Physica **104A** (1980) 495–508 © *North-Holland Publishing Company*

TIME CORRELATION FUNCTIONS AND SPECTRAL DISTRIBUTIONS OF THE DUFFING OSCILLATOR IN A RANDOM FORCE FIELD

M.I. DYKMAN

Institute of Semiconductors, Academy of Sciences of the Ukrainian SSR, Kiev, USSR

M.A. KRIVOGLAZ

Institute of Metal Physics, Academy of Sciences of the Ukrainian SSR, Kiev, USSR

Received 27 June 1980

The Brownian motion of the Duffing oscillator is analyzed in the case when the oscillator damping is small compared with its frequency, whereas the nonlinearity may be arbitrary. The expressions for the time-correlation functions of coordinates are obtained in an explicit form. If the nonlinearity is small, the dynamics of the system is shown to be determined by a relation between the frequency straggling due to fluctuations of the amplitude and damping. At large nonlinearity the correlators do not depend on the damping. The frequency dependences of the spectral representations of the correlators of coordinates are investigated for various ratios between the oscillator parameters.

1. Introduction

The investigation of the fluctuations of dynamical variables for nonlinear oscillating systems is an essentially more difficult problem than that for linear ones. The difficulties are connected with the fact that the presence of nonlinear terms in the equations of motion of the system subject to a field of the random force does not allow us to carry out an averaging over the realizations of an external force in a general case. Therefore a number of approximate methods allowing to investigate spectral characteristics of the random oscillations was developed. One of the most widely used methods is that of equivalent linearization in which a nonlinear oscillating system is replaced by an effective linear one (see, for example, ref. 1). The methods based on the decoupling of the chains of equations of motion for averaged dynamical variables are used also[2–4]). A number of results was obtained by numerical methods, in particular for van der Pol[5]) and Duffing oscillators[6,7]).

Obviously it is interesting to estimate the accuracy of the various approximate methods. From this point of view it is essential to study relatively simple and at the same time nontrivial models of the nonlinear systems. The

Duffing oscillator in the field of a random Gaussian δ-correlated force is one such model. Its motion is described by the well-known Langevin equation:

$$\ddot{q} + \omega_0^2 q + 2\Gamma\dot{q} + \gamma q^3 = f(t);$$
$$\langle f(t) f(t') \rangle = 2B\Gamma\delta(t - t'). \tag{1}$$

Here, B characterizes the intensity of the white noise $f(t)$ (the case of the two-well potential corresponding to $\omega_0^2 < 0$[8]) is eliminated in the present paper although some results obtained in section 3 for the case of strong nonlinearity are also applicable at $\omega_0^2 < 0$).

The approximate methods mentioned above were used to investigate this problem in refs.[1-4,6]. However, the analysis carried out in refs.[2-4]) is actually valid provided the criterion stated in ref.[2]):

$$|\gamma| B \ll \omega_0^3 \Gamma \tag{2}$$

is fulfilled.

Meanwhile, an analysis of the random motion of the Duffing oscillator may be carried out in a substantially wider range of the parameters where the value of the nonlinearity constant γ is arbitrary and the only restriction is the inequality for the damping parameter Γ:

$$\Gamma \ll \omega_0. \tag{3}$$

Then it appears to be possible to obtain the explicit expressions for various time-correlation functions.

The physical picture and the mathematical consideration are essentially different depending on the value of the parameter

$$\alpha = 3\gamma B / 16\omega_0^3 \Gamma. \tag{4}$$

This parameter is determined by a relation between the typical frequency straggling $\delta\omega \simeq |\gamma| \langle q^2 \rangle \omega_0^{-1} \sim |\gamma| B / \omega_0^3$ due to the dependence of the nonlinear oscillator frequency on the amplitude of random oscillations and the uncertainty Γ of the oscillator frequency, connected with damping. Depending on the value of α the main contribution into a broadening of the spectral distributions of the time-correlation functions is due either to the oscillator damping or to its frequency modulation by the fluctuations of the amplitude (modulational broadening). The first mechanism prevails in the case $|\alpha| \ll 1$, i.e. when condition (2) is satisfied. In this case the nonlinear effects may be considered as small perturbations.

In the range

$$\alpha \gg 1 \tag{5}$$

on the contrary, the damping weakly influences the spectral distributions of the time-correlation functions of the coordinates or velocities. In the limit $\Gamma \to 0$ the motion may be considered as a quasiconservative one. An averaging over the realizations of the random force is reduced simply to the averaging of the correlator obtained at $\Gamma = 0$, $f = 0$ over the oscillator energy. Corresponding calculations are performed below for the case $\gamma > 0$ (when the oscillator motion is finite).

The most complicated case for investigation is

$$|\alpha| \sim 1$$

when both mechanisms of the broadening play an essential role. However, in this case the results of ref.[9]) may be used where the solution of a similar problem concerning the vibrations of the nonlinear oscillator interacting with a medium was obtained in the explicit form*. The solution obtained in ref.9) allows us to describe the whole range of parameter α (but at $|\gamma| B \ll \omega_0^4$).

The results of paper[9]) will be generalized below (section 2) for the problem of a Duffing oscillator in a field of arbitrary white noise. Besides, the time-correlation functions for various powers of coordinates and velocities will be obtained by the generating function method for arbitrary α but $|\gamma| B \ll \omega_0^4$. In section 3 the case $\alpha \gg 1$ will be investigated, supposing the ratio of γB and ω_0^4 to be arbitrary. In section 4 a comparison of the present results with the results of papers[1–4]) will be carried out.

2. Time-correlation functions at small frequency straggling due to nonlinearity

When the condition

$$\beta = |\gamma| B / 2\omega_0^4 \ll 1 \tag{6}$$

is fulfilled, the frequency straggling due to the amplitude dependence of the frequency of nonlinear oscillator $\delta\omega \sim \beta\omega_0 \ll \omega_0$. Taking into account that the values $\delta\omega$ and Γ are small compared with ω_0, it is convenient to use a standard method of the asymptotic theory of nonlinear oscillations[13]) and to go from fast oscillating functions $q(t)$ and $\mathrm{d}q/\mathrm{d}t$ to smooth complex functions $u_1(t)$, $u_2(t)$. These functions may be defined as the coefficients which slowly vary in time of the fast oscillating terms with $\exp(\pm \mathrm{i}\omega_0 t)$:

$$q(t) = u_1\, \mathrm{e}^{\mathrm{i}\omega_0 t} + u_2\, \mathrm{e}^{-\mathrm{i}\omega_0 t}; \quad \frac{\mathrm{d}q}{\mathrm{d}t} = \mathrm{i}\omega_0 (u_1\, \mathrm{e}^{\mathrm{i}\omega_0 t} - u_2\, \mathrm{e}^{-\mathrm{i}\omega_0 t});$$

$$u_2(t) = u_1^*(t). \tag{7}$$

* A solution of this problem was also obtained in the quantum theory for both equilibrium[10]) and nonequilibrium[11]) oscillators. In ref.[12]) a case of a quantum nonlinear oscillator with nonlinear friction was investigated.

Then eq. (1) may be rewritten as

$$\ddot{q} + \omega_0^2 q = 2\mathrm{i}\,\omega_0 \dot{u}_1\,\mathrm{e}^{\mathrm{i}\,\omega_0 t} = -2\mathrm{i}\,\omega_0 \dot{u}_2\,\mathrm{e}^{-\mathrm{i}\,\omega_0 t}$$
$$= -2\,\mathrm{i}\omega_0 \Gamma(u_1\,\mathrm{e}^{\mathrm{i}\omega_0 t} - u_2\,\mathrm{e}^{-\mathrm{i}\omega_0 t}) - \gamma(u_1\,\mathrm{e}^{\mathrm{i}\omega_0 t} + u_2\,\mathrm{e}^{-\mathrm{i}\omega_0 t})^3 + f(t). \tag{8}$$

Multiplying these equalities by $\exp(\pm\mathrm{i}\,\omega_0 t)$ and neglecting fast oscillating terms in the right-hand parts we obtain the system of stochastic Langevin equations for the functions u_1, u_2:

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = \frac{3\mathrm{i}\gamma}{2\omega_0} y u_1 - \Gamma u_1 + \frac{1}{2\mathrm{i}\,\omega_0}\exp(-\mathrm{i}\omega_0 t) f(t);$$
$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = -\frac{3\mathrm{i}\gamma}{2\omega_0} y u_2 - \Gamma u_2 - \frac{1}{2\mathrm{i}\omega_0}\exp(\mathrm{i}\omega_0 t) f(t); \tag{9}$$
$$y = u_1 u_2.$$

To solve the nonlinear stochastic equations (9) we may use an artificial method[9]. For this purpose the differential equations (9) are formally transformed into integral ones:

$$u_1(t) = u_1(0)\exp[F(t) - \Gamma t] + \exp[F(t)]x(t);$$
$$x(t) = \frac{1}{2\mathrm{i}\omega_0}\exp(-\Gamma t)\int_0^t \mathrm{d}t_1 \exp[-\mathrm{i}\omega_0 t_1 - F(t_1) + \Gamma t_1] f(t_1);$$
$$F(t) = \frac{3\mathrm{i}\gamma}{2\omega_0}\int_0^t y(t_1)\,\mathrm{d}t_1; \quad u_2(t) = u_1^*(t). \tag{10}$$

The equations of motion in the form of (10) are substantially more convenient for carrying out the averaging over realizations of the random force $f(t)$ in the asymptotic time range $t \gg \omega_0^{-1}$. This is connected with the fact that asymptotically the random process $x(t)$ turns out to be normal and the probability distribution of the values of $x(t) = x'(t) + \mathrm{i}x''(t)$ at various times $t_1, t_2, \ldots, t_m$ is Gaussian:

$$w(\ldots, x'(t_n), x''(t_n), \ldots) = (2\pi)^{-m}|A_{nn'}|^{-1}$$
$$\times \exp\left\{-\frac{1}{2}\sum_{n,n'=1}^{m}(A^{-1})_{nn'}[x'(t_n)x'(t_{n'}) + x''(t_n)x''(t_{n'})]\right\}, \tag{11}$$

where

$$A_{nn'} \equiv A(t_n, t_{n'}) = \tfrac{1}{2}\langle x(t_n)x^*(t_{n'})\rangle. \tag{12}$$

If one put $F(t) = 0$ in eq. (10), the Gaussian shape of the distribution of the random function $x(t)$ would be the direct consequence of the Gaussian distribution of $f(t)$. However, due to the δ-correlation of $f(t)$, the presence of the multiplier $\exp[-F(t_1)]$ in the integrand in the expression for $x(t)$ at $t \gg \omega_0^{-1}$ does not change any moments of the distribution or, therefore, the distribution itself. Really, as far as

$$\langle f(t)\, e^{-F(t_1)} \rangle = 0 \quad \text{at} \quad t_1 \leq t, \tag{13}$$

then e.g. under the double integral over t_1, t_2 that determines the second moment $\langle x(t)x^*(t')\rangle$ according to eq. (10), the δ-function $\delta(t_1 - t_2)$ appears and hence the factors $\exp[-F(t_1)]$ and $\exp[F(t_2)]$ are cancelled. As a result, at $t > 0, t' > 0$

$$\langle x(t)x^*(t')\rangle = \frac{B}{4\omega_0^2} \exp[-\Gamma(t + t')] \times [\exp(2\Gamma t_{\min}) - 1], \quad t_{\min} = \min(t, t'). \tag{14}$$

Similarly, the multipliers $\exp[\pm F(t_n)]$ are also cancelled in the expressions for correlators of higher order.

In the case of the correlators $\langle x(t)x(t')\rangle$ and $\langle x^*(t)x^*(t')\rangle$ the fast oscillating multiplier $\exp(\pm 2i\, \omega_0 t)$ remains under the integral after an averaging. Therefore, such correlators at $|t - t'| \gg \omega_0^{-1}$ are proportional to the small parameter Γ/ω_0 and may be neglected in the approximation adopted here.

Thus, the problem of Brownian motion of the Duffing oscillator is reduced to the averaging of the expressions of the type $\exp[F(t)]$ and $x(t)\exp[F(t)]$ over the random process $x(t)$, which is normal but not δ-correlated according to eq. (11). The analogous problem for the nonlinear oscillator interacting with a medium was solved in ref.[9]). There the property of $F(t)$ to be a quadratic functional of $x(t)$ was used and the calculation of the averages mentioned above was reduced to solving a certain integral equation. This allowed us to express the time-correlation functions of the coordinates in the explicit form (by elementary functions). The formulae obtained in ref.[9]) may be used in the case considered here if one substitutes $\frac{1}{2}B$ for the characteristic of thermal noise kT.

Then we get for the time dependence of the coordinate $\langle u_1(t)\rangle$, at a given initial $u_1(0)$:

$$\langle u_1(t)\rangle = u_1(0)a^2 \exp(\Gamma t)[a\, \mathrm{ch}(at) + \Gamma\, \mathrm{sh}(at)]^{-2} \times \exp\left[\frac{3i\gamma|u_1(0)|^2}{2\omega_0} \frac{\mathrm{sh}(at)}{a\, \mathrm{ch}(at) + \Gamma\, \mathrm{sh}(at)}\right], \tag{15}$$

where

$$a = (\Gamma^2 - 3i\gamma B\Gamma/4\omega_0^3)^{1/2} \equiv \Gamma(1 - 4i\alpha)^{1/2}, \quad \mathrm{Re}\, a > 0. \tag{16}$$

Formula (15) at small nonlinearity takes the form

$$\langle u_1(t)\rangle = u_1(0)\exp(-\Gamma t + 4\mathrm{i}\alpha\Gamma t)$$
$$\times\left[1 - 2\mathrm{i}\alpha\left(1 - \frac{2\omega_0^2|u_1(0)|^2}{B}\right)(1 - \mathrm{e}^{-2\Gamma t})\right], \quad |\alpha| \ll 1. \tag{17}$$

It is seen from eq. (17), that already to the first order of the perturbation theory in γ/Γ the nonlinearity results in a renormalization $4\alpha\Gamma$ of the oscillator frequency and nonexponential decay of $\langle u_1(t)\rangle$. At $|\alpha| \gtrsim 1$ the deviations from the exponential decay law $\exp(-\Gamma t)$ become even more essential in accordance with eq. (15).

The expression (15) allows us to determine easily the time-correlation functions of the oscillator coordinates and momenta. The initial distribution of $u_1(0)$ for the Brownian oscillator is Gibbsian. If the condition (6) of the smallness of the frequency straggling due to nonlinearity is satisfied then this distribution is normal

$$w(u_1(0)) = \frac{4\omega_0^2}{\pi B}\exp\left[-\frac{4\omega_0^2}{B}|u_1(0)|^2\right]. \tag{18}$$

The calculation of the correlator $\langle u_1(t)u_1^*(0)\rangle$ is reduced to an averaging of expression (15) multiplied by $u_1^*(0)$ with the statistical weight $w(u_1(0))$. We obtain as a result:

$$\tilde{Q}(t) = \langle u_1(t)u_1^*(0)\rangle = \frac{B}{4\omega_0^2}\mathrm{e}^{\Gamma t}\psi^{-2}(t);$$
$$\psi(t) = \mathrm{ch}(at) + \frac{\Gamma(1-2\mathrm{i}\alpha)}{a}\mathrm{sh}(at), \quad \beta \ll 1. \tag{19}$$

It is seen from eq. (19) that the correlator $\tilde{Q}(t)$ is expressed by elementary functions. Its dependence on a dimensionless time Γt is determined by the single parameter α. This dependence was analyzed in ref.[9]).

Parallel with the function $\tilde{Q}(t)$ the spectral distribution of the time-correlation function of the coordinates

$$Q(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\omega t}\langle q(t)q(0)\rangle \tag{20}$$

is of interest for a number of applications. In the case under consideration ($\beta \ll 1$) the expression for $Q(\omega)$ near its maximum is reduced to a spectral representation of $\tilde{Q}(t)$:

$$Q(\omega) \simeq \tilde{Q}(\Omega); \quad \tilde{Q}(\Omega) = \frac{1}{\pi}\mathrm{Re}\int_0^{\infty}\mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\Omega t}\tilde{Q}(t),\ \Omega = \omega - \omega_0, \quad |\Omega| \ll \omega_0. \tag{21}$$

Expressions (19) and (21) allow us to obtain $\tilde{Q}(\Omega)$ in the explicit form at $|\alpha| \ll 1$ or $|\alpha| \gg 1$. In the range $|\alpha| \ll 1$, to the first order in α, the distribution $\tilde{Q}(\Omega)$ remains Lorentzian and is only shifted by the value

$$\Omega_{\mathrm{m}} = 4\alpha\Gamma \tag{22}$$

relative to the corresponding distribution for the harmonic oscillator. The terms quadratic in α distort the shape of the distribution although it remains symmetrical. As $|\alpha|$ increases, the distribution peak is shifted more (see fig. 2 in ref.[9])). It becomes essentially asymmetrical and in the range of large $|\alpha|$ it is described by the formula

$$\tilde{Q}(\Omega) = \frac{B}{16\omega_0^2\Gamma^2\alpha^2} |\Omega| \exp\left(-\frac{\Omega}{2\alpha\Gamma}\right)\Theta(\alpha\Omega) \quad \text{at } \omega_0 \gg |\Omega| \gg \Gamma\sqrt{\alpha}, \tag{23}$$

where $\Theta(x) = 1$ at $x > 0$ and $\Theta(x) = 0$ at $x < 0$. In this case the position of the distribution maximum is

$$\Omega_{\mathrm{m}} = 2\alpha\Gamma. \tag{24}$$

The dynamics of the Duffing oscillator may also be investigated by another method based on the solution of the corresponding Fokker–Planck equation. It is shown in the appendix that such a solution at $\beta \ll 1$ may be obtained using the method of generating functions. In particular, it allows one to determine the time-correlation functions for arbitrary powers of the coordinates and momenta.

3. Time-correlation functions for a nonlinear oscillator when the damping is neglected

The method used in the previous section is essentially based on assumption (6), that the frequency straggling $\delta\omega$ caused by the oscillator nonlinearity is small compared with the frequency ω_0. A consideration valid for an arbitrary ratio of $\delta\omega$ and ω_0 will be made below. However, the damping will be assumed to be small not only compared with ω_0 but also compared with $\delta\omega$, i.e., the condition $\alpha \gg 1$ will be used. In this case the broadening of the spectral distribution of the time-correlation function of the coordinates is mainly due to the frequency straggling $\delta\omega$, while the damping Γ causes only insignificant corrections, which may be neglected in zeroth order approximation (at $\beta \ll 1$ this is seen, for example, from eq. (23), reckoning that $\alpha\Gamma$ does not depend on Γ).

The damping and random force at $\Gamma \ll \omega_0$, $\delta\omega$ provide the forming of the distribution of the oscillator over the energy

$$w(E) = Z^{-1}\exp(-2E/B); \quad Z = \int \mathrm{d}q\,\mathrm{d}p\,\exp(-2E/B);$$
$$E = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega_0^2 q^2 + \tfrac{1}{4}\gamma q^4. \tag{25}$$

Since the oscillator motion for the time $\sim(\delta\omega)^{-1}$ at $\Gamma \ll \delta\omega$ is quasiconservative, to calculate the time-correlation functions one may first solve the problem of the free motion of the nonlinear undamped oscillator with a given energy and then average over the energy (and phase) with the weight (25). This procedure may be based strictly mathematically on the grounds of the Fokker–Planck equation.

The solution of the equation of motion for the Duffing oscillator

$$\ddot{q} + \omega_0^2 q + \gamma q^3 = 0 \quad (\gamma \geqslant 0) \tag{26}$$

with a given energy E is (see, for example, ref.[13])

$$q(t) = \frac{\omega_0}{\sqrt{\gamma}}\sqrt{b^2-1}\,\mathrm{cn}\left(\frac{2K}{\pi}\varphi\right); \quad \varphi = \frac{\pi}{2K}b\omega_0 t + \varphi_0, \tag{27}$$
$$k^2 = m = \frac{b^2-1}{2b^2}; \quad b^2 = \sqrt{1+\frac{4\gamma E}{\omega_0^4}}.$$

Here, $\mathrm{cn}\,u \equiv \mathrm{cn}(u \mid m)$ is the Jacobian elliptic cosine, k is the module, (m is the parameter), $K \equiv K(k)$ is the complete elliptic integral of the first kind, φ_0 is the initial phase. The function $\mathrm{cn}\,u$ has a period $4K$. Therefore, the nonlinear oscillator frequency according to (27) is:

$$\omega(E) = \frac{\pi}{2K}b\omega_0. \tag{28}$$

The time-correlation function of the coordinates

$$Q(t) = \langle q(t)q(0)\rangle = \int \mathrm{d}q(0)\,\mathrm{d}p(0)q(t)q(0)w(E) \tag{29}$$

is determined by formulae (25) and (27). To calculate the integral (29) it is convenient to transform to new canonical variables: the action I and phase φ and then to the energy E and phase φ, taking into account the relations

$$\mathrm{d}q\,\mathrm{d}p = \mathrm{d}I\,\mathrm{d}\varphi; \quad \mathrm{d}I = \omega^{-1}(E)\,\mathrm{d}E \tag{30}$$

(the action depends only on the energy). Thus the averaging in eq. (29) over the initial coordinates and momenta reduces to an integration over E and the initial phase φ_0 entering formula (27) for $q(t)$. The averaging over φ_0 may be easily performed using a series expansion of the elliptic cosine in eq. (27) in the

Jacobi parameter (see, for example, ref. 14)

$$\mathrm{cn}\left(\frac{2K}{\pi}\varphi\right)=\frac{2\pi}{kK}\sum_{n=0}^{\infty}q_J^{n+1/2}[1+q_J^{2n+1}]^{-1}\cos(2n+1)\varphi;$$
$$q_J=q_J(k)=\exp[-\pi K'(k)/K(k)];\quad K'(k)=K(\sqrt{1-k^2}). \tag{31}$$

Substituting expressions (27) and (31) for $q(t)$ and $q(0)$ into eq. (29), transforming to the integration over E and φ_0, and taking eqs. (30), (25) and (28) into account, we obtain as a result of integration over φ_0:

$$Q(t)=\frac{4\pi^3\omega_0^2}{\gamma}Z^{-1}\int_0^{\infty}\frac{\mathrm{d}E}{\omega(E)}\exp\left(-\frac{2E}{B}\right)\frac{b^2-1}{\kappa^2K^2}\sum_{n=0}^{\infty}\frac{q_J^{2n+1}}{(1+q_J^{2n+1})^2}\cos[(2n+1)\omega(E)T],$$

$$Z=2\pi\int_0^{\infty}\frac{\mathrm{d}E}{\omega(E)}\exp\left(-\frac{2E}{B}\right),\quad b=b(E),\quad k\equiv k(E). \tag{32}$$

It is seen from eq. (27) that $k^2<1/2$, i.e., $q_J^2<\mathrm{e}^{-2\pi}\sim 2\times 10^{-3}$. Therefore, series (32) converges very fast and it is practically sufficient to retain only the first term in it (if the spectrum in the range $\omega/\omega(B)\gg 1$ is not of interest). Thus the expression for $Q(\omega)$ is reduced to a quadrature.

The spectral distribution of the time-correlation function according to eq. (32) is:

$$Q(\omega)=\frac{8\pi\omega}{\gamma Z}\,\Theta(\omega-\omega_0)\sum_{n=0}^{\infty}\varphi_n(\omega)\simeq\frac{8\pi\omega}{\gamma Z}\,\Theta(\omega-\omega_0)\varphi_0(\omega);$$
$$\varphi_n(\omega)=(2n+1)^{-2}\exp\left(-\frac{2E_n}{B}\right)\frac{b^2(E_n)-1}{b^2(E_n)k_n^2}\left(\frac{\mathrm{d}\omega(E_n)}{\mathrm{d}E_n}\right)^{-1}\frac{q_J^{2n+1}(k_n)}{[1+q_J^{2n+1}(k_n)]^2}; \tag{33}$$

where $k_n\equiv k(E_n)$, and E_n is determined by the equation

$$\omega(E_n)=\frac{\omega}{2n+1}. \tag{34}$$

The function $Q(\omega)$ according to eq. (33) is expressed by well-known special functions. The explicit form of $Q(\omega)$ may be rather easily obtained in the limiting cases for small and large values of the parameter β, eq. (6). The case $\beta\ll 1$ corresponds to the oscillator with the anharmonic part of the potential energy $\frac{1}{4}\gamma q^4$ being small compared with the harmonic one in the actual energy range, whereas in the case $\beta\gg 1$ the harmonic part is negligibly small compared with $\frac{1}{4}\gamma q^4$.

At $\beta\ll 1$ in the actual energy range $E\lesssim B$ we have $b^2\simeq 1$, $k^2\ll 1$. Then expression (33) reduces to formula (23) obtained above by a different method. At $\beta\gg 1$ and $\omega^4\gg\omega_0^4$, according to eq. (27) we have $b^2\gg 1$, $k^2\simeq\frac{1}{2}$, and taking into account eqs. (28) and (34), expression (33) in the range of its maximum (where the term with $n = 0$ may only be retained in eq. (33)) is given by

$$Q(\omega) = \frac{\xi_1 \omega_0}{\gamma \beta^{3/4}} \left(\frac{\omega}{\omega_0}\right)^4 \exp\left[-\frac{\xi_2}{\beta}\left(\frac{\omega}{\omega_0}\right)^4\right] \Theta(\omega - \omega_0), \quad \omega^4 \gg \omega_0^4, \quad \beta \gg 1;$$

$$\xi_1 = \frac{16\sqrt{2}}{\pi^3 \Gamma(\frac{3}{4})} \operatorname{ch}^{-2}(\tfrac{1}{2}\pi) K^3\left(\frac{1}{\sqrt{2}}\right) \simeq 0.60; \quad \xi_2 = \frac{4}{\pi^4} K^4\left(\frac{1}{\sqrt{2}}\right) \simeq 0.49. \tag{35}$$

The position of the maximum of this curve ω_m and its integral width $\delta\omega_i$ are determined by the formulae:

$$\omega_m = (\beta/\xi_2)^{1/4}\omega_0 \simeq 1.2\beta^{1/4}\omega_0; \quad \delta\omega_i \simeq 0.74\beta^{1/4}\omega_0. \tag{36}$$

It is seen from eqs. (36) and (23), that the shift and broadening of the spectral distribution due to the nonlinearity at $|\alpha| \gg 1$ are of the same order of magnitude.

It is not difficult to determine the function $Q(\omega)$ numerically at artibrary values of β. The plots of this function for several values of β are given in fig. 1. It is seen that, as β increases the width of the distribution increases too, and its maximum is shifted to larger frequencies.

It should be noted that one may neglect damping at the evaluation of the spectral distribution if the frequency lies in the range where $Q(\omega)$ is not

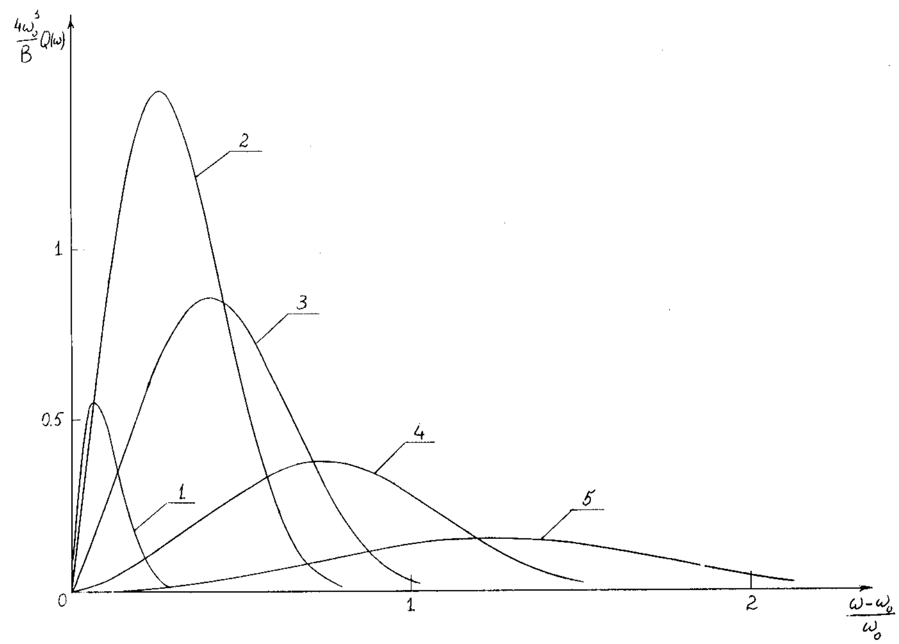

Fig. 1. Plots of the dependence of $Q(\omega)$ on ω/ω_0. Curves 1–5 correspond to the values $\beta = 0.1$; 0.5; 1; 3; 10. Curve 1 is scaled down by a factor 10 along the ordinate axis.

negligibly small. The nonvanishing damping even being small causes the distribution to fall off in the wings as $\Gamma B(\omega^2-\omega_m^2)^{-2}$ at $\beta \leqslant 1$ ($|\omega-\omega_m| \gg \delta\omega_i$). Therefore, the results obtained above are valid only in the range where this value is small compared with expression (33).

4. Discussion of the results

The Brownian motion of the Duffing oscillator according to eq. (1) is determined by four parameters: ω_0, Γ, γ and B. In fact the number of parameters is reduced to two, namely: β sign γ and Γ/ω_0(or β sign γ and $\alpha = \frac{3}{8}(\beta\omega_0/\Gamma)$ sign γ) after a transition to the dimensionless time $\omega_0 t$ and dimensionless coordinate $\omega_0 q/\sqrt{B}$. The results given above allow us to describe the dynamics of the Duffing oscillator and to determine explicitly the time-correlation functions of coordinates and their spectral representations in a wide interval of the values of these parameters:

$$0 \leqslant \beta \ll \alpha < \infty \quad \text{or} \quad 0 \leqslant \beta < \infty, \quad \Gamma \ll \omega_0. \tag{37}$$

In the range $\beta \ll 1$ the results are valid not only for $\gamma > 0$ (i.e., $\alpha > 0$), when the motion is finite for all the energies but for $\gamma < 0$ also (although here the motion is infinite at high energies, the probability to achieve such energies is exponentially small at small β). It should be noticed that in the limiting case $\beta \gg 1$ formula (35) describes in particular the spectral distribution of the oscillator with $\omega_0^2 < 0$, i.e. with the two-well potential, and also the oscillator with a large damping $\beta\omega_0 \gg \Gamma \gtrsim \omega_0$. The range $|\alpha| \sim \beta \sim 1$ cannot be described by the methods used in the present paper. Some results for this range were obtained numerically and using the expansion of the spectral distributions in a continued fraction[6,7]).

In the range $|\alpha| \ll 1$, where the results of papers[2–4]) are valid, the expressions given in section 2 agree with these results. It is seen from eqs. (15)–(19) that the parameter used to carry out the expansion in this range is α. In the range $|\alpha| \ll 1$ the equivalent linearization method[1]) is also applicable, in which the nonlinear oscillator is substituted by a linear one. The effective frequency ω_e of the latter is determined as a square root of the normalized second moment of the spectral distribution $Q(\omega)$.

In the range $\beta \ll 1$ but $|\alpha| \gtrsim 1$, the expansions in the nonlinearity parameter γ (or in α) become inapplicable. When $|\alpha| \gtrsim 1$, the broadening of the spectral distribution due to nonlinearity is of the same order of magnitude as its shift and hence the effective frequency ω_e may substantially differ from the position of the maximum of the distribution $Q(\omega)$. For example, for $\beta \ll 1$ and $|\alpha| \gg 1$ we have $(\omega_e-\omega_0)/(\omega_m-\omega_0) = 2$, whereas for $\beta \gg 1$ we have $\omega_e/\omega_m \simeq 1.4$.

Appendix

Solution of the Fokker–Planck equation for the Duffing oscillator

The Fokker–Planck equation in the complex variables $u_1, u_2 = u_1^*$, corresponding to the Langevin equations (9) and describing the Brownian motion of the Duffing oscillator, when condition (6) is satisfied, is of the form

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial u_1}\left[\left(\Gamma u_1 - \frac{3i\gamma}{2\omega_0} u_1^2 u_2\right) v\right] + \frac{\partial}{\partial u_2}\left[\left(\Gamma u_2 + \frac{3i\gamma}{2\omega_0} u_1 u_2^2\right) v\right] + \frac{B\Gamma}{2\omega_0^2}\frac{\partial^2 v}{\partial u_1 \partial u_2}. \quad \text{(A.1)}$$

Here $v = v(u_1, u_2, t; u_{10}, u_{20}, t_0)$ is the probability density for the oscillator transition from the state with coordinates u_{10}, u_{20} at time t_0 to the state u_1, u_2 at time t. In (A.1) the fast oscillating terms $\exp(\pm 2i\omega_0 t)$ that are inessential in the interesting range $t - t_0 \gg \omega_0^{-1}$ are neglected. Eq. (A.1) should be solved with the initial condition

$$v(u_1, u_2, t_0; u_{10}, u_{20}, t_0) = \delta(u' - u_0')\delta(u'' - u_0''), \quad u_1 = u_2^* = u' + iu''. \quad \text{(A.2)}$$

It is convenient to transform to the variables r, φ, so that

$$u_1 = \sqrt{r}\, e^{i\varphi}, \quad u_2 = \sqrt{r}\, e^{-i\varphi}. \quad \text{(A.3)}$$

Then eq. (A.1) takes the form

$$\frac{\partial v}{\partial t} = 2\Gamma v + \frac{B\Gamma}{2\omega_0^2}\left(r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} + \frac{1}{4r}\frac{\partial^2 v}{\partial \varphi^2}\right) + 2\Gamma r\frac{\partial v}{\partial r} - \frac{3\gamma}{2\omega_0} r \frac{\partial v}{\partial \varphi}. \quad \text{(A.4)}$$

Obviously the function v is periodic in φ. Therefore a solution of eq. (A.4) may be developed into a series:

$$v(r, \varphi, t) \equiv v(r, \varphi, t; r_0, \varphi_0, t_0) = \sum_{n=-\infty}^{\infty} v_n(r, t)\, e^{-in\varphi}. \quad \text{(A.5)}$$

As far as the coefficients in (A.4) do not depend on φ, the equations for $v_n(r, t)$ resulting from (A.4) (with various n) are separated. It is convenient to solve the one-dimensional diffusion equations for the functions $v_n(r, t)$ by the method of generating functions similar to that used in the quantum theory of the nonlinear oscillator interacting with a medium[10,11]). Let us introduce the generating function

$$W_n(x, t) = \int_0^\infty v_n(r, t) r^{n/2} \exp(-xr)\, dr, \quad n \geq 0. \quad \text{(A.6)}$$

According to eqs. (A.4)–(A.6) the function $W_n(x, t)$ satisfies the equation:

$$\frac{\partial W_n}{\partial t} = -\frac{B\Gamma x}{2\omega_0^2}\left[(1+n)W_n + x\frac{\partial W_n}{\partial x}\right] - \Gamma n W_n - 2\Gamma x \frac{\partial W_n}{\partial x} - \frac{3i\gamma n}{2\omega_0}\frac{\partial W_n}{\partial x} \quad \text{(A.7)}$$

with the initial condition

$$W_n(x,0)=\frac{1}{\pi}\,r_0^{n/2}\exp(-xr_0)\,\mathrm{e}^{\mathrm{i}n\varphi_0},\quad r_0=|u_1(0)|^2 \tag{A.8}$$

following from eqs. (A.2), (A.5) and (A.6).

The linear differential equation of the first order (A.7) may be solved by the method of characteristics. Direct substitution allows one to verify that the solution of eq. (A.7) is of the form

$$W_n(x,t)=\frac{1}{\pi}\,r_0^{n/2}\,\mathrm{e}^{\mathrm{i}n\varphi_0}\,a_n^{n+1}\,\mathrm{e}^{\Gamma t}\left[a_n\,\mathrm{ch}(a_nt)+\Gamma\left(1+\frac{Bx}{2\omega_0^2}\right)\mathrm{sh}(a_nt)\right]^{-(n+1)}$$
$$\times\exp\left[-r_0\,\frac{xa_n\mathrm{ch}(a_nt)-(\Gamma x+3\mathrm{i}\gamma n/2\omega_0)\,\mathrm{sh}(a_nt)}{a_n\,\mathrm{ch}(a_nt)+\Gamma(1+Bx/2\omega_0^2)\,\mathrm{sh}(a_nt)}\right], \tag{A.9}$$

where

$$a_n=(\Gamma^2-3\mathrm{i}\gamma B\Gamma n/4\omega_0^3)^{1/2}=\Gamma(1-4\mathrm{i}\alpha n)^{1/2}. \tag{A.10}$$

Expression (A.9) for the generating function permits the straightforward determination of the average values for various powers of the dynamical variables. For example, taking into account that according to eqs. (A.3), (A.5) and (A.6)

$$\langle u_1^n(t)\rangle=\tfrac{1}{2}\int \mathrm{d}r\,\mathrm{d}\varphi r^{n/2}\,\mathrm{e}^{\mathrm{i}n\varphi}\,v(r,\varphi,t)=\pi W_n(0,t), \tag{A.11}$$

one may obtain immediately the value of $\langle u_1^n(t)\rangle$, assuming in eq. (A.9) $x=0$ and $r_0=|u_1(0)|^2$. Similarly, from the equality:

$$\langle u_1^n(t)u_2^m(t)\rangle=\pi\int \mathrm{d}r\,r^{(n-m)/2}r^m v_{n-m}\quad (n\geqslant m) \tag{A.12}$$

it follows that

$$\langle u_1^n(t)u_2^m(t)\rangle=\pi(-1)^m\left(\frac{\partial^m W_{n-m}(x,t)}{\partial x^m}\right)_{x=0}. \tag{A.13}$$

It is seen from eqs. (A.13) and (A.9) in particular, that the nonlinearity does not influence the decay of the values $\langle u_1^n(t)u_2^n(t)\rangle$ (for example the decay of the energy).

The explicit expressions (A.11) for the average values of $\langle u_1^n(t)\rangle$ allow one to determine easily the double-time-correlation functions of the type $\langle u_1^n(t)u_2^n(0)\rangle$ for arbitrary n, by integration over the initial coordinates $u_1(0)$

with the statistical weight (18):

$$\langle u_1^n(t)u_2^n(0)\rangle = \left(\frac{B}{4\omega_0^2}\right)^n \mathrm{e}^{\Gamma t}[\psi_n(t)]^{-(n+1)},$$

$$\psi_n(t) = \mathrm{ch}(a_n t) + \frac{\Gamma}{a_n}(1-2\mathrm{i}\alpha n)\,\mathrm{sh}(a_n t). \tag{A.14}$$

The spectral representation of the correlator (A.14) describes, in particular, a susceptibility of the nonlinear oscillator near the nth overtone. At $n = 1$ formula (A.14) is transformed into eq. (19). Similarly, one may obtain the expressions for the correlation functions of the type $\langle u_1^n(t)u_2^m(t)u_1^m(0)u_2^n(0)\rangle$ that give the correlators of the coordinates and momenta $q(t)$, $\mathrm{d}q(t)/\mathrm{d}t$ according to eq. (7).

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