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Distribution of fluctuational paths in noise-driven systems*

M. I. DYKMAN, V. N. SMELYANSKIY
Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48824, U.S.A.

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Dynamics of a system that performs a large fluctuation to a given state is essentially deterministic: the distribution of fluctuational paths peaks sharply at a certain *optimal* path along which the system is most likely to move. For the general case of a system driven by colored Gaussian noise, we provide a formulation of the variational problem for optimal paths. We also consider the prehistory problem, which makes it possible to analyze the shape of the distribution of fluctuational paths that arrive at a given state. We obtain, and solve in the limiting case, a set of linear equations for the characteristic width of this distribution.

Key words: fluctuations, colored noise, noise-driven systems, optimal paths.

1. Introduction

Large fluctuations, although infrequent, play a fundamental role in a broad range of processes, from diffusion in crystals to nucleation at phase transitions, mutations in DNA sequences, and failures of electronic devices. In many cases the fluctuating systems of interest are far from thermal equilibrium. Examples include lasers, pattern-forming systems [1], trapped electrons which display bistability and switching in a strong periodic field [2, 3], and spatially periodic systems (ratchets) which display a unidirectional current when driven away from thermal equilibrium [4].

It was very clearly shown by Landauer [5, 6] that, whereas for systems in thermal equilibrium the probabilities of fluctuations are known at least in principle, for nonequilibrium systems there is no universal relation from which these probabilities can be obtained: even though the system mostly stays in the vicinity of one of locally stable states, the distribution over these stable states can be found only from global analysis. This distribution may be strongly affected by nonthermal perturbations in the rarely occupied intermediate states, i.e. by the large fluctuations which determine the probabilities of switching between the stable states (Landauer's blowtorch theorem).

The major physical problems in the theory of large fluctuations are not only calculation of the fluctuation probabilities, but also analysis of the *dynamics* of large fluctuations. Understanding this dynamics is particularly important for controlling large fluctuations.

An intuitive approach to the theory of large fluctuations makes use of the *optimal path* concept. The optimal path is the path along which the system is most likely to move when it fluctuates to a given state from the vicinity of the stable state. Although trajectories of a fluctuating system are random, it is clear from Fig. 1 that the probabilities for the system which is found at a point q_f at an instant t_f to have arrived at this state along

* This paper is dedicated to Rolf Landauer, with deep respect and admiration.

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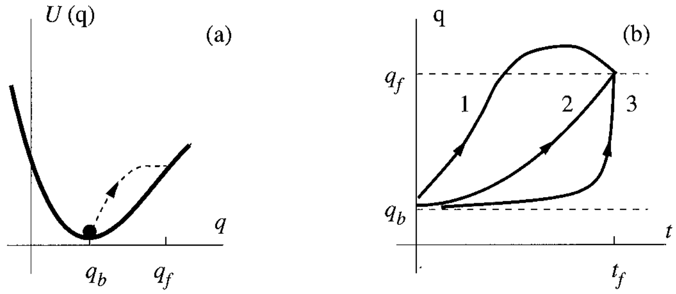

Fig. 1. (a), A particle with a coordinate q fluctuating away from the bottom $q = q_b$ of the potential well $U(q)$ to a point q_f. (b), Various paths along which a particle can reach q_f at a given instant t_f in the course of the fluctuation; the probability densities for moving along different paths are exponentially different. With overwhelming probability the system moves to a point (q_f, t_f) along an *optimal path.*

different paths are very different: e.g., it is unlikely that the system has been staying far from the equilibrium position for a long time, or that it has experienced an extremely large acceleration, as is the case for the paths 1 and 3 in Fig. 1.

Optimal paths and the probability distribution of the fluctuational paths are real physical objects: they have been observed in analog experiments [7–9] and digital simulations [10]. In the theory of large fluctuations, the pattern of optimal paths plays a role similar to that of the phase portrait in nonlinear dynamics.

The fundamental role of the distribution of fluctuational paths was recognized by Onsager and Machlup [11]; in fact, they obtained optimal paths for a linear system in thermal equilibrium with the bath with a short correlation time (the approximation of Brownian motion). For such systems, whether they are linear or nonlinear, the optimal path to a given state is the time-reversed path from this state to the vicinity of the stable state in the neglect of fluctuations (the deterministic path) [9, 12]. This is no longer true for nonequilibrium systems, because, in general, they lack time reversibility. Even for simple nonequilibrium systems the pattern of optimal paths may have singular features [13, 14].

In the present paper we provide a general formulation of the problem of optimal paths for nonequilibrium systems driven by Gaussian noise. In Section 2, based on a path-integral expression for the transition probability density, which allows for a prefactor, we derive an integral variational functional for optimal fluctuational paths. In Section 3 we provide a formulation of the prehistory problem for the distribution of fluctuational paths to a given state. This formulation is reduced to a linear integro-differential equation; the explicit form of this equation depends on the form of the correlation function of the noise. In Section 4 we discuss the solution of this equation in the case where the destination state is close to the stable state of the system. Section 5 contains concluding remarks.

2. Large fluctuations induced by Gaussian noise with an arbitrary power spectrum

Nonlinear systems driven by nonthermal Gaussian noise form an important and fairly general class of nonequilibrium systems. In spite of seeming simplicity, they display a variety of interesting effects, unidirectional current in periodic structures being an example [4]. Substantial progress in the theory of large fluctuations in such systems has been made within the last two decades, particularly by implementing the path integral technique (see [15–19] and references therein). So far this technique has been applied to systems driven by the noise which is a component of a Markov process [19], in which case the reciprocal power spectrum of the noise $1/\Phi(\omega)$ is a polynomial in ω^2, where

$$\Phi(\omega) = \int dt\, e^{i\omega t}\phi(t), \qquad \phi(t) = \langle f(t)f(0)\rangle. \tag{1}$$

It is advantageous to express the characteristics of fluctuations in the system in terms of the noise power spectrum, since $\Phi(\omega)$ can often be measured in experiment. The shape of $\Phi(\omega)$ depends on the source of the noise and the coupling between the dynamical system and this source.

The noise with $\Phi^{-1}(\omega)$ of the form of a polynomial in ω^2, although interesting, is not the most general type of noise. An important example of Gaussian noise with a non-polynomial $\Phi^{-1}(\omega)$ is the noise with a Gaussian power spectrum in the central part

$$\Phi(\omega) = D\exp\left[-(\omega^2 - \Omega_0^2)^2/4\Omega_0^2\sigma^2\right], \qquad |\omega^2 - \Omega_0^2| \lesssim \sigma^2. \tag{2}$$

In particular, noise of this sort is produced by the electric field of inhomogeneously broadened (in particular, Doppler-broadened) radiation.

2.1. Transition probability in a system driven by Gaussian noise

In this section we provide a general formulation of the problem of large fluctuations induced by Gaussian noise. We consider stationary systems or systems in a time-periodic field. To simplify notations, we will assume that the system under consideration is described by one dynamical variable, q. The Langevin equation of motion is then of the form:

$$\dot{q} = K(q;t) + f(t), \qquad K(q;t+T) = K(q;t), \tag{3}$$

where $f(t)$ is the zero-mean stationary noise. We assume that the noise is characterized by a certain correlation time t_{corr} over which its correlation function decays (at least exponentially, in the limit of large time).

For weak noise intensities, over a time t which exceeds t_{corr} and the characteristic relaxation time in the absence of noise t_{rel}, the system will approach the stable state $q^{(0)}(t)$ and will then perform small fluctuations about it. In periodically driven systems the state $q^{(0)}(t)$ is also periodic and is given by the equation

$$\dot{q}^{(0)} = K(q^{(0)};t), \qquad q^{(0)}(t+T) = q^{(0)}(t). \tag{4}$$

(we assume that the period of the state $q^{(0)}$ is the same as that of the periodic driving).

In the course of a large fluctuation, the dynamical system is brought from the attractor to a distant point q_f at the instant t_f (cf. Fig. 1). For this to happen the system should have been subjected to *finite* forcing over certain time. Different realizations of the force $f(t)$ can result in the same final state. The system trajectories $q(t)$ for each realization of $f(t)$ are deterministic rather than random, and they are independent of the characteristic noise intensity

$$D = \max\Phi(\omega). \tag{5}$$

The probability density of realizations of $f(t)$ is given by the functional (cf. [20])

$$\mathcal{P}[f(t)] = \exp\left[-\frac{1}{2D}\int dt\,dt'\,f(t)\hat{\mathcal{F}}(t-t')f(t')\right], \tag{6}$$

where the operator $\hat{\mathcal{F}}(t)$ is related to the correlation function of the noise $\phi(t)$ by the expression

$$\int dt_1\,\hat{\mathcal{F}}(t-t_1)\phi(t_1-t') = D\delta(t-t'). \tag{7}$$

In some cases (in particular, for the noise $f(t)$ being a component of a Markov process) a formal solution of this equation can be written as

$$\hat{\mathcal{F}}(t) = D\delta(t)/\Phi(-id/dt). \tag{8}$$

Here, we have taken into account that the noise correlation function is even, $\phi(t) = \phi(-t)$, as is also the noise power spectrum, $\Phi(\omega) = \Phi(-\omega)$.

One can write the probability density $p(q_f, t_f)$ for the noise-driven system to arrive at the point q_f at the instant t_f, provided it has started from the point q_i at the initial instant t_i, as a path integral

$$p(q_f, t_f) = \left\langle \int_{q_i \approx q^{(0)}(t_i)}^{q_f} \mathcal{D}q(t)\, \delta[q(t) - q_{det}(t; f|q_i)] \right\rangle. \tag{9}$$

Here, $q_{det}(t; f|q_i)$ is the solution of the dynamical equation of motion (3) for a given realization of the noise $f(t)$, and $\delta[q(t) - q_{det}(t)]$ is the functional delta-function: it peaks at the function $q(t)$ equal to $q_{det}(t)$. The averaging $\langle \cdots \rangle$ means integration over $f(t)$ with the probability density functional (6) as a weighting factor [20]. In what follows we assume that $t_i \to -\infty$ and the initial point is close to the attractor, $q_i \approx q^{(0)}(t_i)$. In this case the function p (9) gives the stationary probability distribution, which is periodic in time for the time-periodic force K in (3), $p(q_f, t_f) = p(q_f, t_f + T)$.

It is convenient to perform averaging over f in (9) by writing the δ-function in the form of a path integral over an auxiliary variable $k(t)/D$. Using standard transformations [20, 21, 16] one can show that the expression for $p(q_f, t_f)$ can be written in the following form:

$$p(q_f, t_f) = C \int \mathcal{D}f(t)\, \mathcal{P}[f(t)] \int \mathcal{D}\frac{k(t)}{D} \mathcal{D}q(t)$$
$$\times \exp\left\{ \int_{t_i}^{t_f} dt \left[i\frac{k(t)}{D}[\dot{q} - K(q; t) - f(t)] - \frac{1}{2}K' \right] \right\}, \quad K'(q; t) \equiv \frac{\partial K(q; t)}{\partial q}, \tag{10}$$

where C is the normalization constant, and $\mathcal{P}[f(t)]$ is the probability density functional for the random force (6).

2.2. Variational problem for optimal paths

If D is sufficiently small, as we assume, then, for all $f(t)$ which result in a large fluctuation to a given state, the values of the probability density functional (6) are exponentially small; they also exponentially strongly differ from each other for different $f(t)$. Thus one would expect that there exists one realization $f(t) = f_{opt}(t)$ which is exponentially more probable than the others. This realization provides the maximum to the functional (6) subject to the *constraint* that the system described by eqn (3) is driven to a designated state q_f. Respectively, through (3), there are in fact *two* interrelated optimal paths: that of the system, $q_{opt}(t)$, and that of the force, $f_{opt}(t)$ [15, 19].

Formally, optimal paths can be obtained for small D by evaluating the path integral (10) by the steepest descent method. It follows from eqns (6) and (10) that the optimal paths provide the minimum to the functional

$$\mathcal{R}[q(t), \lambda(t), f(t)] = \tfrac{1}{2} \iint_{-\infty}^{\infty} dt\, dt'\, f(t)\hat{\mathcal{F}}(t - t') f(t') + \int_{t_i}^{t_f} dt\, \lambda(t)[\dot{q} - K(q; t) - f(t)], \tag{11}$$

where $\lambda(t) \equiv -ik(t)$; one can think of $\lambda(t)$ as of a Lagrange multiplier that relates to each other the optimal realization of the random force and the path of the system.

Variational equations for the trajectories that provide an extremum to the functional (11) are of the form

$$\int dt'\, \hat{\mathcal{F}}(t - t') f(t') - \lambda(t) = 0, \qquad \dot{\lambda}(t) + K'(q; t)\lambda(t) = 0, \qquad \dot{q}(t) - K(q; t) - f(t) = 0. \tag{12}$$

In the problem of the stationary probability density for the system to be in the state q_f at the time t_f (this probability density is periodic in t_f with the period T), the boundary conditions for eqns (12) take the form (cf. [19])

$$f(t) \to 0 \quad \text{for } t \to \pm\infty, \qquad \lambda(t) \to 0 \quad \text{for } t \to -\infty, \qquad \lambda(t) = 0 \quad \text{for } t > t_f,$$
$$q(t) \to q^{(0)}(t) \quad \text{for } t \to -\infty, \qquad q(t_f) = q_f. \tag{13}$$

In deriving the boundary conditions (13) for $t \to -\infty$ we took into account that the system fluctuates about the stable state for a long time before the large fluctuation starts (cf. Fig. 1). Respectively, one may set in (9) $t_i \to -\infty$, $q(t_i) = q^{(0)}(t_i)$ (this is consistent with eqns (12)). On the other hand, the motion of the system *after* it has reached the point q_f is not important for the large fluctuation. Therefore, the constraint on $f(t)$ is lifted for $t > t_f$. Clearly, the force decays to zero for $t > t_f$.

The boundary conditions should be modified if one considers the problem of escape from a metastable state. By generalizing the arguments [19] to the case of a non-polynomial reciprocal power spectrum $1/\Phi(\omega)$, one can show that the optimal escape path corresponds to the system approaching the unstable periodic state $q_u(t)$ (the saddle state) for $t_f \to \infty$, and $\lambda(t_f) \to 0$ for $t_f \to \infty$, because in this case $\int_0^T dt\, K'(q_u(t); t) > 0$. The analysis of the escape problem is beyond the scope of the present paper.

Equations (12) and (13) provide a complete set of equations for the interconnected optimal fluctuational paths of the system and the force, $q_{opt}(t|q_f, t_f)$ and $f_{opt}(t|q_f, t_f)$, for reaching the state q_f at the instant t_f. The probability to reach this state, according to eqn (10), is of the form

$$p(q_f, t_f) \propto \exp[-R(q_f, t_f)/D], \qquad R(q_f, t_f) = \mathcal{R}[q_{opt}, \lambda_{opt}, f_{opt}] \equiv \min \mathcal{R}[q, \lambda, f]. \tag{14}$$

where $\lambda_{opt}(t)$ is the optimal Lagrange multiplier as given by eqns (12) and (13).

We note that eqns (12), (13) may have several solutions. In this case, the physically meaningful solution is the one that provides the *absolute minimum* to the functional $\mathcal{R}$. The criterion of applicability of the approach is $R/D \gg 1$—it is this condition that determines how small the noise intensity D should be.

An important feature of fluctuations induced by nonwhite noise, as it is clear from (12) and (13), is that the optimal force $f_{opt}(t)$ does not become equal to zero once the system has reached the state q_f. Time evolution of the optimal force is given by the equation

$$f_{opt}(t) = \int_{-\infty}^{t_f} dt'\, \bar{\phi}(t - t')\lambda_{opt}(t'), \qquad \bar{\phi}(t) = D^{-1}\phi(t). \tag{15}$$

The function $\bar{\phi}(t)$ is the noise correlation function rescaled so that it was independent of the characteristic noise intensity D (the noise correlator $\phi \propto D$, cf. (1), (5)).

Based on eqn (15) one can predict how the system will move, most likely, *after* the state q_f is reached. The trajectory of the system is described just by the equation of motion (3) with the random force given by eqn (15). We note that, even for the state q_f lying in the basin of attraction to the initially occupied stable state from which the fluctuation starts, on the way back from q_f the system may not necessarily come to the same stable state, but to a different state. This is in contrast with what happens in systems driven by white noise, unless q_f lies very close to the basin boundary. An example of such fluctuations in systems driven by colored noise was, in fact, considered in [19], although trajectories of the system after it has fluctuated to a remote state were not analyzed.

2.3. Vicinity of the stable state

To illustrate the solution of the variational problem (12) and (13) we will consider a simple case where the state q_f is close to the stable state $q^{(0)}(t_f)$, so that the force $K(q; t)$ can be linearized in $q - q^{(0)}(t)$, and yet the difference $|q_f - q^{(0)}(t_f)|$ is big enough so that the asymptotic expression (14) applies. With account taken of eqn (7), we obtain:

$$\lambda(t) = u(t, t_f)\lambda_f, \quad q(t) - q^{(0)}(t) = \int_{-\infty}^{t} d\tau\, f(\tau)u(\tau, t), \tag{16}$$
$$u(t, t') = \exp\left[\int_t^{t'} d\tau\, K'(q^{(0)}(\tau); \tau)\right].$$

One can see from eqn (16) that

$$\lambda_f = g^{-1}(t_f, t_f)[q_f - q^{(0)}(t_f)], \qquad g(t, t') = \int_{-\infty}^{t} d\tau \int_{-\infty}^{t'} d\tau' \,\bar{\phi}(\tau - \tau')u(\tau, t)u(\tau', t'). \tag{17}$$

From eqns (15)–(17) one can find the activation energy (14) of reaching the state q_f,

$$R(q_f, t_f) = \tfrac{1}{2} g^{-1}(t_f, t_f)[q_f - q^{(0)}(t_f)]^2. \tag{18}$$

The activation energy (18) is quadratic in the distance between the state q_f and the stable state $q^{(0)}(t_f)$. The proportionality factor $g^{-1}(t_f, t_f)$ depends on the shape of the noise-power spectrum, and also on the dynamics of the system in the absence of noise. We note that, if the regular force $K(q;t)$ is independent of t, the stable state $q^{(0)}$ is independent of t, and the function $g(t_f, t_f)$ is independent of t_f, as expected.

3. The prehistory problem

The distribution of paths for large fluctuations can be investigated and visualized through the analysis of the prehistory probability density, $p_h(q_h, t_h|q_f, t_f)$ [7]. This is the conditional probability density for a system that (i) had been fluctuating about $q^{(0)}(t)$ for a time greatly exceeding the relaxation time of the system t_{rel} and the correlation time of the noise t_{corr}, and (ii) arrived to the point q_f at the instant t_f, to have passed through (and been observed at) the point q_h at the instant t_h, $t_h < t_f$. Using the path-integral formulation (10) one can write the prehistory probability density as

$$p_h(q_h, t_h|q_f, t_f) = M \int \mathcal{D}f(t)\,\mathcal{P}[f(t)] \int \mathcal{D}\frac{k(t)}{D} \int_{q_i}^{q_f} \mathcal{D}q(t)\,\delta[q(t_h) - q_h] \tag{19}$$

$$\times \exp\left\{\int_{t_i}^{t_f} dt\left[i\frac{k(t)}{D}[\dot{q} - K(q;t) - f(t)] - \frac{1}{2}K'(q;t)\right]\right\},$$

$$\int dq\; p_h(q, t|q_f, t_f) = 1. \tag{20}$$

The normalization constant M in the general expression for p_h (19) is defined by the condition (20). Throughout this section we assume that $t_i \to -\infty$.

We expect (and will confirm *a posteriori*) that, for small D, the distribution $p_h(q, t|q_f, t_f)$ peaks sharply for q lying close to the optimal fluctuational path $q_{opt}(t|q_f, t_f)$. Therefore, in evaluating p_h one can expand the exponent in $\mathcal{P}[f]$ and the term with $k(t)$ in the exponent in eqn (19) in the deviations $\delta f(t)$, $\delta q(t)$, $\delta k(t)$ from the optimal realization of the random force $f_{opt}(t)$, the trajectory $q_{opt}(t)$, and $k_{opt}(t) \equiv i\lambda_{opt}(t)$.

In this paper we will not address the problem of singularities of optimal paths in systems driven by colored noise [19], which has been understood only recently for white-noise-driven systems [14]. If the point (q_f, t_f) is far from singularities, it suffices to keep in the aforementioned expansion only the second-order terms in $\delta f(t)$, $\delta q(t)$, $\delta k(t)$. Integrating over $\delta f(t)$ and writing $\delta[q(t_h) - q_h]$ as an integral, one can rewrite eqn (19) in the following form

$$p_h(q_h, t_h|q_f, t_f) = M_1 \int_{-\infty}^{\infty} d\frac{a}{2\pi D} \int \mathcal{D}\frac{k(t)}{D} \int \mathcal{D}q(t)\,\exp(-S[\delta k(t), \delta q(t)]/D), \tag{21}$$

where the quadratic functional S is given by the expression

$$\begin{aligned} S[\delta k, \delta q] = {} & \tfrac{1}{2}\iint_{-\infty}^{t_f} dt\,dt'\,\delta k(t)\bar{\phi}(t - t')\delta k(t') - i\int_{-\infty}^{t_f} dt\,\delta k(t)[\delta\dot{q}(t) - K'(t)\delta q(t)] \\ & - \tfrac{1}{2}\int_{-\infty}^{t_f} dt\,\lambda_{opt}(t)K''(t)\delta q^2(t) + ia[q(t_h) - q_h], \end{aligned} \tag{22}$$

with

$$\delta k(t) = k(t) - k_{opt}(t) \equiv k(t) - i\lambda_{opt}(t), \qquad \delta q(t) = q(t) - q_{opt}(t),$$
$$K'(t) \equiv K'(q_{opt}(t); t), \qquad K''(t) \equiv K''(q_{opt}(t); t) = \partial^2 K/\partial q^2,$$

and with the boundary conditions

$$\delta q(-\infty) = \delta q(t_f) = 0, \qquad \delta q(t_h) = q_h - q_{opt}(t_h), \qquad \delta k(-\infty) = 0. \tag{23}$$

It is convenient to rewrite the expression for S in the matrix form,

$$S[-i\psi_1, \psi_2] = \tfrac{1}{2}\iint_{-\infty}^{t_f} dt\, dt' \big(\psi_1(t), \psi_2(t)\big)\hat{\mathcal{H}}(t,t')\begin{pmatrix} \psi_1(t') \\ \psi_2(t') \end{pmatrix} + ia[q(t_h) - q_h], \tag{24}$$

where $\hat{\mathcal{H}}$ is a Hermitian operator,

$$\hat{\mathcal{H}}(t,t') = \begin{pmatrix} -\bar{\phi}(t-t') & \delta(t-t')(K'(t') - d/dt') \\ \delta(t-t')(K'(t') + d/dt') & -\delta(t-t')\lambda_{opt}(t)K''(t) \end{pmatrix}. \tag{25}$$

In obtaining eqn (25) we took into account the boundary conditions (23). As we will see, of immediate interest is the value of $S[\delta k, \delta q]$ for purely imaginary δk and real δq, which justifies the unusual form of S (24).

3.1. The variance of the prehistory probability distribution

Integration over $\delta k(t)$, $\delta q(t)$ in the expression (21) can be performed by the steepest descent method. In this method, one has to find the extremum of the quadratic functional S, which requires solving the following equations for the extreme values of $\delta k(t) = -i\psi_1(t)$, $\delta q(t) = \psi_2(t)$:

$$\int_{-\infty}^{t_f} dt'\, \hat{\mathcal{H}}(t,t')\begin{pmatrix} \psi_1(t') \\ \psi_2(t') \end{pmatrix} = -ia\delta(t_h - t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \psi_2(t_h) = q_h - q_{opt}(t_h) \tag{26}$$

where the value of a is determined by the boundary conditions (23).

The solution of eqn (26) can be sought in terms of the Green's function $G_{ij}(t,t')$ which provides the solution to the equation

$$\int_{-\infty}^{t_f} dt'\, \hat{\mathcal{H}}_{ii_1}(t,t')G_{i_1 j}(t',t'') = \delta_{ij}\delta(t-t''), \qquad G_{ij}(t,t') = G_{ji}(t',t), \qquad G_{i2}(t,t_f) = 0. \tag{27}$$

Here, summation is performed over repeated subscripts i_1; the subscripts i, i_1, j take on the values 1, 2. The functions ψ_i in eqn (26) are expressed in terms of the function G as

$$\psi_j(t) = [q_h - q_{opt}(t_h)]G_{j2}(t,t_h)/G_{22}(t_h,t_h). \tag{28}$$

Clearly, the functions ψ_j are proportional to the distance of the point q_h, for which the prehistory probability density is sought, to the optimal fluctuational path $q_{opt}(t_h|q_f, t_f)$.

Using eqn (24), the prehistory probability density can be expressed in terms of the functions $\psi_{1,2}$, and then in terms of the Green's function G,

$$p_h(q_h, t_h|q_f, t_f) = (2\pi D\sigma_h^2(t_h|q_f, t_f))^{-1/2}\exp\left(-\frac{[q_h - q_{opt}(t_h|q_f, t_f)]^2}{2D\sigma_h^2(t_h|q_f, t_f)}\right) \tag{29}$$

$$\sigma_h^2(t_h|q_f, t_f) = G_{22}(t_h, t_h).$$

The distribution p_h is Gaussian, with a maximum on the optimal path. It is seen from eqns (28) and (29) that the variance of the distribution is given immediately by the component of the Green's function $G_{22}(t_h, t_h)$. For very large $t_f - t_h$ the variance becomes independent of t_f, and eqn (29) describes the stationary distribution about the stable state.

Alternatively, the problem of the variance of the prehistory probability distribution can be solved in terms of the eigenfunctions and eigenvectors of the appropriate Hamiltonian. For white-noise-driven systems this method was discussed earlier [8].

We note that the above analysis makes it also possible to investigate the 'post-history' probability distribution: the distribution of the paths of the system *after* it has fluctuated to a remote state. For white-noise-driven systems, this distribution peaks at the path along which the system, prepared initially in the state q_f, goes down to the stable state $q^{(0)}$, as clearly demonstrated experimentally by Luchinsky and McClintock [9]. As discussed below eqn (15), colored noise is not turned off once it has driven the system to a given state, and therefore it affects the motion of the system after the state q_f has been reached.

Formulation of the post-history problem requires changing in all above equations to integration over the paths $q(t)$ and the auxiliary field $k(t)$ for t varying from $-\infty$ to ∞ (instead of $-\infty$ to t_f), with the condition that the paths go through the state q_f at the instant t_f. The final answer is again given by eqn (29), and the distribution peaks at the most probable path for the motion of the system after the state q_f has been reached.

4. Prehistory probability distribution close to the stable state

Explicit expressions for the prehistory probability density can be obtained if the final point q_f lies close enough to the stable state $q^{(0)}(t_f)$, in which case the equations for optimal paths of the system and the force are linear. In the prehistory problem, to the lowest order in $|q_f - q^{(0)}(t_f)|$, one can neglect the term in (25) with $\lambda_{opt}K'' \propto q_f - q^{(0)}(t_f)$ (cf. (17)), and can also replace K' by its value for the stable state. Equation (26) may then be immediately integrated. After straightforward but somewhat tedious calculations one obtains the following expression for the reduced variance of the distribution p_h:

$$\sigma_h^2(t_h|q_f, t_f) = [g(t_f, t_f)g(t_h, t_h) - g^2(t_f, t_h)]/g(t_f, t_f) \tag{30}$$

(the function $g(t, t')$ is defined in eqn (17)).

We emphasize that eqn (30) applies for an *arbitrary* shape of the power spectrum of the noise (however, we assume that the function $\bar{\phi}(\tau)$ decays at least exponentially for $|\tau| \to \infty$). It applies also for an arbitrary periodic driving. It may be further simplified in the absence of periodic driving, in which case $K' = -\alpha =$ constant, with $\alpha > 0$, and we obtain from (17)

$$g(t, t') = \frac{1}{2\pi} \int d\omega\, D^{-1}\Phi(\omega)(\alpha^2 + \omega^2)^{-1} e^{i\omega(t-t')}. \tag{31}$$

This expression makes it simple to calculate the variance of the prehistory probability density for an arbitrary shape of the noise power spectrum $\Phi(\omega)$. The results of these calculations for noise with the Gaussian power spectrum centered at zero frequency are shown in Fig. 2. It follows from this figure that the noise color changes the broadening of the prehistory distribution very substantially. An important consequence of eqns (30), (31) is that the variance $\sigma_h^2(t_h|q_f, t_f) \propto (t_f - t_h)^2$ for small $t_f - t_h$. This is qualitatively different from the linear time dependence of σ_h^2 in the case of white noise, known from [7, 8]. The reason is that, for systems driven by colored noise, the mean square displacement over the time Δt, which is small compared with the noise correlation time t_{corr}, is proportional to $(\Delta t)^2$, not to Δt, as for white-noise-induced diffusion.

For large $t_f - t_h$, which exceeds the relaxation time of the system and the correlation time of the noise t_{rel}, t_{corr}, the prehistory probability distribution goes over into the stationary distribution which is described by the function $R(q_h, t_h) = [q_h - q^{(0)}(t_h)]^2/2\sigma_h^2$, where $R(q_h, t_h)$ is given by eqn (18).

5. Conclusions

In contrast to white-noise-driven systems, for colored noise, after the noise has driven the system to a remote state q_f, it does not become small at a time. As the noise decays it drives the system further along a certain path. This path differs from the path which the system would follow if it were prepared in the state q_f 'by hand', not as a result of the fluctuation.

In the present paper we provided a general formulation of the problem of large fluctuations in systems

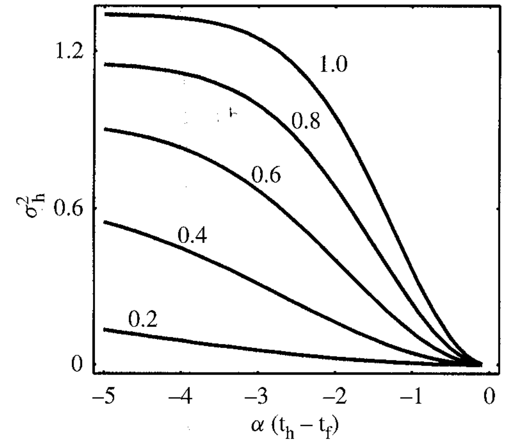

Fig. 2. Time dependence of the scaled variance σ_h^2 of the prehistory probability density in a linear system driven by noise with the Gaussian power spectrum $\Phi(\omega) = D\exp(-\omega^2/2\alpha^2\sigma^2)$, for different values of the dimensionless variance of the noise spectrum σ^2. The time is scaled by the decrement of the system $\alpha = -K'(q^{(0)})$.

driven by Gaussian noise. This formulation makes it possible to describe optimal fluctuational paths of the system, and also to evaluate the width of the tube of fluctuational paths that arrive at a given target state. The latter is done using the prehistory probability distribution for the system to have passed through a given point on its way to the target state. The tube of the fluctuational paths is centered at the optimal path. Evaluation of the width of the tube has been reduced to solution of a linear equation. Explicit results have been obtained for the fluctuations in the linear range close to the stable state, and the effect of noise color in this domain has been analyzed.

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