13 Transition probabilities and spectral density of fluctuations of noise driven bistable systems

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13.1 Introduction

One of the important problems of physical kinetics is the investigation of relaxation and fluctuation phenomena in systems which have two or more stable states. Bi- and multistable systems are studied in various fields of physics, and the causes of multistability and the types of stable states are different in different cases. For systems moving in static potential fields (disregarding the interactions that give rise to relaxation and fluctuations in a system) multistability takes place if a potential has several minima. In this case the stable states are the equilibrium states. A number of systems of this type are investigated in solid state physics; in particular, diffusing atoms and impurity centers that reorient within a unit cell (see Narayanamurti and Pöhl, 1970).

Multistability may also arise in systems driven by an external periodic field. The constrained vibrations correspond to the stable states in this case (the attractors with a more complicated structure may also appear here). In particular, nonlinear oscillators of various physical nature refer to such systems (see Landau and Lifshitz, 1976). It is well known that in a certain frequency range the dependence of the amplitude A of the constrained vibrations of a nonlinear oscillator on the resonant external field amplitude h may be S-shaped (cf. curve (c) in Figure 13.1). In the range of the non-single-valued dependence $A(h)$ the states with the largest and smallest A are stable. In the absence of noise the oscillator appears in one or another state depending on the 'history' of the field amplitude or frequency variation, i.e. the hysteresis is present when the field is varied.

Bi- and multistability of nonlinear systems in an external periodic field is studied intensively at present in nonlinear optics (Gibbs, 1985). Several stable equilibrium or vibrational states may arise also in nonequilibrium nonlinear systems driven by stationary energy sources of other types. The examples of such systems are well known in fluid dynamics, radiophysics, chemical kinetics, laser physics, etc. (see Haken, 1983).

The interaction of a multistable system with a medium (in particular, with a thermal bath) leads to relaxation of the system and to fluctuations in it.

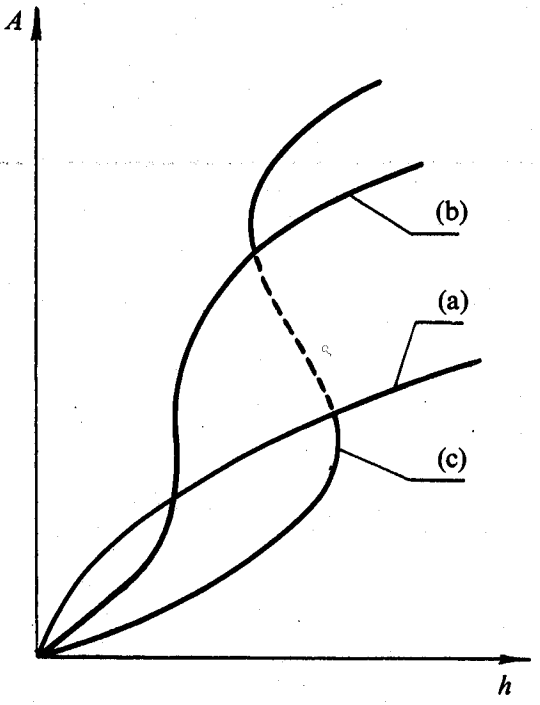

Figure 13.1. Schematic dependence of the amplitude A of the constrained vibrations of a nonlinear oscillator on the amplitude h of an external resonance force. Curves (a)–(c) correspond to different frequency detunings $\omega_h - \omega_0$. (b) corresponds to the critical value of $\omega_h - \omega_0$ starting with which the plot of A versus h becomes S-shaped. The unstable stationary states are shown as a dashed line.

Fluctuations may also arise due to other noise sources acting on a system. The presence of several stable states gives rise to a number of features of relaxation and fluctuation phenomena in a system. These features are due to a great extent to the sufficiently large fluctuations being able to result in transitions between the stable states.* As a result of fluctuational transitions the stationary distribution of a system over the states is established and the dependence of the characteristics of a system on its parameters becomes single-valued. In particular, for sufficiently slow parameter variation the hysteresis described above for a nonlinear oscillator does not arise.

Since large fluctuations are needed for the transitions between the states to occur the transition probabilities W are small for small intensity $\mathscr{B}$ of a random force acting on a system. The value of W in the case of a system performing Brownian motion in a static potential was obtained by Kramers (1940). In this case $\ln W \propto -\Delta U/\mathscr{B}$, where ΔU is the potential barrier height

* We use the term 'stable states' for those states that are stable with respect to small fluctuations.

(when relaxation and fluctuations result from coupling to a thermostat, $\mathscr{B} \propto T$ and $W \propto \exp(-\Delta U/T)$). Transition probabilities may be obtained also for Markov systems in which the potentiality conditions are fulfilled and there are no flows in the steady state.

In the general case of a nonpotential motion or motion in a variable field the calculation of W at small $\mathscr{B}$ is rather a complicated problem. To solve it when random forces are of the white-noise type or, to be more general, are Gaussian it is convenient (Dykman and Krivoglaz, 1979) to use the Feynman approach (Feynman and Hibbs, 1965) to the description of fluctuations in dynamical systems approach. This is based on the path integral method. It allows us to reduce the problem to an investigation of the extreme trajectory of a certain auxiliary dynamical system and to show the transition probability to be given by the expression

$$W = \text{const} \cdot \exp(-P/\mathscr{B}). \tag{13.1.1}$$

The quantity P in (13.1.1) is proportional to the action along the extreme trajectory, and $PT/\mathscr{B}$ plays a role of activation energy. The value of P is determined by the parameters of a dynamical system and does not depend on random force intensity. In particular, for a nonlinear oscillator in a strong resonant field, the quantities P were obtained explicitly (Dykman and Krivoglaz, 1979).

Both P and the constant pre-exponential factor in (13.1.1) may be obtained in a simple explicit form for the problem of an escape from a metastable state near bifurcation point where this state coalesces with an unstable stationary state (Dykman and Krivoglaz, 1980a). As the bifurcation point is approached the escape probability and fluctuations as a whole increase rapidly.

In a bistable system (to be precise we assume in the following that the number of stable states is equal to two) the populations w_1 and w_2 of the stable states 1 and 2 are inversely proportional to the probabilities W_{12} and W_{21} of the transitions from these states. In the general case, the values of P_1 and P_2 in (13.1.1) for the transitions $1 \to 2$ and $2 \to 1$ are different, and $|P_1 - P_2| \gg \mathscr{B}$ for small $\mathscr{B}$. Therefore w_1 and w_2 differ by many times in almost the entire range of the parameters of the system, and one of w_i is close to unity while another is close to zero. Only in the extremely narrow range of parameters where $P_1 \simeq P_2$ the values of w_1 and w_2 are of the same order of magnitude. Owing to the sharp exponential dependence (13.1.1) of W on P, the behavior of the system under its parameters passing through the range where $P_1 \simeq P_2$ is perceived as a smeared first-order phase transition. In particular, in case of a nonlinear oscillator in a strong resonant field the vibration amplitude A changes sharply from the value corresponding to one branch of $A(h)$ to that for another branch (see Figure 13.1), with a corresponding sharp change in the absorption of energy from the resonant field. In the transition region fluctuations in bistable systems acquire a number of characteristic features.

The important characteristic of relaxation and fluctuations in a system is the

spectral density $Q'(\omega)$ of fluctuations of the generalized coordinates. For underdamped systems far from the transition region $Q'(\omega)$ has peaks at the eigenfrequencies of vibrations about that stable state whose population $w \simeq 1$. In the transition region besides the peaks corresponding to vibrations near both stable states $Q'(\omega)$ has an additional extremely narrow peak due to fluctuational transitions between the states. Its width is of the order of the transition probabilities $W_{12} \sim W_{21}$. It is much smaller than inverse characteristic relaxation times. For systems moving in a static bistable potential, the peak induced by fluctuational transitions occurs at zero frequency. In systems performing forced vibrations at the strong-field frequency ω_h the peak lies at the same frequency ω_h. The corresponding extremely narrow peak at ω_h is present also in the spectrum of the absorption of an additional weak field (Dykman and Krivoglaz, 1979).

As the random force intensity $\mathscr{B}$ increases, the form of $Q'(\omega)$ for underdamped systems changes substantially (Dykman and Krivoglaz, 1971, 1984; Dykman, Soskin and Krivoglaz, 1984). Even when $\mathscr{B}$ is rather small, the shape of the peaks due to small-amplitude vibrations about the stable states changes strongly because the vibration nonlinearity reveals itself. The width of the peak induced by fluctuational transitions increases exponentially with $\mathscr{B}$ (see (13.1.1)). In addition, the relatively broad peak of $Q'(\omega)$ caused by features of the motion in the vicinity of the unstable equilibrium state appears.

In what follows, two relatively simple but nontrivial bistable systems, a nonlinear oscillator driven by a sufficiently strong resonant field and an oscillator moving in a static potential with two minima, are analyzed in considerable detail. The results for these model systems describe many properties of the physical systems mentioned above.

In Section 13.2 the probabilities of transitions between stable states of a nonequilibrium system are considered using the path integral method. The spectra of fluctuations at low noise intensities, including the narrow peaks induced by fluctuational transitions, are analyzed in Section 13.3. In Section 13.4 the features of the spectral density of fluctuations in underdamped, essentially nonlinear systems with one and two stable states are investigated.

13.2 Probabilities of transitions between stable states of a nonequilibrium system

In the analysis of transition probabilities for small intensities of a random force acting on a dynamical system the most important thing is to calculate the argument of the exponential in (13.1.1), while it suffices to estimate only the order of magnitude of the pre-exponential factor. The method given in Section 13.2.1 permits the calculation of the transition probabilities to logarithmic accuracy.

We illustrate this method and its application by considering the example of

an underdamped nonlinear Duffing oscillator driven by a relatively strong resonant force $h\cos\omega_h t$ and a weak random force $f(t)$ which presents white noise (Dykman and Krivoglaz, 1979). A Duffing oscillator in a resonant field is frequently used as a model; in particular in the analysis of optical bistability (Flytsanis and Tang, 1980; Goldstone and Garmire, 1984).

The equation of motion for the normal coordinate q of the oscillator is of the form

$$\left.\begin{aligned} &\ddot{q} + 2\Gamma\dot{q} + \omega_0^2 q + \gamma q^3 = h\cos\omega_h t + f(t) \quad (t>0) \\ &\langle f(t)f(t')\rangle = 2\mathscr{B}\delta(t-t'). \end{aligned}\right\} \tag{13.2.1}$$

Here ω_0 is the eigenfrequency of small-amplitude vibrations in the absence of friction and external forces, Γ is the friction coefficient, and γ is the nonlinearity parameter. We suppose the oscillator to be underdamped,

$$\Gamma \ll \omega_0, \tag{13.2.2}$$

and that the periodic force is resonant and not too strong, so that $|\omega_h - \omega_0|$, $\gamma\langle q^2\rangle \ll \omega_h$. (We assume hereafter that $\gamma > 0$; the results may be generalized immediately also to the case $\gamma < 0$.)

When these conditions are fulfilled it is convenient to transform from the fast oscillating variables $q, \dot{q}$ to the slowly varying (over the time $\sim \omega_h^{-1}$) dimensionless envelopes u_1, u_2,

$$\left.\begin{aligned} q &= (3\gamma/8\omega_h\Gamma)^{-1/2}(u_1\cos\omega_h t - u_2\sin\omega_h t), \\ \dot{q} &= -\omega_h(3\gamma/8\omega_h\Gamma)^{-1/2}(u_1\sin\omega_h t + u_2\cos\omega_h t). \end{aligned}\right\} \tag{13.2.3}$$

Using the ideas of the averaging method from nonlinear vibration theory (see Bogolyubov and Mitropolsky, 1961) and neglecting small fast oscillating corrections to u_1, u_2 one can write the equations for u_1, u_2 in the form

$$\left.\begin{aligned} &\dot{u}_n = v_n(\mathbf{c};\mathbf{u}) + f_n(t), \quad \langle f_n(t)f_m(t')\rangle = 2\alpha\Gamma\delta_{nm}\delta(t-t'), \\ &\alpha = 3\gamma\mathscr{B}/16\omega_h^3\Gamma^2. \end{aligned}\right\} \tag{13.2.4}$$

Here $\mathbf{u} \equiv (u_1, u_2), \mathbf{c} \equiv (c_1, c_2)$. The functions v_1, v_2 are cubic polynomials in u_1, u_2 (the explicit expression for $v(u_1 + iu_2, u_1 - iu_2) = (v_1 + iv_2)/\Gamma$ is given by Dykman and Krivoglaz, 1979). The quantities c_1 and c_2 determine the characteristic dimensionless parameters of the dynamical system,

$$c_1 = \Gamma/(\omega_h - \omega_0), \quad c_2 = 3\gamma h^2/32\omega_h^3|\omega_h - \omega_0|^3. \tag{13.2.5}$$

The dimensionless parameter α, characterizing noise intensity in (13.2.4), is supposed to be small.

The bistability of the nonlinear oscillator can arise (in the absence of a random force) due to the dependence of the effective vibration frequency $\omega_{\text{eff}} \simeq \omega_0 + \frac{3}{8}\gamma A^2/\omega_0$ on the vibration amplitude A. As a consequence of this dependence, forced vibrations of both large amplitude (for which ω_{eff} is close to ω_h and the resonance condition is fulfilled very well) and also of small

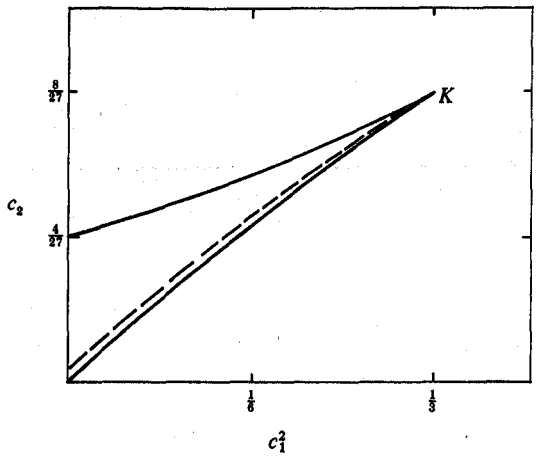

Figure 13.2. Range of coexistence of two stable states of a nonlinear oscillator (solid lines are the loci of the bifurcation points, K is the spinode). The dashed line corresponds to the parameter values at which the probabilities of the transitions $1 \to 2$ and $2 \to 1$ are equal and thus the kinetic phase transition occurs (Dykman and Krivoglaz, 1979).

amplitude (with considerably larger $|\omega_h - \omega_{\text{eff}}|$) may turn out to be stable and 'self-consistent' provided that $\gamma(\omega_h - \omega_0) > 0$. The region of the parameters c_1^2, c_2 where the bistability occurs is bounded by the full curves in Figure 13.2.

Equations of the type (13.2.4) also describe the dynamics of more complicated (than an oscillator) systems. In this case the number of the variables u_n may exceed two, while the intensities of the random forces may differ for different n.

The phase portrait of an oscillator in the variables u_1 and u_2 for the parameter range where the bistability occurs is shown schematically in Figure 13.3. The dashed line in the figure is the separatrix between the attraction regions of the foci f_1 and f_2. The saddle point, s, lies on this line. Such a phase portrait is typical for a wide class of bistable systems. In the absence of a random force, the system, located at the initial instant at some general position point, will approach, over a characteristic relaxation time $t_r (t_r \sim \Gamma^{-1})$, that focus (or node) in whose attraction region it was located initially. (Examples of phase trajectories are shown in Figure 13.3.)

In the presence of a weak random force the system moves, with overwhelming probability, practically along the same trajectory. On approaching the focus, the system stays near it for a long time, greatly exceeding t_r, undergoes small fluctuations. Ultimately it will experience a sufficiently large fluctuation, as a result of which the phase trajectory will cross the separatrix. After this the system will approach another focus over a time $\sim t_r$ and then will fluctuate near it. This just means a transition to a new stable state.

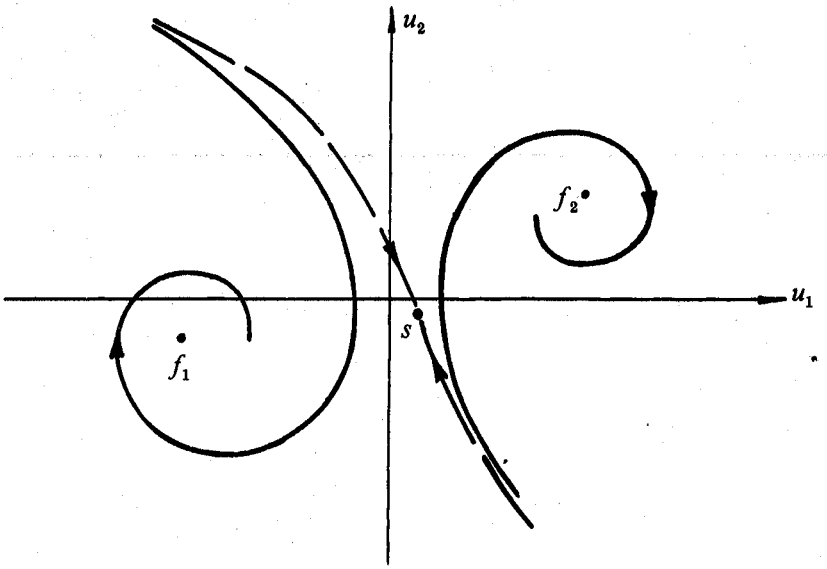

Figure 13.3. Schematic phase portrait of a nonlinear dynamical system with two stable states of the focus type. The dashed line is the separatrix, s is the saddle point, f_1 and f_2 are foci. Arrows indicate the direction of motion.

13.2.1 *Application of the path-integral method to the calculation of transition probabilities*

With the aim of calculating the probabilities W_{ij} of fluctuational transitions we shall consider the probability density $w(\mathbf{u}_b; \mathbf{u}_a; t_b - t_a)$ of the transition of the system from a certain point $\mathbf{u}_a$ (in the phase space), at which the system was located at instant t_a, to a point $\mathbf{u}_b$ at instant t_b. To logarithmic accuracy the value of, say, W_{12} is determined by the function w for $\mathbf{u}_a$ located near the focus f_1 and $\mathbf{u}_b$ located near the separatrix.

It is convenient to write $w(\mathbf{u}_b; \mathbf{u}_a; t_b - t_a)$ in the form of a path integral

$$w(\mathbf{u}_b; \mathbf{u}_a; t_b - t_a) = \int_{\mathbf{u}(t_a)=\mathbf{u}_a} \mathscr{D}\mathbf{f}(t)\mathscr{P}[\mathbf{f}(t)]\delta(\mathbf{u}(t_b) - \mathbf{u}_b) \times \left\{\int \mathscr{D}\mathbf{f}(t)\mathscr{P}[\mathbf{f}(t)]\right\}^{-1}, \quad \mathscr{D}\mathbf{f}(t) \equiv \prod_n \mathscr{D}f_n(t). \tag{13.2.6}$$

In the calculation of the path integral it is assumed that at $t = t_a$ the system is located at the point $\mathbf{u}_a$. As seen from (13.2.6), contributions to the path integral are made only by such realizations of the random force $\mathbf{f}(t)$ which transfer the system from $\mathbf{u}_a$ to $\mathbf{u}_b$ over the time $t_b - t_a$.

The functional $\mathscr{P}[\mathbf{f}]$ determines the probability distribution of the random

function $\mathbf{f}(t)$. It is known (Feynman and Hibbs, 1965) that for a random function of the white-noise type with correlators (13.2.4)

$$\mathscr{P}[\mathbf{f}(t)] = \exp\left[-\frac{1}{4\alpha\Gamma}\int dt \sum_n f_n^2(t)\right]. \tag{13.2.7}$$

To find the transition probability W_{12} it is necessary to determine the probability density (13.2.6) in the range of time $t_r \ll t_b - t_a \ll W_{12}^{-1}$. Over the time $\sim t_r$ the system approaches the focus f_1 and 'forgets' the initial state $\mathbf{u}_a$. The 'jump' into the point $\mathbf{u}_b$ from a certain point in the vicinity of f_1 is also rapid, within the time $\sim t_r$. It is evident, therefore, that the function $w(\mathbf{u}_b; \mathbf{u}_a; t_b - t_a)$ depends neither on $\mathbf{u}_a$ nor on $t_b - t_a$ (for points $\mathbf{u}_b$ and $\mathbf{u}_a$ located in attraction region of one and the same stable state). This function thus determines the quasistationary population of a point $\mathbf{u}_b$ in the considered time range.

For $\mathbf{u}_b$ located near the separatrix the transition probability density (13.2.6) is exponentially small. It contains a small parameter α in the denominator of the expression in the exponent. Performing the calculation with logarithmic accuracy, we shall determine only the exponent and ignore the pre-exponential factor, which depends on α weakly. To do this, it is convenient, following Feynman, to change in (13.2.6) from integration over the random force trajectories $\mathscr{D}\mathbf{f}(t)$ to integration over the trajectories of the system $\mathscr{D}\mathbf{u}(t)$,

$$\mathscr{P}[\mathbf{f}(t)]\mathscr{D}\mathbf{f}(t) = \tilde{\mathscr{P}}[\mathbf{u}(t)]\mathscr{D}\mathbf{u}(t). \tag{13.2.8}$$

The functional $\tilde{\mathscr{P}}[\mathbf{u}(t)]$ determines the probability distribution of the random function $\mathbf{u}(t)$. According to (13.2.4), (13.2.7) and (13.2.8)

$$\tilde{\mathscr{P}}[\mathbf{u}(t)] = \exp(-S/4\alpha\Gamma)J[\mathbf{u}(t)],$$
$$S = \int dt\, \mathscr{L}(\mathbf{u}, \dot{\mathbf{u}}), \quad \mathscr{L}(\mathbf{u}, \dot{\mathbf{u}}) = \sum_n [\dot{u}_n - v_n(\mathbf{c}; \mathbf{u})]^2, \tag{13.2.9}$$

where $J[\mathbf{u}]$ is the Jacobian for the transformation (13.2.8). It is obvious from (13.2.4) that $J[\mathbf{u}]$ is independent of α and influences only the pre-exponential factor in $w(\mathbf{u}_b; \mathbf{u}_a; t_b - t_a)$.

Within the adopted accuracy, it suffices to single out in the integral over $\mathscr{D}\mathbf{u}(t)$ the main exponential factor, which corresponds to the extremal path $\mathbf{u}(t)$. This path is evident from (13.2.6), (13.2.8) and (13.2.9) to be determined by the condition that the functional S be minimum. At instant t_b the extremal path passes through a point $\mathbf{u}_b$, while it starts at $\mathbf{u}_{f_1}$ (according to the physical picture described above). We note that the functional S can be regarded as the action of a certain auxiliary particle, and $\mathscr{L}$ is its Lagrangian.

Within the logarithmic accuracy the transition probability W_{12} is determined by the maximum value of $w(\mathbf{u}_b; \mathbf{u}_a; t_b - t_a)$ for the points $\mathbf{u}_b$ located on the separatrix (on approaching the separatrix, the system with the probability $\sim \frac{1}{2}$ will go to another stable state). The extremum with respect to $\mathbf{u}_b$ is reached in the saddle point, $\mathbf{u}_b = \mathbf{u}_s$, and the transition probability

is given by

$$W = \text{const} \cdot \exp(-R/\alpha), \quad R = \frac{1}{4\Gamma} \min \int_0^t \mathrm{d}t\, \mathscr{L}(\mathbf{u}, \dot{\mathbf{u}}),$$

$$\mathbf{u}(0) \simeq \mathbf{u}_f, \quad \mathbf{u}(t) \simeq \mathbf{u}_s \tag{13.2.10}$$

(Dykman and Krivoglaz, 1979). The quantity R in (13.2.10) is calculated under the additional condition

$$\mathscr{E}(\mathbf{u}, \dot{\mathbf{u}}) = 0, \quad \mathscr{E} = -4\Gamma \frac{\partial R}{\partial t} = \sum_n [\dot{u}_n^2 - v_n^2(\mathbf{c}; \mathbf{u})]. \tag{13.2.11}$$

The equality $\mathscr{E} = 0$ is obviously the condition of R be extremal with respect to the duration of the motion along the path $\mathbf{u}(t)$. The criterion of the applicability of (13.2.10) is the inequality

$$R \gg \alpha. \tag{13.2.12}$$

In the case of thermal fluctuations in a system (which are due to coupling to a thermostat), $\alpha \propto T$ and the quantity RT/α is independent of T and equals the activation energy of the transition.

It is obvious from (13.2.9)–(13.2.11) that the suggested method reduces the calculation of the probabilities of transitions between stable states for a nonlinear oscillator driven by a resonant field, or for a system of a more general type, to the solution of the variational problem of finding a minimum of the action S (13.2.9) of an auxiliary particle which moves from the point $\mathbf{u}_f$ to $\mathbf{u}_s$. The function $\mathscr{E}(\mathbf{u}, \dot{\mathbf{u}})$ (13.2.11), is the energy of this particle (Landau and Lifshitz, 1976). The auxiliary particle has twice as many degrees of freedom as the initial dynamical system (its coordinates are u_n and the momenta are $\dot{u}_n - v_n$).

13.2.2 *Fluctuational transitions between stable states in concrete systems*

The explicit expression for R may be obtained easily in the case when the functions v_n in (13.2.4) satisfy the 'potentiality condition' (see Haken, 1983), $v_n = -\partial U/\partial u_n$. In this case the equations of motion for an auxiliary particle are solved by the substitution $\dot{u}_n = -v_n$, and as a result $R = \Gamma^{-1}[U(\mathbf{u}_s) - U(\mathbf{u}_f)]$.

Equations (13.2.10) and (13.2.11) permit us to find the transition probabilities in the nontrivial case of a system whose motion is nonpotential. An example of such a system is a Duffing oscillator in a resonant field. The values of the effective 'activation energy' R may be obtained here in the explicit form in a number of limiting cases (Dykman and Krivoglaz, 1979, 1980a). The dependences of R_1 and R_2 on the characteristic field intensity $c_2 \propto h^2$ at relatively large frequency detuning, $\omega_h - \omega_0 \gg \Gamma(c_1 \ll 1)$, are shown in Figure 13.4 (the states 1 and 2 correspond to the smaller and the larger

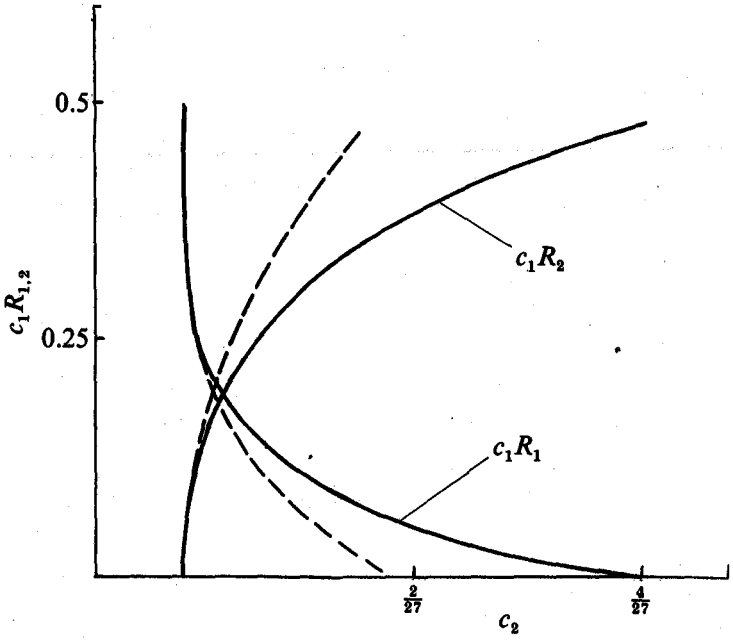

Figure 13.4. Dependences of the characteristic 'activation energies' R_1 and R_2 on c_2 for the transitions $1\to2$ and $2\to1$ between the stable states of a nonlinear oscillator in a strong resonant field at relatively large frequency detuning, $|c_1|\ll1$ (Dykman and Krivoglaz, 1979).

amplitudes of the constrained oscillations). For small c_2 (but $c_2\gg c_1^2$) these dependences are steep,

$$c_1R_1=\tfrac{1}{2}-\zeta' c_2^{1/4},\quad c_1R_2=2c_2^{1/2},\quad c_1^2\ll c_2\ll1;\ \zeta'\simeq0.98. \tag{13.2.13}$$

The dashed curves in Figure 13.4 are plots of the expressions (13.2.13). Similar results were obtained recently for $c_1\ll1$ (Dmitriev and Dyakonov, 1986; Maslova, 1986) by another approach.

The arguments $R_{1,2}/\alpha$ of the exponentials in (13.2.10) for the transition probabilities at $c_1\ll1$ are seen from (13.2.5), (13.2.13) and from Figure 13.4 to depend on the external field amplitude h only through the parameter $c_2\propto h^2|\omega_h-\omega_0|^{-3}$, and R_1 decreases monotonically with increasing h, while R_2 increases. The frequency detuning $\omega_h-\omega_0$ enters in both c_1 and c_2, and $R\propto|\omega_h-\omega_0|$ for fixed c_2. In addition, $R/\alpha\propto(\alpha\Gamma)^{-1}$. If friction and noise result from the coupling of an oscillator to a thermostat, then $\alpha\propto T\Gamma^{-1}$ and thus $R/\alpha\propto T^{-1}$, while Γ drops out from R/α.

The approach to the calculation of transition probabilities stated above may be generalized directly to the case when random forces acting on a dynamical system depend on dynamical variables (in a nonsingular way) or are Gaussian, but not δ-correlated (in the latter case the expressions of the type (13.2.9) and

(13.2.10) for the action S and R include retardation). The approach may be also generalized easily to systems described by higher-order equations.

We shall illustrate the latter by taking as an example a system performing Brownian motion in a static potential $U(q)$,

$$\ddot{q} + 2\Gamma\dot{q} + \frac{\mathrm{d}U}{\mathrm{d}q} = f(t), \quad \langle f(t)f(t')\rangle = 2\mathscr{B}\delta(t-t'). \tag{13.2.14}$$

Writing the transition probability density $w(q_b, \dot{q}_b; q_a, \dot{q}_a; t_b - t_a)$ in the form (13.2.6) and changing from integration over $\mathscr{D}f(t)$ to integration over $\mathscr{D}q(t)$ similarly to (13.2.8), we obtain, that within the logarithmic accuracy the probability of the transition from an equilibrium position $q = q_f$ to a saddle point $q = q_s$ is

$$W = \text{const}\cdot\exp(-P/\mathscr{B}), \quad P = \tfrac{1}{4}\min\int_0^t \left(\ddot{q} + 2\Gamma\dot{q} + \frac{\mathrm{d}U}{\mathrm{d}q}\right)^2 \mathrm{d}t,$$

$$q(0) = q_f, \quad q(t) = q_s, \quad \dot{q}(0) = \dot{q}(t) = 0. \tag{13.2.15}$$

The extremal trajectory of the functional P, which satisfies the boundary conditions and the condition $\partial P/\partial t = 0$, is described by the equation

$$\ddot{q} - 2\Gamma\dot{q} + \frac{\mathrm{d}U}{\mathrm{d}q} = 0 \tag{13.2.16}$$

(i.e. it corresponds to a motion with negative friction). Substituting (13.2.16) into (13.2.15) we obtain the well-known result

$$W = \text{const}\cdot\exp\left\{-\frac{2\Gamma}{\mathscr{B}}[U(q_s) - U(q_f)]\right\}. \tag{13.2.17}$$

13.2.3 Transition probabilities near bifurcation points

The additive-noise induced fluctuations in dynamical systems acquire peculiar features if the system is near a bifurcation point, i.e. if its parameters are close to values at which, e.g., new equilibrium positions or limit cycles appear or coalesce. These features are quite universal, they depend not on details of a system but on a type of bifurcation. The problem of fluctuations near bifurcation points was first considered for the bifurcation points corresponding to soft excitation of a limit cycle within a model of the Van der Pol oscillator (see, e.g., Lax, 1967, 1968; Risken, 1970; Rytov, 1955).

Besides the bifurcation points where the roots $\Lambda_{1,2}$ of the characteristic equation for a dynamical system pass through the imaginary axis and a limit cycle is excited, as in the case of the Van der Pol oscillator, there are also quite general bifurcation points of the marginal type, where $\Lambda_1 = 0$ ($\Lambda_i \neq 0$ at $i \neq 1$) and two singular points (e.g. a node and a saddle) mutually annihilate (or arise) in phase space. The value of the set of the system parameters $\mathbf{c} = (c_1, c_2, \ldots)$ corresponding to a marginal point will be denoted by $\mathbf{c}_M$.

In the range of $\mathbf{c}$ close to $\mathbf{c}_M$ the inequality $|\Lambda_1| \ll |\mathrm{Re}\,\Lambda_2|, |\mathrm{Re}\,\Lambda_3|, \ldots$ holds, i.e. one of the motions in the system (say that described by the variable u_1) is slow (u_1 is a 'soft mode'). This results in the strong increase of fluctuations, if the system is located in the corresponding region of phase space. The smallness of Λ_1 makes it possible to use an adiabatic approximation for the description of the fluctuations and to reduce the multidimensional problem, generally speaking, to a one-dimensional one if the random forces are small (Dykman and Krivoglaz, 1980a).

In the region of phase space adjacent to the singular points emerging at $\mathbf{c} = \mathbf{c}_M$ the 'fast' dynamical variables of the system $u_2, u_3, \ldots$ within a time $\sim t_c (t_c = \max(|\mathrm{Re}\,\Lambda_2|^{-1}, |\mathrm{Re}\,\Lambda_3|^{-1}, \ldots), |\Lambda_1| t_c \ll 1$ relax to their equilibrium values (for a given u_1) and then fluctuate about these values, following adiabatically the slow variation of u_1. The equation of motion for $u \equiv u_1$ near the marginal point is of the form

$$\dot{u} = -\frac{dU_M}{du} + f(t), \quad U_M(u) = u(\tfrac{1}{3}bu^2 - \varepsilon),$$

$$\langle f(t) f(t') \rangle = 2\mathscr{B}\delta(t-t') \quad (u \equiv u_1, f(t) \equiv f_1(t)). \tag{13.2.18}$$

The parameter ε in (13.2.18) characterizes a distance to the bifurcation point in space of the parameters $\mathbf{c}$ (ε, b, $\mathscr{B}$ may be easily expressed in terms of the parameters of the initially multidimensional system (cf. Dykman and Krivoglaz, 1980a)). Only the main terms of the expansion of $U_M(u)$ in u are kept in (13.2.18). The adiabatic approximation used in the derivation of (13.2.18) is accurate to corrections $\sim (\mathscr{B}b^2)^{1/3} t_c \propto \mathscr{B}^{1/3}$ in the most favorable case, $\varepsilon = 0$.

At $\varepsilon b > 0$ in the region of small $|u|$ the system has a stable equilibrium point $u^{(0)}$ and a saddle point $u^{(s)}$,

$$u^{(0)} = (\varepsilon/b)^{1/2} \operatorname{sign} b, \quad u^{(s)} = -(\varepsilon/b)^{1/2} \operatorname{sign} b, \tag{13.2.19}$$

which correspond to a local minimum and maximum of the potential $U_M(u)$ (see Figure 13.5). At $\varepsilon = 0$ the points $u^{(0)}$ and $u^{(s)}$ merge, and at $\varepsilon b < 0$ the system has no stable states with small $|u|$. Obviously, the point $u^{(0)}$ corresponds to a metastable state at sufficiently small ε.

The one-dimensional Markov process $u(t)$ can be investigated by standard methods. In particular, when $(\varepsilon/b)^{1/2} |\varepsilon| \gg \mathscr{B}$ it is of interest to determine the probability W_M of the escape of the system from the metastable state (in this case W_M is small compared with the reciprocal time $\Lambda_1 \sim |b|(\varepsilon/b)^{1/2}$ characterizing the motion of the system near the equilibrium position). With allowance for physical picture of a motion, the problem of calculation of the escape probability W_M for the one-dimensional process $u(t)$ reduces in a natural way to the well-known (Kolmogorov and Leontovich, 1933) first passage time problem. This makes it possible to determine explicitly not only the exponent, but also the pre-exponential factor in the expression for W_M (Dykman and

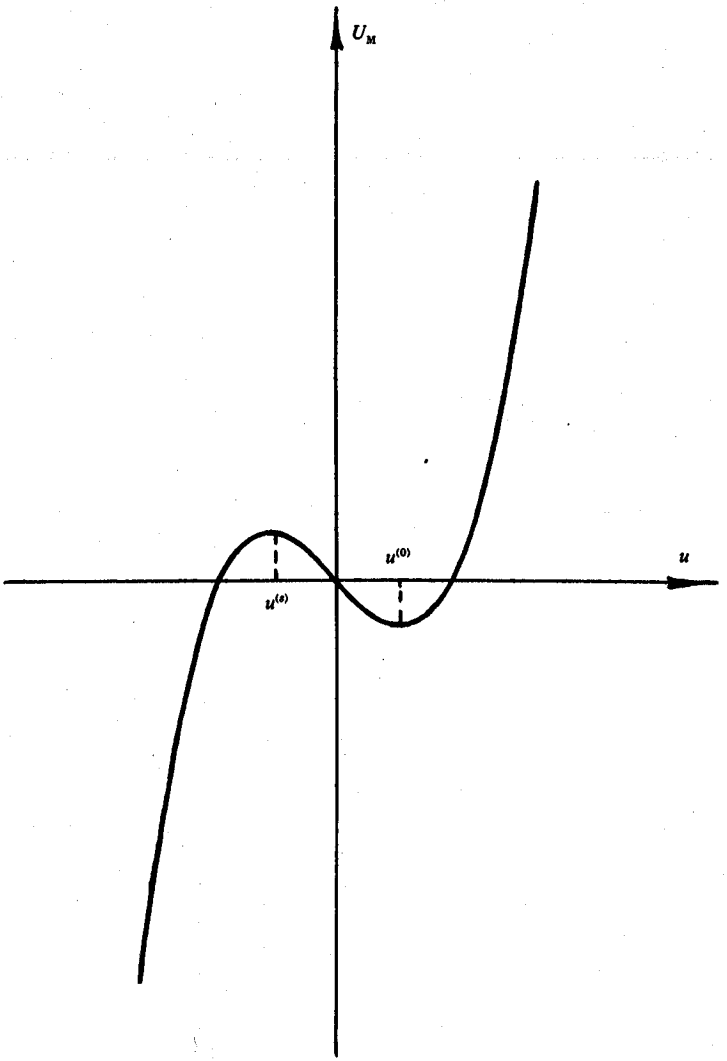

Figure 13.5. The potential for the 'slow' variable of a system near the marginal point.

Krivoglaz, 1980a):

$$W_M = \frac{1}{\pi}(\varepsilon b)^{1/2} \exp(-P_M/\mathscr{B}),$$

$$P_M = \tfrac{4}{3}(\varepsilon/b)^{1/2}|\varepsilon| \qquad (P_M \gg \mathscr{B}). \tag{13.2.20}$$

According to (13.2.20), the escape probability W_M near a marginal point depends on the noise intensity $\mathscr{B}$ in an activation way. The 'activation energy'

P_M equals the potential difference for a saddle and stable points, $P_M = U_M(u^{(s)}) - U_M(u^{(0)})$. It varies with the distance to the bifurcation point along the axis ε as $|\varepsilon|^{3/2}$ and, thus, decreases rapidly (while W_M increases) with approaching bifurcation point. The argument of the exponential in (13.2.20) may be found also by the direct calculation of the quantity $R/4\alpha\Gamma$ (13.2.10) near the bifurcation point. The result coincides with that in (13.2.20). The pre-exponential factor in (13.2.20) depends on the small 'distance' to the bifurcation point ε as $|\varepsilon|^{1/2}$.

In two-parameter systems, $\mathbf{c} = (c_1, c_2)$, the curves in the parameter plane, which are the loci of the bifurcation points of the marginal type, can have singular points of the form of spinodes. An example of the spinode is the point K in Figure 13.2 which refers to an oscillator driven by a resonant field. Another important example is the spinode that occurs under polarization optical bistability as a consequence of symmetry properties (Dykman, 1986).

The shape of the bifurcation curve in the vicinity of the spinode K ($c_1 = c_{1K}$, $c_2 = c_{2K}$) in the general case coincides with that shown in Figure 13.2. In the parameter region bounded by the solid curves in Figure 13.2 the system has two stable states and one unstable equilibrium state – a saddle point. As the point K is approached (in parameter space) these states come closer together (in phase space), the rigidity of the system decreases, and as a result the fluctuations increase sharply. In a certain sense, the point K is analogous to the critical point on the gas–liquid phase-transition curve.

To describe the fluctuations near the point K one may also use the adiabatic approximation and thus reduce the problem to a one-dimensional one. The error due to this approximation is $\propto \mathscr{B}^{1/4}$ here. The equation for the 'slow' variable $u \equiv u_1$ is similar to (13.2.18), but the potential $U_K(u)$ is the polynomial of fourth degree,

$$U_K(u) = u(\tfrac{1}{4}du^3 - \tfrac{1}{2}\varepsilon_1 u - \varepsilon_2) \quad (d > 0). \tag{13.2.21}$$

In the range

$$\varepsilon_1 > 0, \quad 3|\varepsilon_2|(3d)^{1/2}/2\varepsilon_1^{3/2} \leqslant 1 \tag{13.2.22}$$

the potential $U_K(u)$ has two minima separated by a region in which $U_K(u)$ has a local maximum. The extrema of $U_K(u)$ correspond to stable states and a saddle point. They coalesce at $\varepsilon_1 = \varepsilon_2 = 0$.

Not too close to the point K the probabilities W_{ij} of transitions between the stable states are small compared with the reciprocal relaxation times. To logarithmic accuracy they are given by

$$W_{ij} = \text{const} \cdot \exp\{-\mathscr{B}^{-1}[U_K(u^{(s)}) - U_K(u^{(0i)})]\}, \quad i, j = 1, 2, \tag{13.2.23}$$

where $u^{(0i)}$ is the position of the ith minimum of $U_K(u)$.

As the point K is approached the quantities W_{ij} increase sharply. In the immediate vicinity of the point K the concept of the transition probability

becomes meaningless, since all relaxation and fluctuation processes have the same time scale (see Section 13.3.3).

The results for the transition probabilities near the marginal and spinode points (the bifurcation points of codimension 1 and 2, cf. Arnold, 1978) were applied to the problem of a nonlinear oscillator in a resonant field (Dykman and Krivoglaz, 1980a). The dependences of these probabilities (in particular, of the activation energies) on the oscillator parameters in the vicinity of the bifurcation curves shown in Figure 13.2 were obtained in the explicit form.

It should be noted that the probability of reaching the phase-space point far from points corresponding to stable states was investigated by another method in a mathematical paper by Ventsel and Freidlin (1970) for a certain type of Markov system. The approach presented above in Section 13.2.2 makes it possible to obtain simply the results of this investigation and to generalize them to include, in particular, the case of Markov processes of more general type, as well as the case of non-Markov processes. It allowed also to obtain the expression for the probabilities of transitions between stable states in the closed form and to calculate these probabilities for a concrete physical system, a Duffing oscillator in a strong resonant field.

To calculate transition probabilities and to find a stationary distribution of a system whose motion presents Markov process the Einstein–Fokker–Planck equation may be used as well. This method is particularly suited to the case of potential motion. It was shown recently that both in this case and in the case of nonpotential motion, the determination of a stationary distribution of a bistable system and the calculation of transition probabilities to logarithmic accuracy may be reduced for small noise intensities to the solution of a nonlinear partial differential equation (Ben-Jacob, Bergman, Matkowsky and Schuss, 1982; Graham and Shenzle, 1981). This equation is the Hamilton–Jacobi equation for the action S, (13.2.9), of an auxiliary particle (see Section 13.2.2). A similar equation arises also when transition probabilities are calculated by solving the first passage time problem for Markov process (Matkowsky and Schuss, 1983; Schuss, 1980; Shenoy and Agarwal, 1984; Talkner and Hänggi, 1984). The latter method permits to find also the pre-exponential factor in the expression for W.

13.3 Spectral density of fluctuations in bistable systems at low noise intensities

The time correlation functions $\langle L(t)M(t')\rangle$ ($\langle\ldots\rangle$ denotes the ensemble averaging) of the dynamical variables L, M and their spectral distributions are analyzed for bistable systems hereafter. These quantities not only characterize quite completely the relaxation and fluctuations, but describe also the generalized susceptibilities (cf. Landau and Lifshitz, 1980) and, thus, may be determined directly by experiment. We note however that when dissipation and a random force acting on a system are not connected (or not only

connected) with coupling of a system to a thermostat or when a system is driven by a periodic field, the fluctuation–dissipation relations become more complicated and lose the universality. For Markov systems these relations are considered in detail by Risken (1984).

In the absence of external periodic fields the correlators $\langle L(t)M(t')\rangle$ under stationary conditions depend on $t-t'$ only. Supposing a system to be ergodic we can write the time correlation function $Q_{LM}(t)$ of variables L, M in the form

$$Q_{LM}(t)=\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}d\tau[L(t+\tau)-\langle L(t+\tau)\rangle]\times[M(\tau)-\langle M(\tau)\rangle]. \tag{13.3.1}$$

For systems subjected to a periodic force $h\cos\omega_h t$ the distribution function under stationary conditions depends periodically on time (such systems are, generally speaking, nonergodic) and therefore the correlators $\langle L(t+\tau)M(\tau)\rangle$ depend periodically on τ. However, for a number of applications it is of interest (cf. Dykman, 1978; Dykman and Krivoglaz, 1984) to find the addend in $\langle L(t+\tau)M(\tau)\rangle$ that does not depend on τ. This addend is given by (13.3.1) as well.

Besides $Q_{LM}(t)$ we shall consider also the spectral distributions

$$Q_{LM}(\omega)=\frac{1}{\pi}\int_0^\infty dt\,e^{i\omega t-\varepsilon t}Q_{LM}(t)\quad(\varepsilon\to+0). \tag{13.3.2}$$

The quantity $\mathrm{Re}\,Q_{LL}(\omega)$ is the spectral density of fluctuations of a dynamical variable L.

In Sections 13.3.1 and 13.3.2 we suppose that the intensity of noise acting on a dynamical system is small, so that the probabilities of transitions between stable states are much smaller than all reciprocal relaxation times (in the absence of noise). In this case within an overwhelming part of time a system fluctuates about equilibrium positions, and it is convenient to write $Q_{LM}(\omega)$ in the form

$$Q_{LM}(\omega)=\sum_i w_i Q_{LM}^{(i)}(\omega)+Q_{LM}^{(\mathrm{tr})}(\omega). \tag{13.3.3}$$

Here w_i is the stationary population of the ith stable state, and $Q_{LM}^{(i)}(\omega)$ is the partial spectrum formed by small fluctuations about the ith state. The term $Q_{LM}^{(\mathrm{tr})}(\omega)$ at small noise intensities is formed by transitions between the states (see Section 13.3.2).

The populations w_i are determined by the balance equation, and in the case when transition probabilities are given by (13.2.10), $W_{ij}\propto\exp(-R_i/\alpha)$, we obtain for a bistable system

$$\frac{w_1}{w_2}=\frac{W_{21}}{W_{12}}=\mathrm{const}\cdot\exp\left(\frac{R_1-R_2}{\alpha}\right),\quad\alpha\ll R_{1,2}. \tag{13.3.4}$$

It is evident from (13.3.4) that for almost all values of the dynamical system parameters, excluding the narrow range where $R_1 \simeq R_2$, the population ratio w_1/w_2 is either exponentially small or large, and only one addend contributes to $\sum_i w_i Q^{(i)}_{LM}(\omega)$ in (13.3.3) (for this addend $w_i \simeq 1$).

13.3.1 Contribution from small fluctuations about stable states

To calculate $Q^{(j)}(\omega)$ in the limit of very small noise intensities (see below) one may linearize the equations of motion of a system in the vicinity of the jth stable equilibrium state. The respective linearized equations may be easily solved. For systems described by (13.2.4) we obtain with account taken of (13.3.2)

$$Q^{(j)}_{u_n u_m}(\omega) = \alpha \sum_p (\Gamma/\mathrm{Re}\,\Lambda^{(j)}_p)(\Lambda^{(j)}_p + i\omega)^{-1}(S^{(j)}_{pn})^* S^{(j)}_{pm}. \tag{13.3.5}$$

Here $\Lambda^{(j)}_p$ are the eigenvalues of the matrix $\|\partial v_n(\mathbf{c};\mathbf{u})/\partial u_m\|$ calculated for the jth stable state, $\hat{S}^{(j)}$ is the unitary transformation that diagonalizes this matrix.

The form of the function $Q^{(j)}(\omega)$ is seen from (13.3.5) to depend on the relation between the real and imaginary parts of the roots $\Lambda^{(j)}_p$ of the characteristic equation (note that for stable states $\mathrm{Re}\,\Lambda^{(j)}_p < 0$). At $|\mathrm{Re}\,\Lambda^{(j)}_p| \gg |\mathrm{Im}\,\Lambda^{(j)}_p|$ the distribution (13.3.5) is smooth, and the spectral densities of fluctuations $\mathrm{Re}\,Q^{(j)}_{u_n u_n}(\omega)$ are of the form of the Lorentzian peak with the maximum at low frequency (as compared with $|\mathrm{Re}\,\Lambda^{(j)}_p|$) and the halfwidth $|\mathrm{Re}\,\Lambda^{(j)}_p|$ (or of the superposition of such peaks, if the number of the dynamical variables u_n exceeds two).

If for some p the opposite condition is fulfilled,

$$|\mathrm{Re}\,\Lambda^{(j)}_p| \ll |\mathrm{Im}\,\Lambda^{(j)}_p|, \tag{13.3.6}$$

the spectral densities of fluctuations $\mathrm{Re}\,Q^{(j)}_{u_n u_n}(\omega)$ have distinct peaks at frequencies $\mathrm{Im}\,\Lambda^{(j)}_p$. The shape of these peaks is also Lorentzian, and their halfwidth is $|\mathrm{Re}\,\Lambda^{(j)}_p|$.

For a nonlinear oscillator performing constrained vibrations in a resonant field the values of $\Lambda^{(j)}_{1,2}$ may be obtained explicitly with account taken of (13.2.2)–(13.2.4), $\Lambda^{(j)}_{1,2} = -\Gamma[1 \pm i(\lambda_j^2 - 1)^{1/2}]$, where λ_j are given in eqn. (25) of the paper by Dykman and Krivoglaz (1979). The 'weak damping' condition (13.3.6) is fulfilled for a relatively large detuning of the field frequency relative to the eigenfrequency of the oscillator, $|\omega_h - \omega_0| \gg \Gamma$ (then $\lambda_j^2 \gg 1$).

The spectral densities of fluctuations of the oscillator coordinates and momenta in the range of resonance, $\omega_h \simeq \omega$, are expressed according to (13.2.3) in terms of the functions $Q_{u_n u_m}(\omega - \omega_h)$. In terms of these functions one can express also the coefficient $\mu(\omega)$ of the absorption (amplification) of an additional weak resonant field at frequency ω by the oscillator. At small noise intensities the quantity $\mu(\omega)$ may be put into a form similar to that of (13.3.3). When condition (13.3.6) is satisfied, the partial contributions to $\mu(\omega)$ have distinct Lorentzian peaks with a halfwidth Γ (see Dykman and Krivoglaz,

1979, 1984 for details). The emergence of peaks of this type was shown by Bonifacio and Lugiato (1978) (see also Lugiato, 1984) when analyzing light transmission spectra for bistable optical systems.

If the weak damping condition (13.3.6) is satisfied, the range of noise intensities, where the expression (13.3.5) holds, appears to be much narrower than that determined by the inequality $\alpha \ll R$, (13.3.4). Indeed, we have obtained (13.3.5) allowing only for the terms linear in the displacements from the equilibrium positions. In this approximation the frequencies $|\mathrm{Im}\,\Lambda_p^{(j)}|$ of vibrations about equilibrium positions are independent of the vibration amplitudes. This does not, of course, hold true once the nonlinearity of the motion becomes significant. The dependence of the vibration frequency on the amplitude results in the frequency being modulated by fluctuations of the amplitude. This gives rise to the specific modulational broadening of the peaks of the spectral densities of fluctuations (Dykman and Krivoglaz, 1971).

It is evident that the approximation (13.3.5) describes the spectrum at the frequency $\simeq \mathrm{Im}\,\Lambda_p^{(j)}$ correctly when the characteristic frequency straggling due to fluctuations is small compared with the frequency uncertainty due to damping $|\mathrm{Re}\,\Lambda_p^{(j)}|$. In most cases for underdamped systems this condition reduces to the inequality

$$\frac{\alpha}{R_j} \ll \left|\frac{\mathrm{Re}\,\Lambda_p^{(j)}}{\mathrm{Im}\,\Lambda_p^{(j)}}\right| \ll 1 \tag{13.3.7}$$

(see Section 13.4.1 for details). The restriction (13.3.7) is substantially stronger than (13.3.4). The effects arising when the left inequality in (13.3.7) does not hold are discussed below.

13.3.2 *Narrow spectral peaks caused by noise-induced transitions between stable states*

In contrast with the addends $Q_{LM}^{(i)}(\omega)$ in (13.3.3), which are formed by small fluctuations, the addend $Q_{LM}^{(\mathrm{tr})}(\omega)$ is formed by large fluctuations causing transitions between the states of a system. At small noise intensities, when $\alpha \ll R$, the probability of such fluctuations is small. The peaks of $Q_{LM}^{(i)}(\omega)$ and $Q_{LM}^{(\mathrm{tr})}(\omega)$ are formed within substantially different time intervals, i.e. the different ranges of integration over t in (13.3.2) contribute to $Q_{LM}^{(i)}(\omega)$ and $Q_{LM}^{(\mathrm{tr})}(\omega)$. The peaks of $Q_{LM}^{(i)}(\omega)$ are formed within a characteristic relaxation time $|\mathrm{Re}\,\Lambda_p^{(i)}|^{-1}$ (cf. (13.3.5)), while those of $Q_{LM}^{(\mathrm{tr})}(\omega)$ relate, in effect, to the times corresponding to the transition probabilities, $t \sim W_{12}^{-1} + W_{21}^{-1} \gg |\mathrm{Re}\,\Lambda_p^{(i)}|^{-1}$.

The values of the time correlation function $Q_{LM}(t)$ in the region $t \gg |\mathrm{Re}\,\Lambda_p^{(i)}|^{-1}$ and, consequently, the values of $Q_{LM}^{(\mathrm{tr})}(\omega)$ differ qualitatively for the values of the system parameters far from the range of the kinetic phase transitions and for those close to this range. It is easy to see that in the first case,

when $|R_1 - R_2| \gg \alpha$, the function $Q_{LM}(t)$ is exponentially small for corresponding t. Indeed, in this case with a probability close to unity the system fluctuates about the certain stable states, while an average time passed in the vicinity of another stable state is exponentially small. The correlation of such fluctuations decays within a time $\sim |\mathrm{Re}\,\Lambda_p^{(i)}|^{-1}$ (with the proper i).

For the parameter region where $|R_1 - R_2| \lesssim \alpha$ and the populations w_1, w_2 are thus of the same order of magnitude the form of $Q_{LM}(t)$ is quite different. In this region a system located initially, e.g., in the state 1, within a time $t \sim W_{12}^{-1} \sim W_{21}^{-1}$ changes to state 2 with an appreciable probability. The transition results in the finite change of the values of dynamical variables L, M (from those corresponding to the state 1, $L \simeq L_1$, $M \simeq M_1$, to those corresponding to state 2, $L \simeq L_2$, $M \simeq M_2$). The probability of the opposite transition is of the same order of magnitude. Taking the transitions into account, we obtain

$$Q_{LM}(t) = w_1 w_2 (L_1 - L_2)(M_1 - M_2)\exp[-(W_{12} + W_{21})t],$$
$$t \gg |\mathrm{Re}\,\Lambda_p^{(1,2)}|^{-1}$$

and respectively,

$$Q_{LM}^{(\mathrm{tr})}(\omega) = \frac{1}{\pi} w_1 w_2 (W_{12} + W_{21} - \mathrm{i}\omega)^{-1}$$
$$\times (L_1 - L_2)(M_1 - M_2). \tag{13.3.8}$$

In the more general case, when a system is subjected to a field $h\cos\omega_h t$ and the values of $L(t)$, $M(t)$ in the stable states depend periodically on time,

$$L_j(t) = \sum_n L_{jn} \exp(\mathrm{i}n\omega_h t),$$
$$M_j(t) = \sum_n M_{jn} \exp(\mathrm{i}n\omega_h t), \quad j = 1, 2$$

the function $Q_{LM}^{(\mathrm{tr})}(\omega)$ takes the form

$$Q_{LM}^{(\mathrm{tr})}(\omega) = \frac{1}{\pi} w_1 w_2 \sum_n [W_{12} + W_{21} - \mathrm{i}(\omega - n\omega_h)]^{-1}$$
$$\times (L_{1n}^* - L_{2n}^*)(M_{1n} - M_{2n}). \tag{13.3.9}$$

It is obvious from (13.3.8) and (13.3.9) that in bistable systems the spectral density of fluctuations $\mathrm{Re}\,Q_{LL}^{(\mathrm{tr})}(\omega)$ has the extremely narrow Lorentzian peak at zero frequency and similar peaks at the external field frequency ω_h and multiple frequencies. The halfwidth of the peaks $W_{12} + W_{21}$ equals the sum of the transition probabilities. It is much smaller than the reciprocal relaxation time of a dynamical system. The intensity of the peak is proportional to

$$w_1 w_2 = W_{12} W_{21} (W_{12} + W_{21})^{-2} \sim \exp[-|R_1 - R_2|/\alpha].$$

It is exponentially small everywhere excluding the range of the smeared phase transition.

In the case of a nonlinear oscillator in a strong resonant field the most intensive peak of the spectral density of fluctuations of the coordinates (momenta) is located at the field frequency ω_h. The corresponding extremely narrow peak exists as well in the spectrum $\mu(\omega)$ of the absorption (amplification) of an additional weak field by an oscillator (Dykman and Krivoglaz, 1979, 1984). The intensity of this peak is also proportional to $\exp[-|R_1 - R_2|/\alpha]$. The emergence of such a specific peak in the vicinity of the first-order kinetic phase transition in a system far from thermal equilibrium may underlie a way of detecting the transition itself. The peak of $\mu(\omega)$ makes it possible also to compare with high accuracy the frequencies of the strong and weak fields. The frequency error is $\sim W$ here, and thus can be smaller by many orders of magnitude than not only the strong field frequency ω_h itself, but also the small friction coefficient Γ of the nonlinear oscillator.

The relation between the parameters of a nonlinear oscillator and strong resonant field that corresponds to the kinetic phase transition may be obtained by solving the variational problem (13.2.10), (13.2.11) and (13.2.4) for R_1, R_2. The results of Sections 13.2.2 and 13.2.3 (see also Dykman and Krivoglaz, 1979) give this relation in the explicit form in the region of a comparatively large frequency detuning, $|\omega_h - \omega_0| \gg \Gamma$, and also in the parameter region near the spinode K in Figure 13.2. For the dimensionless parameters c_1 and c_2, (13.2.5), it takes the form: $c_2 \simeq 0.013$ at $c_1 \to 0$, and $c_2 = c_{2K}[1 - \frac{3}{2}\sqrt{3}(c_{1K} - c_1)]$ at $c_1 \simeq c_{1K} = \sqrt{3}$, $c_2 \simeq c_{2K} = 8/27$. Allowing for these results and for the monotonic character of the 'phase-transition curve' $c_2(c_1^2)$, which is obvious from qualitative arguments, we have interpolated this curve by that shown dashed in Figure 13.2.

13.3.3 *Fluctuations in close vicinity to the spinode point on the bifurcation curve*

The spectral density of fluctuations has the characteristic narrow peak in the case when the parameters of the system lie in the immediate vicinity of the spinode point K on the bifurcation curve (see Figure 13.2). Since the two stable states and the saddle point are very close to one to another in this parameter range (see Section 13.2.3), the system is 'soft' and the relaxation times are large (the so-called 'critical slowing down' occurs). At point K itself the damping is substantially nonexponential in the absence of fluctuations (the dynamics is obvious from (13.2.18) and (13.2.21) to be described by the equation $\dot{u} = -du^3$). Respectively, for the parameter values close enough to the point K even a weak random force causes such large fluctuations, that the probabilities of transitions between the stable states appear to be of the same order of magnitude as the reciprocal relaxation times, and thus the concept of a transition probability becomes meaningless. The fluctuations may be characterized here with the aid of the time correlation function $Q_{uu}(t)$ of the slow variable u and its spectral distribution $Q_{uu}(\omega)$.

The evolution of the correlator $Q_{uu}(t)$ is determined by the set of the damping decrements $\lambda_n \mathscr{B}^{1/2}$,

$$Q_{uu}(t) = \sum_n a_n \exp(-\lambda_n \mathscr{B}^{1/2} t) \quad (t > 0, \lambda_n > 0). \tag{13.3.10}$$

The values of λ_n are independent of the random force intensity $\mathscr{B}$, and thus the 'rate' of damping is proportional to the small quantity $\mathscr{B}^{1/2}$. The behavior of $Q_{uu}(t)$ at large times is determined by the lowest nonzero decrement $\lambda_1 \mathscr{B}^{1/2}$,

$$Q_{uu}(t) \simeq a_1 \exp(-\lambda_1 \mathscr{B}^{1/2} t), \quad \lambda_1 \mathscr{B}^{1/2} t \gg 1. \tag{13.3.11}$$

The quantities λ_n may be obtained by reducing the Einstein–Fokker–Planck equation for the random process $u(t)$ in (13.2.18) and (13.2.21) to the eigenvalue problem (see, e.g., Lax, 1967; Tomita, Ito and Kidachi, 1976; Van Kampen, 1965). The dependence of λ_1 on the parameters of the system obtained in this manner is shown in Figure 13.6 (Dykman and Krivoglaz, 1980a, 1984). The parameters g_1 and g_2 in Figure 13.6 are proportional to the parameters ε_1 and ε_2 in (13.2.21), which characterize the distance to the spinode point,

$$g_1 = \mathscr{B}^{-1/2} \varepsilon_1 d^{-1/2}, \quad g_2 = \mathscr{B}^{-1/4} \varepsilon_2 d^{-1/4}. \tag{13.3.12}$$

In particular, at the spinode point itself ($\varepsilon_1 = \varepsilon_2 = 0$) we have $\lambda_1 \simeq 1.37 d^{1/2}$. It is seen from Figure 13.6 that the dependence of λ_1 on ε_1 is, generally speaking, nonmonotonous and has rather sharp maximum for sufficiently large $|\varepsilon_2|$. The asymptotic expressions for λ_1 may be obtained in the explicit form (Dykman and Krivoglaz, 1984).

Some other questions concerning fluctuations near bifurcation points were considered by Mangel (1979). The fluctuation phenomena arising in the case of a multiplicative noise were analyzed in detail by Horsthemke and Lefever (1984).

13.4 Spectral density of fluctuations in underdamped systems

It follows from the results given above that the form of the spectral density of fluctuations in a bistable system changes extremely strongly with increasing intensity of a random force acting on a system. The features of the spectra caused by the bistability are manifested most distinctly when the characteristic frequencies of a system (in particular, the frequencies of vibrations about stable states) exceed substantially the characteristic reciprocal relaxation times, i.e. when dissipation is small. In this case the system motion is quasiconservative, there is a certain quantity of the type of energy (or action) that varies in time relatively slowly (the adiabatic invariant). Just this case is investigated in the following.

Since the considered features of the spectral density of fluctuations are quite general we shall analyze them within the simple model of a one-dimensional

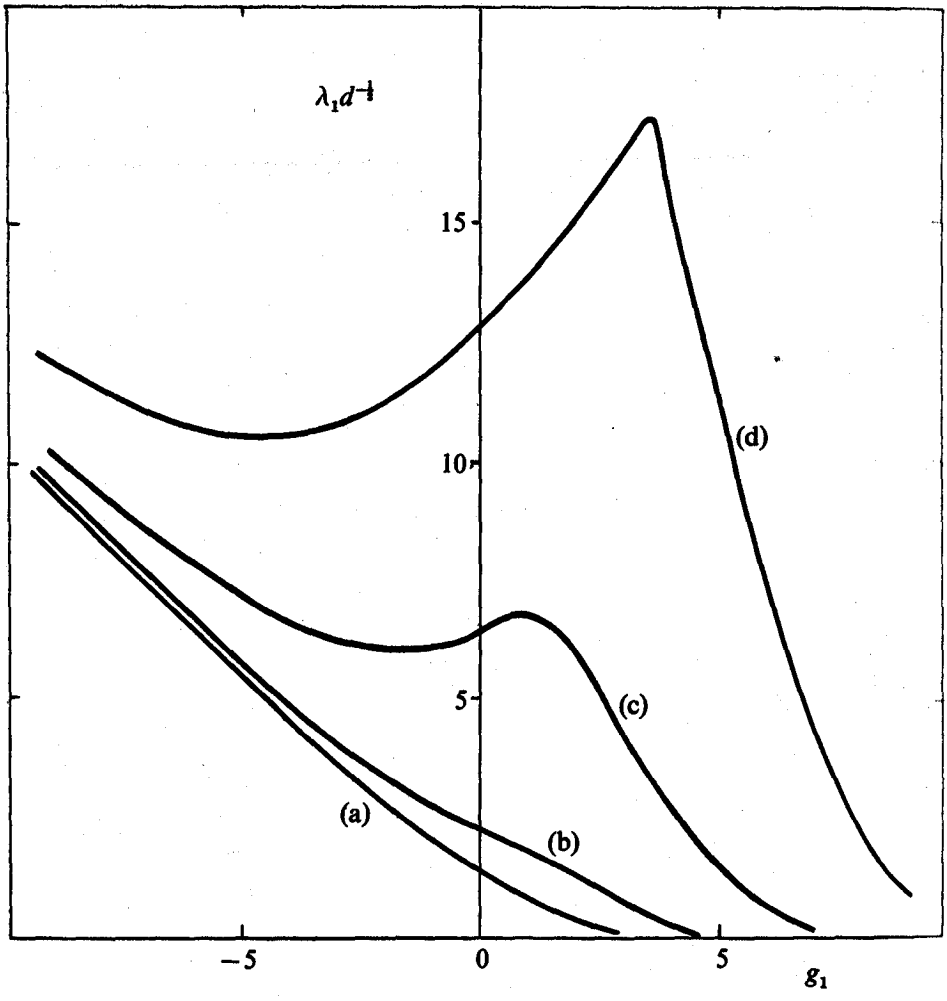

Figure 13.6. Dependence of the parameter λ_1, which determines the lowest nonzero decrement $\lambda_1 \mathscr{B}^{1/2}$ in the immediate vicinity of the spinode point K, on g_1. Curves (a)–(d) correspond to the values $g_2 = 0$, 2, 5, and 10 (Dykman and Krivoglaz, 1980a, 1984).

oscillator performing Brownian motion in a static potential $U(q)$ which has two minima and a local maximum between them (see Figure 13.7). The known example of such a potential is the potential of the double-well Duffing oscillator,

$$U_D(q) = -\tfrac{1}{2}\kappa^2 q^2 + \tfrac{1}{4}\gamma q^4 \quad (\gamma > 0). \tag{13.4.1}$$

This potential is symmetric, its wells have equal depths and curvatures.

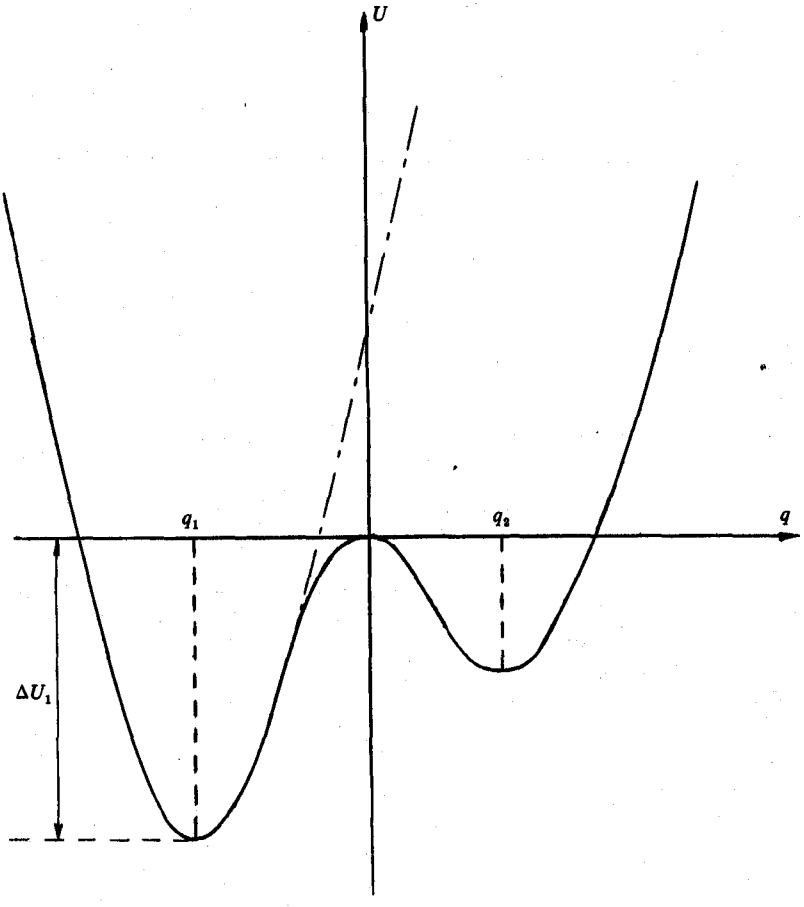

Figure 13.7. The double-well potential. For an underdamped bistable oscillator at low noise intensities the peak in the spectrum of fluctuations due to small-amplitude vibrations about q_1 coincides in shape with that for an oscillator moving in the single-well potential, whose part to the right of q_1 is shown as a dot–dashed line.

In the general case the potential $U(q)$ is asymmetric and parameters of the wells are different. It is essential that, as a rule, near minima, $q = q_i$ $(i = 1, 2)$, and near the local maximum, $q = 0$, the potential $U(q)$ is parabolic,

$$U(q) \simeq U_i + \tfrac{1}{2}\omega_i^2 (q - q_i)^2, \quad q \simeq q_i, \quad U_i \equiv U(q_i),$$
$$U(q) \simeq -\tfrac{1}{2}\kappa^2 q^2, \quad |q| \ll |q_i|. \tag{13.4.2}$$

Here ω_i are the eigenfrequencies of small-amplitude vibrations about the

equilibrium positions. At the local maximum, $q = 0$, the potential is supposed to equal zero.

The Brownian motion of the oscillator with a potential $U(q)$ is described by (13.2.14). We shall suppose the friction coefficient to be small,

$$\Gamma \ll \omega_1, \omega_2, \kappa. \tag{13.4.3}$$

If friction and a random force are neglected the oscillator energy

$$E = \tfrac{1}{2}p^2 + U(q) \tag{13.4.4}$$

is the integral of motion. Dissipation (friction) and fluctuations cause the energy to vary in time, and as a result the stationary distribution w_{st} is worked out in a system,

$$w_{st} = w_{st}(E) = Z^{-1} \exp(-2E\Gamma/\mathscr{B}),$$
$$Z = \iint dq\, dp \exp(-2E\Gamma/\mathscr{B}). \tag{13.4.5}$$

If friction and fluctuations are due to coupling of a system to a thermostat, then $\mathscr{B} = 2\Gamma T$ and the distribution (13.4.5) is Gibbsian, $w_{st}(E) = Z^{-1}\exp(-E/T)$.

For the sake of concreteness we shall consider below the time correlation function of the coordinates

$$Q(t) \equiv Q_{qq}(t) = \langle [q(t) - \langle q \rangle][q(0) - \langle q \rangle] \rangle \tag{13.4.6}$$

and the spectral density of fluctuations of the coordinates

$$Q'(\omega) \equiv \mathrm{Re}\, Q_{qq}(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty dt \exp(i\omega t) Q(t). \tag{13.4.7}$$

The general structure of the function $Q'(\omega)$ in the limit of low noise intensity $\mathscr{B}$ was considered in Section 13.3. It follows from the results of Section 13.3 that for the potential shown in Figure 13.7 at

$$F \gg 1, \quad F = 2\Delta U_1 \Gamma/\mathscr{B}, \quad \Delta U_1 = U(0) - U(q_1) = -U(q_1) \tag{13.4.8}$$

$Q'(\omega)$ has the distinct peak caused by vibrations about the equilibrium position q_1 which corresponds to the lowest minimum of $U(q)$. This peak is located at frequency $\simeq \omega_1$.

The shape of the peak in question within the range $|\omega - \omega_1| \ll \omega_1$ at $\exp(-F) \ll 1$ is determined by relatively small displacements $q - q_1$, $|q - q_1| \ll |q_1|$, and therefore is determined by the form of the potential $U(q)$ near the equilibrium position q_1. In fact, this shape coincides with that of the peak of the spectral density of fluctuations $\tilde{Q}(\Omega_1)$ ($\Omega_1 = \omega - \omega_1$) of the displacements $q - q_1$ of an oscillator moving in a single-well potential coinciding with $U(q)$ near the minimum (but differing from $U(q)$ at large

$|q-q_1|$, cf. the dot–dashed line in Figure 13.7),

$$Q'(\omega) \simeq w_1 \tilde{Q}(\Omega_1), \quad \Omega_1 = \omega - \omega_1, \quad |\Omega_1| \ll \omega_1,$$

$$w_1 = \frac{\pi \mathscr{B}}{\omega_1 \Gamma} w_{st}(U_1), \tag{13.4.9}$$

where

$$\tilde{Q}(\Omega_1) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty \mathrm{d}t \exp(\mathrm{i}\omega t) \langle [q(t) - q_1][q(0) - q_1] \rangle_1,$$

$$\Omega_1 = \omega - \omega_1. \tag{13.4.10}$$

($\langle \cdots \rangle_1$ denotes statistical averaging for a single-well oscillator; in the 'degenerate' case of a double-well potential with symmetric wells the r.h.s. of (13.4.9) should be doubled.)

The spectrum of fluctuations of a substantially nonlinear oscillator with a single-well potential is of interest not only in connection with the problem of a bistable oscillator, but is of great interest also in itself.

13.4.1 *Features of the spectrum of fluctuations of a nonlinear oscillator with a single-well potential*

To analyze the spectral density of fluctuations $\tilde{Q}(\Omega_1)$ of an underdamped oscillator at not too low noise intensities the nonlinear in $q - q_1$ terms in the expansion of $U(q)$ should be taken into account,

$$U(q) \simeq U_1 + \tfrac{1}{2}\omega_1^2 (q-q_1)^2 + \tfrac{1}{3}\zeta_1 (q-q_1)^3 + \tfrac{1}{4}\gamma_1 (q-q_1)^4 + \cdots. \tag{13.4.11}$$

These terms result, in particular, in the dependence of the effective frequency ω_{eff} of eigenvibrations on the vibration amplitude (or energy):

$$\omega_{\mathrm{eff}} \simeq \omega_1 + \frac{3}{4\omega_1} \tilde{\gamma}_1 \overline{(q-q_1)^2} \simeq \omega_1 + \tfrac{3}{4}\tilde{\gamma}_1 \omega_1^{-3} (E - U_1)$$

$$(\tilde{\gamma}_1 = \gamma_1 - \tfrac{10}{9}\zeta_1 \omega_1^{-2}). \tag{13.4.12}$$

Here the bar denotes averaging over the period of vibrations.

Random-force-induced fluctuations of the oscillator energy are seen from (13.4.12) to result in fluctuations of the vibration frequency. The characteristic frequency spread $\delta\omega$ is determined by the width of the distribution in energy, $\delta\omega = |3\tilde{\gamma}_1/4\omega_1^3| \mathscr{B}/\Gamma$ according to (13.4.5).

The shape of $\tilde{Q}(\Omega_1)$ is determined by competition between the two spectrum-broadening mechanisms, the broadening due to friction and that due to frequency modulation by a random force. Thus $\tilde{Q}(\Omega_1)$ depends on the ratio of the modulational broadening $\delta\omega$ to the friction coefficient Γ, i.e. on the

parameter

$$\rho = \tfrac{3}{16}\tilde{\gamma}_1 \mathscr{B}/\Gamma^2\omega_1^3. \tag{13.4.13}$$

The quantity ρ is of the order of the ratio of two small parameters, $F^{-1} = \mathscr{B}/2\Gamma\Delta U_1$ and Γ/ω_1. Therefore even when (13.4.8) is fulfilled $|\rho|$ may be more or less than unity.

At $|\rho| \ll 1$ the spectrum broadening is due mainly to friction and the function $\tilde{Q}(\Omega_1)$ is described by the Lorentzian distribution with the halfwidth Γ. The corrections caused by nonlinearity can be treated here by perturbation theory (Ivanov, Kvashnina and Krivoglaz, 1965; Krivoglaz and Pinkevich, 1970). The perturbation theory in oscillator nonlinearity applicable for $|\rho| \ll 1$ was developed by Sture, Nordholm and Zwanzig (1974) and by Rodriguez and Van Kampen (1976).

The analysis of the oscillator relaxation and of the spectrum of fluctuations is most complicated in the region $|\rho| \sim 1$, where both broadening mechanisms (damping and frequency modulation) make contributions of the same order of magnitude. The spectrum $\tilde{Q}(\Omega_1)$ in this region is substantially non-Lorentzian, and the time correlation function of the displacements from the equilibrium position $q - q_1$ decays substantially nonexponentially.

The displacement correlator and the spectrum of fluctuations were calculated (Dykman and Krivoglaz, 1971) by the special method based on the averaging method and some properties of the Gaussian random processes (we note that the process $q(t)$ itself at $|\rho| \gtrsim 1$ is essentially non-Gaussian; see also Dykman and Krivoglaz, 1984). The time correlation functions were also obtained by another method based on the solution of the Einstein–Fokker–Planck equation for a nonlinear oscillator (Dykman and Krivoglaz, 1980b). The same solution in a somewhat different form was obtained recently by Renz (1985).

The resulting expression for $\tilde{Q}(\Omega_1)$ takes on the form of an integral of an elementary function,

$$\left.\begin{aligned} &\tilde{Q}(\Omega_1) = \frac{1}{\pi} G_1 \operatorname{Re} \int_0^\infty dt \exp(i\Omega_1 t)\tilde{Q}^*(t), \quad G_1 = \mathscr{B}/4\Gamma\omega_1^2, \\ &\tilde{Q}(t) = \exp(\Gamma t)\psi^{-2}(t), \ \psi(t) = \cosh at + \frac{\Gamma}{a}(1 - 2i\rho)\sinh at, \\ &a = \Gamma(1 - 4i\rho)^{1/2} \quad (\operatorname{Re} a > 0). \end{aligned}\right\} \tag{13.4.14}$$

It is obvious from (13.4.14) that the shape of the spectrum $G_1^{-1}\tilde{Q}(\Omega_1)$ as a function of Ω_1/Γ is determined by the single parameter ρ. At $\rho \to 0$ (13.4.14) goes over into the Lorentzian distribution $(G_1/\pi)\Gamma/(\Gamma^2 + \Omega_1^2)$. At small $|\rho|$ the maximum of $\tilde{Q}(\Omega_1)$ shifts by $\simeq 4\rho\Gamma$, while the asymmetric part of $\tilde{Q}(\Omega_1)$ is of the order of ρ^3. As $|\rho|$ increases the deviation of $\tilde{Q}(\Omega_1)$ from a Lorentzian distribution becomes more and more pronounced. At $|\rho| \gg 1$ the shape of the

peak of $\tilde{Q}(\Omega_1)$ is substantially asymmetric,

$$\left.\begin{aligned}
&\tilde{Q}(\Omega_1) \simeq \tilde{Q}_0(\Omega_1),\\
&\tilde{Q}_0(\Omega_1) = G_1 \frac{|\Omega_1|}{\bar{\Omega}_1^2} \exp\left(-\frac{\Omega_1}{\bar{\Omega}_1}\right)\theta\left(\frac{\Omega_1}{\bar{\Omega}_1}\right),\\
&\rho\Omega_1 \gg \Gamma|\rho|^{3/2};\\
&\tilde{Q}(\Omega_1) \sim \left|\frac{\omega_1}{\tilde{\gamma}_1}\right|^{3/2} G_1^{-1/2}\Gamma^{1/2}, \quad |\Omega_1| \lesssim \Gamma|\rho|^{1/2};\\
&\bar{\Omega}_1 = 2\rho\Gamma, \quad |\rho| \gg 1.
\end{aligned}\right\} \tag{13.4.15}$$

($\theta(x)$ is the step function.)

According to (13.4.15) $\tilde{Q}(\Omega_1)$ near the maximum is determined at $|\rho| \gg 1$ by only the mechanism of frequency modulation (it is evident in particular that $\bar{\Omega}_1 = \langle E - U_1 \rangle_1 \partial\omega_{\text{eff}}/\partial E$). The terms due to dissipation are, however, essential in the wings of $\tilde{Q}(\Omega_1)$.

More detailed analytic expressions for the spectral density of fluctuations of a single-well oscillator in the limiting cases, as well as the detailed numerical results, are given by Dykman and Krivoglaz (1971); cf. also the computations by Renz and Marchesoni (1985). The respective change of shape and shift of the maximum of the peak in the spectrum with varying ρ were observed for an underdamped single-well oscillator by Fronzoni, Grigolini, Mannella and Zambon (1985, 1986) in analog experiments and were ascribed to the modulational broadening mechanism.

13.4.2 *Spectral distribution of a bistable oscillator in the absence of dissipation*

If the damping Γ is much smaller than the characteristic modulational frequency straggling $\delta\omega$, the spectral density $Q'(\omega)$ near the maximum (or the maxima, but excluding the maximum at zero frequency, see below) is determined just by the modulational mechanism. In this case within a time $\sim (\delta\omega)^{-1}$ needed to form a peak of $Q'(\omega)$ the dissipation effects do not succeed is manifesting themselves and may be neglected. The corresponding 'dissipationless' approximation allows us to investigate the spectrum of both the single- and double-well oscillators over a wide range of frequencies ω and noise intensities $\mathscr{B}$ (note that this approximation neglects the terms $\sim \Gamma/\delta\omega$, while $\mathscr{B}/\Gamma$ is regarded as finite).

In the absence of dissipation and noise a bistable oscillator performs periodic (but, generally speaking, anharmonic) vibrations with a given energy E either within one of the wells ($j = 1, 2$) or over the barrier ($j = 0$):

$$q(t) = \sum_{n=-\infty}^{\infty} q_{nj}(E)\exp[in(\omega_j(E)t + \varphi_j)], \quad j = 0, 1, 2. \tag{13.4.16}$$

Here $\omega_j(E)$ is the energy-dependent vibration frequency (in (13.4.2) $\omega_j \equiv \omega_j(U_j)$), $2|q_{nj}|$ is the amplitude of vibrations at the nth harmonic.

The determination of the coordinate correlator (13.4.6), neglecting dissipation, reduces to the averaging of the product $[q(t)-\langle q\rangle]\cdot[q(0)-\langle q\rangle]$, calculated by (13.4.16), over the phase and energy with the weight $\omega_j^{-1}(E)w_{st}(E)$ and to the subsequent summation over j. Performing then the Fourier-transform (13.4.7) we get the following expession for $Q'(\omega) = Q'_0(\omega)$ in this approximation:

$$Q'_0(\omega) = \sum_{n,j} Q^{(nj)}(\omega) + \mathscr{M}\delta(\omega),$$

$$Q^{(nj)}(\omega) = 2\pi\omega^{-1} w_{st}(E_{nj})|q_{nj}(E_{nj})|^2 |\mathrm{d}\omega_j(E_{nj})/\mathrm{d}E_{nj}|^{-1}, \quad (13.4.17)$$

where the energies E_{nj} are determined by the equation

$$n\omega_j(E_{nj}) = \omega, \quad (13.4.18)$$

while

$$\mathscr{M} = 2\pi \sum_j \int \mathrm{d}E\, w_{st}(E)\omega_j^{-1}(E) q_{0j}^2(E) - \left[2\pi \sum_j \int \mathrm{d}E\, w_{st}(E)\omega_j^{-1}(E) q_{0j}(E)\right]^2. \quad (13.4.19)$$

Equations (13.4.17)–(13.4.19) may be easily understood: in the absence of dissipation the contribution to the spectral density of fluctuations at a frequency ω is made only by those vibrations whose eigenfrequency $\omega_j(E)$ or its overtones are equal to ω. This contribution is proportional to the squared amplitude of the corresponding vibrations $|q_{nj}(E_{nj})|^2$, to the occupation factor $w_{st}(E_{nj})$ and to the spectral density of vibrations at ω/n, $|\mathrm{d}\omega_j(E_{nj})/\mathrm{d}E_{nj}|^{-1}$ (if it diverges, additional peaks of $Q'(\omega)$ appear: Soskin, 1987). The singular term $\mathscr{M}\delta(\omega)$ in (13.4.17) is connected with the terms q_{0j} in (13.4.16) that do not vary in time but depend on E and j. The term $\mathscr{M}\delta(\omega)$ is smeared when relaxation is taken into account (see below; it should be noted that if (13.4.18) has several solutions for given j, n, the summation over these solutions ought to be carried out in (13.4.17)).

At $F = 2\Gamma\Delta U_1/\mathscr{B} \gg 1$ the oscillator performs mainly small-amplitude vibrations about the equilibrium position q_1, and the main contribution to (13.4.17) in the range $\omega \simeq \omega_1$ is made by the term $Q^{(11)}(\omega)$ which is due to the fundamental tone of these vibrations (if $|U_1 - U_2| \lesssim \mathscr{B}/\Gamma$, and ω_1 and ω_2 are closely spaced, the term $Q^{(12)}(\omega)$ is also essential). For small $\omega - \bar{\omega}_1(U_1)$ in solving (13.4.18) $\omega_1(E_{11}) = \omega$ it suffices to retain only the linear in $E - U_1$ term in the expansion of $\omega_1(E)$. To the lowest order in F^{-1}, the expression for $Q^{(11)}(\omega)$ goes over into that for $w_1\tilde{Q}_0(\Omega_1)$ (see (13.4.9), (13.4.15); apparently, the result of the dissipationless approximation (13.4.17) is valid near the maximum of $Q'(\omega)$ provided $|\rho| \gg 1$). However, in contrast to (13.4.14) and (13.4.15), the expressions (13.4.17)–(13.4.19) are not limited to the range of

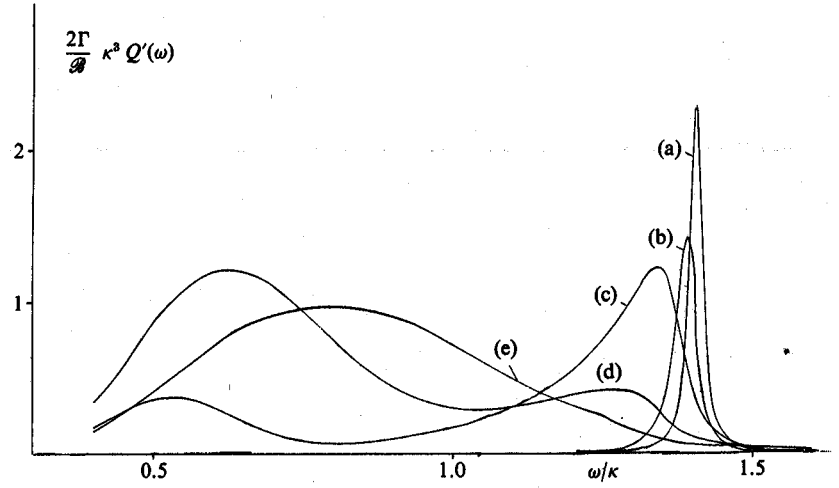

Figure 13.8. Spectral density of fluctuations of an underdamped Duffing oscillator performing Brownian motion in the double-well potential (13.4.1). The low-frequency parts of the spectra, which contain the zero-frequency peak, are not plotted. Curves (a)–(e) correspond to the values $F^{-1} = 2\gamma\mathscr{B}/\Gamma\kappa^4 = 0.02$, 0.06, 0.24, 0.6, 1.6; $\Gamma/\kappa = 0.01$. The ordinates of curves (a) and (b) are decreased by a factor of three.

small F^{-1}, $|\omega - \omega_1|/\omega_1$, and thus give the essential corrections to (13.4.14) and (13.4.15) when these parameters are not too small.

For many actual double-well potentials $\omega_1 \equiv \omega_1(U_1)$ is the highest eigenfrequency of the vibrations in well 1. Respectively, $Q^{(11)}(\omega)$ vanishes for $\omega \geqslant \omega_1$. The important contribution to $Q'(\omega)$ in this frequency range is due to the effects of dissipation. In particular, at $|\omega - \omega_1| \lesssim \Gamma|\rho|^{1/2}$, according to (13.4.15), $Q'(\omega)$ depends nonanalytically on the noise intensity $\mathscr{B}$ (and on the friction coefficient Γ for fixed $\mathscr{B}/\Gamma$), $Q'(\omega) \propto (\mathscr{B}/\Gamma)^{-1/2}\Gamma^{1/2}$. On the whole the fluctuation spectrum $Q'(\omega)$ near the frequency ω_1 at $F \gg 1$ may be shown to be described accurately up to terms of order Γ/ω_1 by the expression

$$Q'(\omega) = Q'_0(\omega) + w_1[\tilde{Q}(\Omega_1) - \tilde{Q}_0(\Omega_1)]\left(1 + \frac{\omega - \omega_1}{2\omega_1}\right)^{-2},$$

$$|\omega - \omega_1| \ll \omega_1. \tag{13.4.20}$$

In the case of the double-well Duffing oscillator the second addend in the r.h.s. of (13.4.20) should be doubled. For this case the evolution of the considered peak of $Q'(\omega)$ (it is located at $\omega \simeq \omega_1 = \kappa\sqrt{2}$) with increasing $\mathscr{B}/\Gamma\,\Delta U_1$ is seen from Figure 13.8 (cf. in particular curves (a)–(c)). Curve (a) is close to the Lorentzian ($\rho \simeq -0.25$ here). Curve (b) ($\rho \simeq -0.75$) has a considerably higher width and is slightly asymmetric. Curve (c) ($\rho \simeq -3$) is strongly asymmetric in the range of the considered peak.

13.4.3 *Features of the spectral density of fluctuations due to motion near the local maximum of a potential*

Bistable systems with stable stationary states have, as a rule, an unstable stationary state. For an oscillator with a double-well potential $U(q)$ the unstable state corresponds to the point ($q = 0$) in which $U(q)$ has a local maximum (see Figure 13.7).

Near an unstable stationary state an underdamped system suffers a characteristic slowing down in the absence of fluctuations. For an oscillator this slowing down manifests itself in the period of vibrations diverging while the frequency $\omega_j(E)$ tends to zero as the vibration energy approaches the value of a potential in the unstable state $U(0) = 0$. In particular, for the overbarrier vibrations ($E > 0$)

$$\omega_0(E) = \pi \left\{ \int_{q_t^{(1)}}^{q_t^{(2)}} dq [2E - 2U(q)]^{-1/2} \right\}^{-1}$$
$$\simeq \pi\kappa \ln^{-1}(C_0/E), \quad E \to 0, \tag{13.4.21}$$

where q_t are the turning points, $U(q_t^{(1,2)}) = E$, $C_0 \sim \kappa^2 q_{1,2}^2$.

It is evident from (13.4.17), (13.4.18) and (13.4.21) that in the dissipationless approximation at $e^{-F} \ll 1$ the features of the oscillator motion near the unstable stationary state give rise to the specific quite universal form of $Q_0'(\omega)$ in the range $\omega \ll \pi\kappa$, ω_1:

$$Q_0'(\omega) = w_{st}(0) C_0' \exp\left[-\frac{\pi\kappa}{\omega} - \frac{2\Gamma C_0}{\mathscr{B}} \exp\left(-\frac{\pi\kappa}{\omega} \right) \right],$$
$$\exp\left(-\frac{\pi\kappa}{\omega} \right) \ll 1, \tag{13.4.22}$$
$$C_0' \equiv C_0'(\omega) \sim \frac{C_0^2}{\omega\kappa^3}, \quad w_{st}(0) \propto \exp\left(-\frac{2\Gamma \Delta U_1}{\mathscr{B}} \right) \ll 1$$

(the detailed derivation of (13.4.22) and the explicit form of C_0, C_0' for the Duffing oscillator are given by Dykman, Soskin and Krivoglaz, 1984).

The intensity of the spectrum (13.4.22) is proportional to the population of the oscillator states near the top of the barrier and thus $\propto \exp(-F)$. The shape of the spectrum is determined by the competition between two factors, the reciprocal density of vibration frequencies $|d\omega_0(E)/dE|^{-1}$ and the state occupation factor $w_{st}(E)$ (see (13.4.17)). With rising frequency $\omega_0(E) = \omega$ the former increases exponentially (see (13.4.22)), while the latter decreases extremely sharply (cf. (13.4.5); $E \propto \exp[-\pi\kappa/\omega_0(E)]$).

As a result of this competition a quite narrow peak of $Q_0'(\omega)$ is formed. The position of its maximum is given by

$$\omega_m = \pi\kappa [\ln(2\Gamma C_0/\mathscr{B})]^{-1}, \quad \ln(2\Gamma C_0/\mathscr{B}) \gg 1. \tag{13.4.23}$$

Such a peak for the double-well Duffing oscillator is seen in Figure 13.8(c) (the dissipation was neglected when calculating the spectral curves (c)–(e) in the range $\omega \leqslant \kappa$).

As the noise intensity $\mathscr{B}$ increases the intensity of the peak of $Q'(\omega)$ due to the overbarrier motion increases exponentially (see (13.4.22)). The width of the peak increases rapidly also. For sufficiently high $\mathscr{B}/\Gamma\Delta U_1$ this peak practically overlaps the peaks of $Q'(\omega)$ due to vibrations about the minima of $U(q)$ and a common peak is formed in $Q'(\omega)$ (see Figure 13.8).

The spectral density in the dissipationless approximation $Q'_0(\omega)$ for the Duffing oscillator was calculated by Onodera (1970). In Onodera's paper the term $\mathscr{M}\delta(\omega)$ in (13.4.17), which describes the peak of $Q'_0(\omega)$ at zero frequency was omitted; see Dykman, Soskin and Krivoglaz (1984). In the latter paper it was shown also, in particular, that the slowing down of the motion near the local maximum of a potential means that the combined influence of a weak random force, together with friction, is able to substantially modify the character of motion; the corresponding additions to $Q'(\omega)$ in the low-frequency range are proportional to $\Gamma^{1/2}(\mathscr{B}/\Gamma)^{1/2}w_{st}(0)$. The numerical calculation of $Q'(\omega)$ for the Duffing oscillator for several values of $\mathscr{B}/\Gamma\Delta U_1$ and Γ/κ was carried out by Voigtlaender and Risken (1985).

13.4.4 *The range of very small frequencies*

As mentioned in Section 13.3.2, one of the features of bistable systems driven by low-intensity noise is the extremely narrow peak of $Q'(\omega)$ at zero frequency. This peak is due to fluctuational transitions between stable states. For an oscillator moving in a double-well potential it has noticeable intensity when $|U(q_1)-U(q_2)|\Gamma/\mathscr{B} \lesssim 1$ and its width $\sim \Gamma\exp(-F)$ (see (13.3.8) and (13.4.8)).

The shape of the zero-frequency peak of the spectral density of fluctuations is determined, however, not only by fluctuational transitions, i.e. by large fluctuations, but also by fluctuations with other scales. In particular in the wing of the peak, $\omega \sim \Gamma \gg \Gamma\exp(-F)$, where the contribution to $Q'(\omega)$ made by fluctuational transitions is exponentially small, the spectrum is formed by the small fluctuations about the equilibrium positions, namely by the comparatively slow (with a characteristic time scale $\sim \Gamma^{-1} \gg \omega_j^{-1}$) fluctuations of the oscillator coordinate averaged over the vibration period, $q_{0j}(E)$ (see (13.4.16)). Such fluctuations are connected with the energy fluctuations. This mechanism yields the following term in $Q'(\omega)$:

$$\left.\begin{aligned} &\delta Q'(\omega) = \frac{1}{\pi} D \frac{2\Gamma}{\omega^2 + 4\Gamma^2}, \\ &D = \left(\frac{\mathscr{B}}{2\Gamma}\right)^2 \sum_{j=1,2} w_j [(\mathrm{d}q_{0j}/\mathrm{d}E)_{E=U_j}]^2, \\ &\omega_{1,2} \gg \omega \gg \Gamma\exp(-F), \quad \exp(-F) \ll 1, \; F = 2\Gamma\Delta U_1/\mathscr{B}. \end{aligned}\right\} \tag{13.4.24}$$

The spectral density (13.4.24) is of the form of the Lorentzian peak with a halfwidth 2Γ. This peak is much less intensive than that at $\omega \simeq \omega_1$, the ratio of their intensities $D/G \sim F^{-1} \ll 1$ (see (13.4.24), (13.4.14)). In essence the emergence of the peak of the type (13.4.24) was shown by Krivoglaz and Pinkevich (1966) in considering the single-well quantum oscillator with an asymmetric potential. It is just the asymmetry of the potential $U(q)$ near the minima $q = q_j (j = 1, 2)$ that gives rise to the dependence of q_{0j} on E.

At $|U(q_1) - U(q_2)| \lesssim \mathscr{B}/\Gamma$ the spectrum $Q'(\omega)$ in the intermediate frequency range $\Gamma \exp(-F) < \omega < \Gamma$ goes over from the extremely narrow peak to the function (13.4.24). With rising noise intensity $\mathscr{B}$ the width of the peak increases sharply and the shape of the peak changes as a whole. The integral intensity of the peak is determined by the parameter $\mathscr{M}$, see (13.4.19).

Thus, depending on the noise intensity $\mathscr{B}$ and the relation between the parameters of the potential wells, the spectral density of fluctuations can have different number of peaks. At small $\mathscr{B}$ and different well depths $(U(q_2) - U(q_1) \gg \mathscr{B}/\Gamma) Q'(\omega)$ has a peak near the frequency ω_1 of the vibrations about the lowest minimum of $U(q)$ (and, generally speaking, weak peaks with relative intensities $\propto \mathscr{B}$ at frequencies $\simeq n\omega_1; n = 0, 2, 3, \ldots$). At close values of the well depths $Q'(\omega)$ has an intensive extremely narrow peak at zero frequency. At somewhat larger values of $\mathscr{B}$ the function $Q'(\omega)$ has also a distinct peak at the frequency (13.4.23) which is caused by the overbarrier vibrations. With increasing $\mathscr{B}$ the intensity and width of this peak increase, and it shifts and coalesces with the peaks due to the intrawell vibrations. Experimentally such transformation of the spectrum with rising $\mathscr{B}$, including the emergence of three distinct peaks of $Q'(\omega)$ in a certain interval of $\Gamma \Delta U_1/\mathscr{B}$, was confirmed recently by Mannella, McClintock and Moss (1987; see also Chapter 9 of Volume 3) using the analog electronic circuit which simulated an underdamped double-well Duffing oscillator.

Finally, we note that in terms of quantum theory an anharmonicity of a vibration subsystem results in nonequidistance of the energy levels and therefore in a difference in the Bohr frequencies. If this difference exceeds the damping the spectral density of fluctuations has a fine structure (Ivanov, Kvashnina and Krivoglaz, 1965). For an oscillator coupled to a bath, $Q'(\omega)$ may be calculated in the explicit form with allowance for a fine structure (Dykman and Krivoglaz, 1973). In bistable systems quantum effects lead also to a change of the form of $Q'(\omega)$ at small ω. In particular, the narrow peak of $Q'(\omega)$ turns out to lie at a finite frequency which depends on a tunneling probability. The tunneling in a subsystem also causes transitions between stable states. The probabilities of such transitions in the case of a nonlinear oscillator in a strong, resonant field were obtained by Dmitriev and Dyakonov (1986) neglecting relaxation. The quantum theory of transitions between the stable states of a nonequilibrium nonlinear oscillator coupled to a medium is developed by Dykman and Smelyanskii (1988).

References

Arnold, V. I. 1978. *Additional Chapters of the Theory of Ordinary Differential Equations.* Moscow: Nauka. (In Russian.)

Ben-Jacob, E., Bergman, D. J., Matkowsky, B. J. and Schuss, Z. 1982. *Phys. Rev. A* **26**, 2805–16.

Bogolyubov, N. N. and Mitropolsky, Y. A. 1961. *Asymptotic Methods in the Theory of Nonlinear Oscillators.* New York: Gordon and Breach.

Bonifacio, R. and Lugiato, L. A. 1978. *Phys. Rev. Lett.* **40**, 1023–7.

Dmitriev, A. P. and Dyakonov, M. I. 1986. *Zh. Eksp. Teor. Fiz.* **90**, 1430–40.

Dykman, M. I. 1978. *Fiz. Tverd. Tela (Leningrad)* **20**, 2264–72. [*Sov. Phys.-Solid State* **20**, 306 (1978).]

Dykman, M. I. 1986. *Zh. Eksp. Teor. Fiz.* **91**, 1518–30.

Dykman, M. I. and Krivoglaz, M. A. 1971. *Phys. Stat. Sol. (b)* **48**, 497–512.

Dykman, M. I. and Krivoglaz, M. A. 1973. *Zh. Eksp. Teor. Fiz.* **64**, 993–1005. [*Sov. Phys.-JETP* **37**, 506 (1973).]

Dykman, M. I. and Krivoglaz, M. A. 1979. *Zh. Eksp. Teor. Fiz.* **77**, 60–73. [*Sov. Phys.-JETP* **50**, 30–7 (1979).]

Dykman, M. I. and Krivoglaz, M. A. 1980a. *Physica* **104A**, 480–94.

Dykman, M. I. and Krivoglaz, M. A. 1980b. *Physica* **104A**, 495–508.

Dykman, M. I. and Krivoglaz, M. A. 1984. In *Soviet Physics Reviews* (I. M. Khalatnikov, ed.), vol. 5, pp. 265–441. New York: Harwood.

Dykman, M. I. and Smelyanskii, V. N. 1988. *Zh. Eksp. Theor. Fiz.* **94** (9).

Dykman, M. I., Soskin, S. M. and Krivoglaz, M. A. 1984. Preprint: Institute of Metal Physics 4.84 (in Russian). [*Physica* **133A**, 53–73 (1985).]

Feynman, R. P. and Hibbs, A. R. 1965. *Quantum Mechanics and Path Integrals.* New York: McGraw-Hill.

Flytsanis, C. and Tang, C. L. 1980. *Phys. Rev. Lett.* **45**, 441–5.

Fronzoni, L., Grigolini, P., Mannella, R. and Zambon, B. 1985. *J. Stat. Phys.* **41**, 553–79.

Fronzoni, L., Grigolini, P., Mannella, R. and Zambon, B. 1986. *Phys. Rev. A* **34**, 3293–3303.

Gibbs, H. 1985. *Bistability: Controlling Light with Light.* New York: Academic Press.

Goldstone, J. A. and Garmire, E. 1984. *Phys. Rev. Lett.* **53**, 910–13.

Graham, R. and Schenzle, A. 1981. *Phys. Rev. A* **23**, 1302–21.

Haken, H. 1983. *Synergetics*, 3rd edn. Berlin: Springer.

Horsthemke, W. and Lefever, R. 1984. *Noise-Induced Transitions.* Berlin: Springer.

Ivanov, M. A., Kvashnina, L. B. and Krivoglaz, M. A. 1965. *Fiz. Tverd. Tela (Leningrad)* **7**, 2047–57. (*Sov. Phys.-Solid State* **7**, 1652 (1965).]

Kolmogorov, A. and Leontovich, M. 1933. *Phys. Z. Sowjetunion* **4**, 1.

Kramers, H. A. 1940. *Physica* **7**, 284–304.

Krivoglaz, M. A. and Pinkevich, I. P. 1966. *Zh. Eksp. Teor. Fiz.* **51**, 1151–62.

Krivoglaz, M. A. and Pinkevich, I. P. 1970. *Ukr. Fiz. Zhurn.* **15**, 2039–49.

Landau, L. D. and Lifshitz, E. M. 1976. *Mechanics.* London: Pergamon.

Landau, L. D. and Lifshitz, E. M. 1980. *Statistical Physics*, 3rd edn., part I (revised

by E. M. Lifshitz and L. P. Pitaevskii). New York: Pergamon.
Lax, M. 1967. *Phys. Rev.* **160**, 290–307.
Lax, M. 1968. In *Statistical Physics, Phase Transitions and Superfluidity* (M. Chrétien, E. P. Gros and S. Deser, eds.). New York: Gordon and Breach.
Lugiato, L. A. 1984. *Progr. Optics* **21**, 69–216.
Mangel, M. 1979. *Physica* **97A**, 597–615, 616–31.
Mannella, R., McClintock, P. V. E. and Moss, F. 1987. *Europhys. Lett.* **4**, 511.
Maslova, N. S. 1986. *Zh. Eksp. Teor. Fiz.* **91**, 715–27.
Matkowsky, B. J. and Schuss, Z. 1983. *Phys. Lett.* **95A**, 213–15.
Narayanamurti, V. and Pohl, R. O. 1970. *Rev. Mod. Phys.* **42**, 201–36.
Onodera, Y. 1970. *Prog. Theor. Phys.* **44**, 1477.
Renz, W. 1985. *Z. Phys. B* **59**, 91–102.
Renz, W. and Marchesoni, F. 1985. *Phys. Lett.* **112A**, 124–8.
Risken, H. 1970. *Progr. Optics* **8**, 239–94.
Risken, H. 1984. *The Fokker–Planck Equation*. Berlin: Springer.
Rodriguez, R. F. and Van Kampen, N. G. 1976. *Physica* **85A**, 347–62.
Rytov, S. M. 1955. *Zh. Eksp. Teor. Fiz.* **29**, 304–14, 315–28.
Schuss, Z. 1960. *SIAM Rev.* **22**, 119–55.
Shenoy, S. R. and Agarwal, G. S. 1984. *Phys. Rev. A* **29**, 1315–25.
Soskin, S. M. 1987. Preprint N7 IPAS Ukr.SSR, Kiev.
Sture, K., Nordholm, J. and Zwanzig, R. 1974. *J. Stat. Phys.* **11**, 143–58.
Talkner, P. and Hänggi, P. 1984. *Phys. Rev. A* **29**, 768–73.
Tomita, H., Ito, A. and Kidachi, H. 1976. *Prog. Theor. Phys.* **56**, 786–800.
Van Kampen, N. G. 1965. In *Fluctuation Phenomenon in Solids* (R. E. Butgess, ed.), pp. 139–77. New York: Academic Press.
Ventsel, A. D. and Freidlin, M. I. 1970. *Usp. Mat. Nauk.* **25**, 3–55. [*Russ. Math. Surv.* **25**, 1–50 (1970).]
Voigtlaender, L. and Risken, H. 1985. *J. Stat. Phys.* **40**, 397–429.