two parameters: Ω and β. In a certain range of Ω and β the equations of a stationary point, $v' = 0, v'' = 0$, at $\gamma(\omega - \omega_0) > 0$, have three solutions while outside this range there is only one solution. The boundaries of this range are determined by the expressions

$$\beta = \beta_{\mathrm{B}}^{(1,2)}(\Omega) = \frac{2}{27}[1 + 9\Omega^{-2} \mp (1 - 3\Omega^{-2})^{3/2}], \quad \Omega \geq \sqrt{3}. \tag{24}$$

The dependences $\beta_{\mathrm{B}}^{(1)}(\Omega^{-1}), \beta_{\mathrm{B}}^{(2)}(\Omega^{-1})$ are given schematically by the curves in fig. 1. The spinode point corresponds to $\Omega_{\mathrm{K}} = \sqrt{3}$, $\beta_{\mathrm{K}} = \frac{8}{27}$.

The phase of the field h in eq. (21) is chosen in such a way that the terms linear in u', u'' in eq. (23) for $\mathrm{d}u''/\mathrm{d}\tilde{\tau}$ are eliminated at the bifurcation point i.e. the variable u'' is slow. Therefore eqs. (23) in the range under study may be easily reduced to the standard form (1), (2) or (1), (10).

Near bifurcation points far from the point K such a reduction leads to the expression (2) with the coefficients

$$\begin{aligned} &\epsilon = \frac{\beta - \beta_{\mathrm{B}}(\Omega)}{2\sqrt{\beta_{\mathrm{B}}(\Omega)}}; \quad b = b(\Omega) = \frac{p}{\sqrt{\beta_{\mathrm{B}}}}[5p - 3 + 3(2p-1)^2(p-1)\Omega^2]; \\ &p = p^{(1,2)}(\Omega) = \tfrac{2}{3}[1 \pm \tfrac{1}{2}(1 - 3\Omega^{-2})^{1/2}]. \end{aligned} \tag{25}$$

Substitution of (25) into (8) shows that the exponent Q in the expression for the probability W of the escape from the metastable equilibrium state may be determined for the nonlinear oscillator as

$$Q = \frac{G(\Omega)}{\mathscr{D}}|\beta - \beta_{\mathrm{B}}(\Omega)|^{3/2}; \quad G(\Omega) = \frac{\sqrt{2}}{3}\frac{\Omega^2}{|b(\Omega)|^{1/2}[\beta_{\mathrm{B}}(\Omega)]^{3/4}}. \tag{26}$$

The factor $G(\Omega)$ as a function of the frequency detuning Ω is shown in fig. 4 where the curves 1 and 2 refer to the dependences $\beta_{\mathrm{B}}^{(1)}(\Omega)$ and $\beta_{\mathrm{B}}^{(2)}(\Omega)$, respectively. At large Ω, $G^{(1)}(\Omega) \simeq 2\Omega^3/3$, $G^{(2)}(\Omega) \simeq 4.5\Omega$ (asymptotically), while with the approach to the point K, $G(\Omega)$ decreases strongly. Eq. (26) for $Q^{(1)}$ at $\Omega \to \infty$ coincides with the result obtained in ref. 11 for this particular case by means of an essentially different method.

In order to consider the motion of a nonlinear oscillator at the parameter values within the neighbourhood of the point K it is convenient to go from u', u'' to the variables

$$x_1 = u'' - u''_{\mathrm{K}}, \quad x_2 = u' - u'_{\mathrm{K}} + \frac{1}{\sqrt{3}}(u'' - u''_{\mathrm{K}}); \quad u'_{\mathrm{K}} = -\frac{1}{\sqrt{6}}, \quad u''_{\mathrm{K}} = -\frac{1}{\sqrt{2}}. \tag{27}$$

Then the equations of motion take the form of (1), (10), and the slow motion

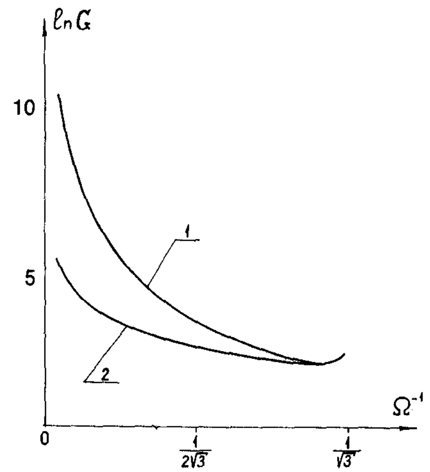

Fig. 4. The dependence of the coefficient $G(\Omega)$ in the expression for the transition probability of the Duffing oscillator on the frequency detuning Ω. The growth of $G(\Omega)$ in the immediate vicinity of the point K is due to the vanishing of the coefficient $b(\Omega)$ in eq. (26) at $\Omega = 1/\sqrt{3}$.

is described by the EFP equation (11) with

$$c = \frac{4}{3\sqrt{3}}; \quad g_1 = \frac{\Omega - \Omega_K}{\sqrt{3\mathcal{D}}}; \quad g_2 = -\frac{9\sqrt{3}(\beta - \beta_K) + 4(\Omega - \Omega_K)}{2^{3/2}3^{1/4}\mathcal{D}^{3/4}}. \tag{28}$$

The formulae (15)–(17) and fig. 3 along with eqs. (28), (23) describe the oscillator behaviour near the point K as a function of a departure in the external field amplitude and frequency from their critical values. It should be noted that eqs. (16) and (28) coincide with eq. (23) of ref. 11 obtained in another way.

Appendix

The EFP equation for a random process (1) is of the form

$$\frac{\partial v}{\partial t} = \sum_{ij} \alpha_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} - \sum_i \frac{\partial}{\partial x_i}(P_i v). \tag{A.1}$$

It is supposed here that $\alpha_{ij}(0,0) \equiv \alpha_{ij} \neq 0$ and that simultaneously the coefficients α_{ij} are small enough. This allows to neglect the terms proportional to derivatives of $\alpha_{ij}(x, x')$ on x, x' in the range of small $|x_i|$.

It follows from eqs. (1), (2), (10) and (A.1) that the probability distribution over fast variables with $i > 1$ is formed over short times $\sim t_0 = \max(A_i')^{-1}$ and has a characteristic width $\sim \alpha^{1/2}$ (it is supposed that $|\alpha_{ij}|$ do not exceed α in the order of magnitude). The distribution v has a maximum at quasistationary values $x_i = X_i(a, x_1)$, where X_i corresponding to a given x_1 are determined by the equations

$$P_i(a, x_1, X_j) = 0, \quad |X_i| \ll |x_1| \ll 1. \tag{A.2}$$

Here and below i, j take the values 2, 3, ...

The distribution over x_1

$$w(x_1, t) = \int v(x, t)\, dx_2\, dx_3 \ldots \tag{A.3}$$

is formed essentially longer and appears to be much more broad. To find it in the time range $t \gg t_0$ the method of moments may be used. Integrating eq. (A.1) over $x_2, x_3, \ldots$ one obtains

$$\begin{gathered}\frac{\partial w}{\partial t} = \alpha \frac{\partial^2 w}{\partial x_1^2} - \frac{\partial}{\partial x_1}\left[P_1(a, x_1, 0)w + \sum_{i>1} (\partial P_1/\partial x_1)_{x_2=x_3=\cdots=0}\langle x_i\rangle + \cdots\right],\\ \langle x_i\rangle = \int x_i v(x, t)\, dx_2\, dx_3 \ldots; \quad \alpha \equiv \alpha_{11}.\end{gathered} \tag{A.4}$$

Multiplication of (A.1) by x_i and integration lead to the equation for the moments $\langle x_i\rangle$:

$$\begin{aligned}\frac{\partial\langle x_i\rangle}{\partial t} &= \alpha \frac{\partial^2\langle x_i\rangle}{\partial x_1^2} - 2\alpha_{1i}\frac{\partial w}{\partial x_1} + P_i(a, x_1, 0)w - A_i\langle x_i\rangle - \cdots\\ &\quad - \frac{\partial}{\partial x_1}\left[P_1(a, x_1, 0)\langle x_i\rangle + \sum_{j>1} (\partial P_1/\partial x_j)_{x_2=x_3=\cdots=0}\langle x_i x_j\rangle + \cdots\right].\end{aligned} \tag{A.5}$$

Here the expansion (2) of $P_i(a, x)$ in x_j has been used. The equations for successive moments are of a similar form.

The chain of the equations for moments may be decoupled at times $t \gg t_0$ if x_1 is small ($|x_1| \ll 1$) and the function $w(x_1, t)$ is smooth enough and varies slowly with time:

$$\begin{gathered}|x_1| \ll 1; \quad \alpha|\partial^2 w/\partial x_1^2| \ll w; \quad |\partial w/\partial t| \ll w;\\ |\alpha\partial^2\langle x_i\rangle/\partial x_1^2| \ll |\langle x_i\rangle|, \quad |\partial\langle x_i\rangle/\partial t| \ll |\langle x_i\rangle|, \quad t \gg t_0.\end{gathered} \tag{A.6}$$

In this case eq. (A.5) has an approximate solution

$$\langle x_i\rangle = A_i^{-1}\left[P_i(a, x_1, 0)w - 2\alpha_{1i}\frac{\partial w}{\partial x_1}\right] \approx X_i(x_1)w - \frac{2\alpha_{1i}}{A_i}\frac{\partial w}{\partial x_1}, \tag{A.7}$$

where eq. (A.2) is taken into account. Substituting eq. (A.7) into eq. (A.4) and neglecting the term with α_{1i} (it is small as compared with the first addend in the right-hand side of eq. (A.4)), we get the EFP equation for the one-dimensional motion

$$\frac{\partial w}{\partial t} = \alpha \frac{\partial^2 w}{\partial x_1^2} - \frac{\partial}{\partial x_1}[Pw],$$
$$P = P(x_1) = P_1(a, x_1, X_i(x_1)). \tag{A.8}$$

Near the usual bifurcation point (of the codimension 1), according to eqs. (2) and (A.2), we may put $X_i = 0$ neglecting in P the quantities of the higher order of smallness. Then, going from x_1, t to new variables y, τ and allowing for eq. (2), we may write the EFP equation (A.8) as

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial y^2} + \frac{\partial}{\partial y}[(b_{11}y^2 - \epsilon\alpha^{-2/3})w];$$
$$y = \alpha^{-1/3}x_1, \quad \tau = \alpha^{1/3}t. \tag{A.9}$$

In the vicinity of a bifurcation point of the codimension 2 (the point K in fig. 1) the functions $X_i(x_1)$ and $P(x_1)$ may be determined by expanding $P_1(\boldsymbol{a}, \boldsymbol{x})$ and $P_i(\boldsymbol{a}, \boldsymbol{x})$ in powers of $\boldsymbol{x}$ and $\boldsymbol{a} - \boldsymbol{a}_K$ with allowance for the relation (10) for the expansion coefficients. Then it follows from eqs. (A.2), (A.8) and (10) that

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial y^2} + \frac{\partial}{\partial y}[(cy^3 - g_1 y - g_2)w], \quad y = \alpha^{-1/4}x_1, \quad \tau = \alpha^{1/2}t, \tag{A.10}$$

where

$$c = -\frac{1}{6}\frac{\partial^3 P_1}{\partial x_1^3} - \frac{1}{2}\sum_{i>1} A_i^{-1}(\partial^2 P_1/\partial x_1 \partial x_i)(\partial^2 P_i/\partial x_1^2) > 0;$$

$$g_1 = \alpha^{-1/2}(\partial^2 P_1/\partial \boldsymbol{a}\partial x_1)(\boldsymbol{a} - \boldsymbol{a}_K) + \alpha^{-1/2}\sum_{i>1} A_i^{-1}(\partial^2 P_1/\partial x_1 \partial x_i)(\partial P_i/\partial \boldsymbol{a})(\boldsymbol{a} - \boldsymbol{a}_K);$$

$$g_2 = \alpha^{-3/4}(\partial P_1/\partial \boldsymbol{a})(\boldsymbol{a} - \boldsymbol{a}_K). \tag{A.11}$$

All the derivatives are taken here at $\boldsymbol{a} = \boldsymbol{a}_K$ and $\boldsymbol{x} = 0$. The condition $c > 0$ is a condition of the stability of the equilibrium state.

From the adiabaticity criterion, $|\partial w/\partial t| \ll w$ and eq. (A.8) it follows that $|P| \ll 1$ and $|dP/dx_1| \ll 1$. This means that for the adiabatic approximation to be applicable the condition $|\boldsymbol{a} - \boldsymbol{a}_B| \ll 1$ must be fulfilled.

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