PHYS851 Quantum Mechanics I, Fall 2009
HOMEWORK ASSIGNMENT 10

Topics Covered: Tensor product spaces, change of coordinate system, general theory of angular momentum
Some Key Concepts: Angular momentum: commutation relations, raising and lowering operators, eigenstates and eigenvalues.

1. [10 pts] Consider the position eigenstate $|\vec{r}\rangle$. In spherical coordinates, this state is written as $|r\theta\phi\rangle$, where $\vec{R}|r\theta\phi\rangle = r\vec{e}_r(\theta,\phi)|r\theta\phi\rangle$. In cartesian coordinates, the same state is written $|xyz\rangle$, where $\vec{R}|xyz\rangle = (x\vec{e}_x + y\vec{e}_y + z\vec{e}_z)\,|xyz\rangle$. Evaluate the following:
   (a) $\langle r\theta\phi|r'\theta'\phi'\rangle$
   (b) $\langle r\theta\phi|xyz\rangle$
   (c) $\langle r\theta\phi|p_x p_y p_z\rangle$
   (d) $\langle r\theta\phi|\vec{R}|r'\theta'\phi'\rangle$
   (e) $\langle r\theta\phi|Z|r'\theta'\phi'\rangle$
   (f) $\langle r\theta\phi|P_z|r'\theta'\phi'\rangle$

2. [10 pts] Consider a system consisting of two spin-less particles with masses m_1 and m_2, and charges q_1 and q_2.
   (a) Write the quantum mechanical Hamiltonian that describes this system.
   (b) Define a suitable tensor-product-state basis to describe the system.
   (c) Evaluate the expression $\langle b|H|\psi\rangle$, where $|\psi\rangle$ is an arbitrary state of the system, and $|b\rangle$ should be replaced by one of your basis states.
   (d) Do the same for N identical particles of mass m and charge q.

3. [20 pts] Let $\vec{L} = \vec{R} \times \vec{P}$, where $\vec{L}$, $\vec{R}$, and $\vec{P}$ are the three-dimensional vector operators for angular momentum, position, and linear momentum, respectively. For $\mu, \nu \in \{x, y, z\}$, evaluate the following expressions:
   (a) $[R_\mu, R_\nu]$
   (b) $[P_\mu, P_\nu]$
   (c) $[R_\mu, P_\nu]$

   Use these results to prove explicitly that $[J_x, J_y] = i\hbar J_z$, then use a symmetry argument to obtain similar expressions for the commutators $[J_y, J_z]$ and $[J_x, J_z]$.

4. [10 pts] Show explicitly that $J^2 = J_x^2 + J_y^2 + J_z^2$ commutes with J_z, then use a symmetry argument to show that J^2 must also commute with J_x and J_y. Then, answer the following (be sure to explain your reasoning):

   (a) Do simultaneous eigenstates of J_x and J_z exist?

   (b) Do simultaneous eigenstates of J^2 and J_z exist?

   (c) Do simultaneous eigenstates of J^2 and J_y exist?

   (d) Do simultaneous eigenstates of J^2, J_z, and J_y exist?

   (e) Do simultaneous eigenstates of J^2 and J_x^2 exist?

   (f) Do simultaneous eigenstates of J_z and $J_x^2 + J_y^2$ exist?

5. [20 pts] With $J_\pm = J_x \pm iJ_y$, express J_+J_- and J_-J_+ in terms of the operators J^2 and J_z, then compute the the following commutators:

   (a) $[J_+, J_-]$

   (b) $[J_\pm, J^2]$

   (c) $[J_\pm, J_z]$

   (d) $[J_\pm, J_x]$

   (e) $[J_\pm, J_y]$

6. [20 pts] Let $|j, m\rangle$ be the standard simultaneous eigenstate of J^2 and J_z. (a) What are $J^2|j, m\rangle$ and $J_z|j, m\rangle$ in terms of j and m? (b) What are the allowed values of j? (c) For a given j-value, what are the allowed values of m? First, re-write your answers to (a), (b), and (c) ten times, then compute the following matrix elements:

   (d) $\langle j, m|J^2|j', m'\rangle$

   (e) $\langle j, m|J_z|j', m'\rangle$

   (f) $\langle j, m|J_\pm|j'm'\rangle$

   (g) $\langle j, m|J_x|j'm'\rangle$

   (h) $\langle j, m|J_y|j'm'\rangle$

7. [10 pts] Consider a system described by the Hamiltonian

$$H = \Delta J_z + U(J_x^2 + J_y^2) \tag{1}$$

   where J_x, J_y, and J_z are the three components of a generalized angular momentum operator, and Δ and U are constants. What are the energy levels of this system?