Stochastic resonance in the linear and nonlinear responses of a bistable system to a periodic field

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The validity of the theory of the linear response (the satisfaction of the standard spectral relations) during a stochastic resonance is demonstrated. A stochastic resonance in relatively strong fields is studied. The reason for the appearance of a stochastic resonance and the region of parameter values in which it is seen are pointed out.

A fundamental new phenomenon in bistable and multistable systems is a stochastic resonance: an increase and a subsequent decay of the signal-to-noise ratio R with increasing level of the external noise. The signal-to-noise ratio here is the ratio of the peak height on the spectral density of the fluctuations, $Q(\omega)$, at the frequency of the external field, Ω, to the value of the spectral density of the fluctuations in the absence of a field, $Q^{(0)}(\Omega)$ (there are also other, essentially equivalent, definitions of a stochastic resonance; see Refs. 1 and 2 and the bibliography there). Stochastic resonances have been observed in ring lasers[1] and in numerical and analog simulations of various bistable systems.[1-3] In studies of the stochastic resonance, its nonlinear nature is usually stressed.

In the present letter we show that in a weak periodic field a stochastic resonance can be described completely by the theory of the linear response. It thus becomes simple to see the reason for the occurrence of a stochastic resonance and to determine how R depends on the parameters. For definiteness, an analysis is carried out for a Brownian particle in a two-well potential $U(q)$ (interest in this model has revived because of, in particular, research on Josephson-junction systems[4]):

$$\ddot{q} + 2\Gamma\dot{q} + U'(q) = A\cos\Omega t + f(t), \quad \langle f(t)f(t')\rangle = 4\Gamma T\delta(t-t'). \tag{1}$$

As a result of forced oscillations, which are determined at a small field amplitude A by the linear susceptibility $\chi(\Omega)$, i.e., $\langle q(t)\rangle = A\,\mathrm{Re}[\chi(\Omega)\exp(-i\Omega t)] + \mathrm{const}$, a δ-function peak arises at the field frequency Ω on the spectra fluctuation density of the system,

$$Q(\omega) = \lim_{\tau\to\infty} (4\pi\tau)^{-1} \Big|\int_{-\tau}^{\tau} dt e^{i\omega t} q(t)\Big|^2 . \tag{2}$$

The ratio of its integral intensity (the area under it) to $Q^{(0)}(\Omega)$, i.e., to the value of the spectral fluctuation density at $A=0$, is evidently

$$R = \frac{1}{4} A^2 |\chi(\Omega)|^2 / Q^{(0)}(\Omega) \tag{3}$$

[$Q(\omega)$ in (2) is equal to the Fourier transform of the correlation function $\langle q(t+t_1) q(t_1)\rangle$, averaged over t_1; it corresponds to the spectral density of fluctuations which is ordinarily measured experimentally]. For a system which is at thermodynamic equilibrium at $A = 0$ [or at a quasiequilibrium if the noise $f(t)$ is of a nonthermal nature], $\chi(\omega)$ can be expressed in terms of $Q^{(0)}(\omega)$ in the standard way:

$$\operatorname{Re}\chi(\omega) = \frac{2}{T}\int_0^\infty d\omega_1 Q^{(0)}(\omega_1)\omega_1^2(\omega_1^2-\omega^2)^{-1}, \quad \operatorname{Im}\chi(\omega) = \frac{\pi\omega}{T} Q^{(0)}(\omega). \tag{4}$$

The quantitative agreement between the values of R measured[5] through an analog simulation of system (1), on the one hand, and the values of R calculated from Eqs. (3) and (4) with the help of the spectral density of fluctuations $Q^{(0)}(\omega)$ measured for the same system at $A = 0$, on the other, is demonstrated by Fig. 1. These results show that the fluctuation-dissipation theorem and the dispersion relations hold under conditions corresponding to a stochastic resonance.

The appearance of a stochastic resonance ~~with~~ is due to a periodic modulation[6] of the probabilities for fluctuational interwell transitions W_{ij} and of the populations of the potential wells, w_i ($i = 1,2$), by an alternating field (cf. Ref. 1) [*sic*]. In the absence of a field, fluctuational transitions at $T \ll \Delta U_{1,2}$ (ΔU_i is the depth of well i) lead to a narrow peak $Q_{\rm tr}^{(0)}(\omega)$ in $Q^{(0)}(\omega)$ (Ref. 7):

$$Q_{\rm tr}^{(0)}(\omega) = \frac{1}{\pi} w_1 w_2 (q_1 - q_2)^2 W/(W^2+\omega^2), \quad W = W_{12} + W_{21} \tag{5}$$

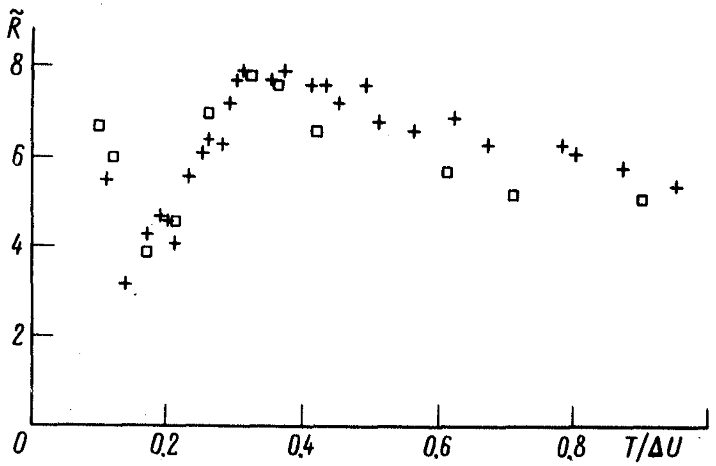

FIG. 1. Values of $\tilde{R} = 6.51\times 10^{-4} R$ for the potential $U(q) = -\frac{1}{2}q^2 + \frac{1}{4}q^4$; $\Omega = 0.0695$, $A = 0.1$, $\Gamma = 0.125$; $\Delta U = 1/4$. □—Direct measurements; +—calculated from measured values of $Q^{(0)}(\omega)$.

($q_{1,2}$ are equilibrium positions). If only the term in (5) is retained in $Q^{(0)}(\omega)$, it is clear from (3) and (4) that we have

$$R = \frac{1}{4}\pi A^2 (q_1 - q_2)^2 w_1 w_2 W/T^2 \propto \exp(-\Delta U_{max}/T), \quad \Delta U_{max} = max(\Delta U_1, \Delta U_2). \tag{6}$$

In other words, the signal-to-noise ratio R increases exponentially with increasing T, both for equal well depths[1-3] and for different well depths.

The range of applicability of (6) and of the existence of a stochastic resonance is found from the condition that $Q^{(0)}(\omega)$ is close to $Q^{(0)}{}_{\rm tr}$, i.e., the condition that the parts of $Q^{(0)}(\omega)$ due to vibrations with respect to the equilibrium positions, $Q_i^{(0)}(\omega) \approx 2/\pi \Gamma T w_i [(\omega^2 - \omega_i^2)^2 + 4\Gamma^2\omega^2]^{-1}$, where $\omega_i = [U''(q_i)]^{1/2}$ $(T \ll \Delta U_i)$ are small. Since $W_{ij} \propto \exp(-\Delta U_i/T) \ll 1$, a stochastic resonance is clearly manifested only at very low frequencies $\Omega \ll \omega_{1,2}, \omega_{1,2}^2/\Gamma$. In systems with a slight damping, $\Gamma \ll \omega_{1,2}$ the increment $\delta Q_i^{(0)} \approx (4\Gamma^2 + \omega^2)^{-1}{}_{\Gamma}/_{2\pi} w_i [TU'''(q_i)\omega_i^{-4}]^2$ becomes important[7] in the region $\omega \lesssim 2\Gamma$, even at comparatively small values of T. This circumstance seriously limits the interval of Ω and T values in which a stochastic resonance is observed, as was verified in the present experiments. It is clear from this discussion that for a given Ω and for very small values of T, under the condition $W \ll \Omega$, the ratio R decreases with T (as $1/T$ at $\Gamma \gtrsim \omega_{1,2}$). At large T, an increase in R then occurs, as was observed in Ref. 1 (cf. Fig. 1).

At low values of T, the response of a bistable system to a low-frequency field ($\Omega \ll \Gamma$, $\omega_{1,2}^2/\Gamma$) is very nonlinear even at small values of A, under the conditions $\Delta U_{1,2} \gg A\,|q_{1,2} - q_s| \gg T$ [q_s is the position of a local maximum of $U(q)$], since the modulation of the well populations due to the field is strong. Solving the balance equations for the populations, and incorporating the periodic increment in ΔU_i in the expression for the transition probability $W_{ij} \propto \exp(-\Delta U_i/T)$, we can show that the term associated with the transitions, $\langle q(t)\rangle_{\rm tr}$, in $\langle q(t)\rangle$ described by a nearly rectangular crest. In the simplest case of a symmetric potential, $U(q) = U(-q)$, $q_2 = -q_1$, we would have

$$\langle q(t)\rangle_{tr} \approx 2\bar{q} \sum_{n=-\infty}^{\infty} \left[\theta\left(t - \frac{2\pi n}{\Omega}\right) - \theta\left(t - \frac{\pi(2n+1)}{\Omega}\right)\right] - \bar{q},$$

$$\bar{q} = q_1 \tanh\lambda, \quad \lambda = \left(\frac{2\pi T}{|Aq_1|}\right)^{1/2} \frac{W_{12}}{2\Omega} \exp\left(\frac{|Aq_1|}{T}\right) \quad (W_{12} = W_{21}). \tag{7}$$

Under the condition $\lambda \gg 1$ (note that we have $1 - \tanh\lambda < 0.1$ for $\lambda \geqslant 1.5$), we have $\bar{q} \approx q_1$, and $\bar{q}$ is essentially independent of T. The peak intensity $Q(\omega)$ in (2) at the frequency Ω is thus also nearly independent of T. A stochastic resonance arises in the region $\Omega < W = 2W_{12}$ (in which the condition $\lambda \gg 1$ definitely holds). It results from a decrease in $Q^{(0)}(\Omega)$ with increasing T, as a result of a spreading of the peak in (5): $Q_{\rm tr}^{(0)}(\Omega) \propto \exp(\Delta U_1/T)$ at $\Omega \ll W$. It can be seen from (2) and (7) that the $Q(\omega)$

peaks at the overtones of Ω were found to be small in the regime of a strong nonlinearity in Refs. 2 and 3 because of a numerical reason: The intensity of the peak at the frequency $(2k+1)\Omega$ is proportional to $(2k+1)^{-2}$.

It follows from the results of the present study and those of Refs. 6 and 7 that a stochastic resonance should occur both at low frequencies and at frequencies close to the frequency of the strong field in systems with several stable forced-oscillation regimes in an intense periodic field (optically bistable and multistable systems, electrons excited by a field at the cyclotron frequency,[8] etc.).

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