PHYS851 Quantum Mechanics I, Fall 2008

FINAL EXAM

NAME:

1. **Heisenberg Picture** Consider a single-particle system described by the Hamiltonian

$$H = i\hbar\chi(A - A^\dagger),$$

where

$$A = \frac{1}{\sqrt{2}}\left(\frac{1}{\lambda}X + i\frac{\lambda}{\hbar}P\right)$$

so that $[A, A^\dagger] = 1$.

   (a) Derive the Heisenberg equations of motion for $A_H(t)$ and $A_H^\dagger(t)$.

   (b) Solve these equations and give the solutions in terms of A_S and $A_S^\dagger$.

   (c) From these solutions, express $X_H(t)$ and $P_H(t)$ in terms of X_S and P_S.

   (d) At $t = 0$ the wavefunction of the particle is known to be $\psi(x, 0) = \mathcal{N}e^{-((x/\sigma)\sin(k_0 x))^3}$, where $\mathcal{N}$ is a normalization constant, and σ and k_0 are arbitrary constants. What is the wavefunction at any later time t? Hint: recall that $T(d) = e^{-\frac{i}{\hbar}dP}$.

2. Consider a pair of identical spin-1/2 particles in a uniform magnetic field. Neglect the motion of the particles, and consider only their spin degrees of freedom. The Hamiltonian is then

$$H = -\gamma \left(\vec{S}_1 \cdot \vec{B} + \vec{S}_2 \cdot \vec{B} \right) ,$$

where $\vec{S}_1$ and $\vec{S}_2$ are the spin operators of particles 1 and 2, respectively, and γ is a constant. Take the z-axis to lie along the magnetic field, so that $\vec{B} = B_0 \vec{e}_z$.

(a) Using the concept of a tensor product space, construct a suitable basis for this system where each basis vector is a simultaneous eigenstate of S_{1z} and S_{2z}.

(b) What are the energy eigenvalues of the system?

(c) What is the degeneracy of each energy level?

(d) What are the energy eigenstates?

3. Let $|1\rangle$, $|2\rangle$, and $|3\rangle$ be a set of three orthonormal state vectors. Consider a system described by the Hamiltonian

$$H = \hbar\Omega(|1\rangle\langle 2| + |2\rangle\langle 1|) + \hbar\omega_3|3\rangle\langle 3|,$$

where $\omega_3 \neq \Omega$.

Let $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)$.

(a) What are the eigenvalues and normalized eigenvectors of H?

(b) What is $|\psi(t)\rangle$?

(c) What is the probability that the system is in state $|3\rangle$ at time t?

(d) What is the probability that the system is in state $|2\rangle$ at time t?

4. (a) Make a sketch of the energy level diagram for a hydrogen atom, with energy on the vertical axis and angular momentum on the horizontal. (i.e. draw a separate level for each state with well defined n and ℓ eigenvalues, even though some of them may be degenerate.) Show all levels up to $n = 4$.

   (b) Indicate the total degeneracy of each n level, as well as the individual degeneracies of each n,ℓ state.

   (c) Write a formula for the wavelength of the photon emitted when the atom decays from level $n = 3, \ell = 2$ to level $n = 1, \ell = 0$.

   (d) Note that the magnetic dipole moment of the hydrogen atom is $\vec{\mu} = \frac{e}{2m}\vec{L}$. What is the paramagnetic term (i.e. the term linear in B) that must be added to the Hydrogen Hamiltonian when a uniform magnetic field is applied?

   (e) Indicate on your diagram what happen to each n,ℓ level in a uniform magnetic field (neglect diamagnetic effects). Clearly indicate which of the new levels are degenerate, and give the complete degeneracy of each energy eigenstate of the atom+field system.

5. Consider two annihilation operators A and B which satisfy $[A, A^\dagger] = [B, B^\dagger] = 1$ and $[A, B] = [A, B^\dagger] = 0$. Show that the operators

$$J_x = \frac{\hbar}{2}(A^\dagger B + B^\dagger A)$$

$$J_y = i\frac{\hbar}{2}(A^\dagger B - B^\dagger A)$$

$$J_z = \frac{\hbar}{2}(B^\dagger B - A^\dagger A)$$

satisfy the angular momentum commutation relations.

What are the energy eigenvalues and corresponding degeneracies of the following Hamiltonian

$$H = \hbar\Omega(2A^\dagger A B^\dagger B + A^\dagger A + B^\dagger B)?$$

Hint: write out the terms in $J_x^2 + J_y^2$.