PHYS851 Quantum Mechanics I, Fall 2009
HOMEWORK ASSIGNMENT 10: Solutions

Topics Covered: Tensor product spaces, change of coordinate system, general theory of angular momentum
Some Key Concepts: Angular momentum: commutation relations, raising and lowering operators, eigenstates and eigenvalues.

1. [10 pts] Consider the position eigenstate $|\vec{r}\rangle$. In spherical coordinates, this state is written as $|r\theta\phi\rangle$, where $\vec{R}|r\theta\phi\rangle = r\vec{e}_r(\theta,\phi)|r\theta\phi\rangle$. In cartesian coordinates, the same state is written $|xyz\rangle$, where $\vec{R}|xyz\rangle = (x\vec{e}_x + y\vec{e}_y + z\vec{e}_z)\,|xyz\rangle$. Evaluate the following:
   Most of these can be evaluated in many different ways, I am just giving one possibility for each:
   (a) $\langle r\theta\phi|r'\theta'\phi'\rangle \quad = \frac{1}{r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')$
   (b) $\langle r\theta\phi|xyz\rangle \quad = \delta(r\sin\theta\cos\phi - x)\delta(r\sin\theta\sin\phi - y)\delta(r\cos\theta - z)$
   (c) $\langle r\theta\phi|p_xp_yp_z\rangle \quad = [2\pi\hbar]^{-3/2}\exp\left[\frac{i}{\hbar}\left(p_xr\sin\theta\cos\phi + p_yr\sin\theta\sin\phi + p_zr\cos\theta\right)\right]$
   (d) $\langle r\theta\phi|\vec{R}|r'\theta'\phi'\rangle \quad = \frac{1}{r\sin\theta}\vec{e}_r(\theta,\phi)\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')$
   (e) $\langle r\theta\phi|Z|r'\theta'\phi'\rangle \quad = \frac{\cot\theta}{r}\vec{e}_r(\theta,\phi)\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')$
   (f) $\langle r\theta\phi|P_z|r'\theta'\phi'\rangle \quad = -i\hbar\partial_z\langle r\theta\phi|r'\theta'\phi'\rangle$

   now $\partial_z = \frac{\partial r}{\partial z}\partial_r + \frac{\partial\theta}{\partial z}\partial_\theta + \frac{\partial\phi}{\partial z}\partial_\phi = \sec\theta\partial_r - \frac{1}{r\sin\theta}\partial_\theta$

   so that

$$\begin{aligned}\langle r\theta\phi|P_z|r'\theta'\phi'\rangle &= 2i\hbar\frac{\sec\theta\csc\theta}{r^3}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') - i\hbar\frac{\sec\theta\csc\theta}{r^2}\delta'(r-r')\delta(\theta-\theta')\delta(\phi-\phi') \\ &\quad -i\hbar\frac{\cos\theta\csc^3\theta}{r^3}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') + i\hbar\frac{\csc^2\theta}{r^3}\delta(r-r')\delta'(\theta-\theta')\delta(\phi-\phi')\end{aligned}$$

2. [10 pts] Consider a system consisting of two spin-less particles with masses m_1 and m_2, and charges q_1 and q_2.

   (a) Write the quantum mechanical Hamiltonian that describes this system.

$$H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{R}_1 - \vec{R}_2|}$$

   (b) Define a suitable tensor-product-state basis to describe the system.

   Basis: $\{|\vec{r}_1, \vec{r}_2\rangle^{(12)}\}$ where $|\vec{r}_1, \vec{r}_2\rangle^{(12)} = |\vec{r}_1\rangle^{(1)} \otimes |\vec{r})2\rangle^{(2)}$

   (c) Evaluate the expression $\langle b|H|\psi\rangle$, where $|\psi\rangle$ is an arbitrary state of the system, and $|b\rangle$ should be replaced by one of your basis states.

$$\langle \vec{r}_1, \vec{r}_2|H|\psi\rangle = \left[-\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_1 - \vec{r}_2|} \right] \psi(\vec{r}_1, \vec{r}_2)$$

   (d) Do the same for N identical particles of mass m and charge q.

   Basis: $\{|\vec{r}_1, \dots, \vec{r}_N\rangle\}$, where $|\vec{r}_1, \dots, \vec{r}_N\rangle = |\vec{r}_1\rangle^{(1)} \otimes \dots \otimes |\vec{r}_N\rangle^{(N)}$

$$\langle \vec{r}_1, \dots, \vec{r}_N|H|\psi\rangle = \sum_{j=1}^{N} \left[-\frac{\hbar^2}{2m_j}\nabla_j^2 + \sum_{k=j+1}^{N} \frac{q_j q_k}{4\pi\epsilon_0 |\vec{r}_j - \vec{r}_k|} \right] \psi(\vec{r}_1, \dots, \vec{r}_N)$$

3. [20 pts] Let $\vec{L} = \vec{R} \times \vec{P}$, where $\vec{L}$, $\vec{R}$, and $\vec{P}$ are the three-dimensional vector operators for angular momentum, position, and linear momentum, respectively. For $\mu, \nu \in \{x, y, z\}$, evaluate the following expressions:

   (a) $[R_\mu, R_\nu] = 0$

   (b) $[P_\mu, P_\nu] = 0$

   (c) $[R_\mu, P_\nu] = i\hbar\delta_{\mu,\nu}$

   Use these results to prove explicitly that $[L_x, L_y] = i\hbar L_z$, then use a symmetry argument to obtain similar expressions for the commutators $[L_y, L_z]$ and $[L_x, L_z]$.

   We start from $L_x = YP_z - ZP_y$, $L_y = ZP_x - XP_z$, and $L_z = XP_y - YP_x$, so that

$$
\begin{aligned}
[L_x, L_y] &= [YP_z - ZP_y, ZP_x - XP_z] \\
&= [YP_z, ZP_x] - [YP_z, XP_z] - [ZP_y, ZP_x] + [ZP_y, XP_z] \\
&= YP_x[P_z, Z] - 0 - 0 + P_yX[Z, P_z] \\
&= -i\hbar YP_x + i\hbar XP_y \\
&= i\hbar L_z
\end{aligned}
$$

   For any right-handed triplet of unit-vectors, the choice which to call $\vec{e}_x$, is arbitrary, but once the choice is made, the labels of the other two are fixed due to handedness. Thus cyclic permutations of x, y, z are all equivalent to different labeling choices. This means that with different choices, our result would become $[L_y, L_z] = i\hbar L_x$ or $[L_z, L_x] = i\hbar L_y$.

4. [10 pts] Show explicitly that $J^2 = J_x^2 + J_y^2 + J_z^2$ commutes with J_z, then use a symmetry argument to show that J^2 must also commute with J_x and J_y. Then, answer the following (be sure to explain your reasoning):

$$\begin{aligned}[J_x^2, J_z] &= J_x^2 J_z - J_z J_x^2 \\ &= J_x^2 J_z - J_x J_z J_x + J_x J_z J_x - J_z J_x^2 \\ &= J_x[J_x, J_z] + [J_x, J_z]J_x \\ &= -i\hbar J_x J_y - i\hbar J_y J_x\end{aligned}$$

$$\begin{aligned}[J_y^2, J_z] &= J_y^2 J_z - J_z J_y^2 \\ &= J_y^2 J_z - J_y J_z J_y + J_y J_z J_y - J_z J_y^2 \\ &= J_y[J_y, J_z] + [J_y, J_z]J_y \\ &= i\hbar J_y J_x - i\hbar J_x J_y\end{aligned}$$

$$[J_z^2, J_z] = 0$$

Adding these results together gives

$$[J^2, J_z^2] = 0$$

(a) Do simultaneous eigenstates of J_x and J_z exist?

No, because J_x and J_z do not commute.

(b) Do simultaneous eigenstates of J^2 and J_z exist?

Yes, because J^2 and J_z commute.

(c) Do simultaneous eigenstates of J^2 and J_y exist?

Yes, because if J^2 commutes with J_z, then by symmetry, it must commute with J_y.

(d) Do simultaneous eigenstates of J^2, J_z, and J_y exist?

No, because J_z and J_y do not commute.

(e) Do simultaneous eigenstates of J^2 and J_x^2 exist?

Yes, simultaneous eigenstates of J^2 and J_x exist because J^2 and J_x commute. Clearly an eigenstate of J_x is also an eigenstate of J_x^2.

(f) Do simultaneous eigenstates of J_z and $J_x^2 + J_y^2$ exist?

Yes, because $J_x^2 + J_y^2 = J^2 - J_z^2$, and J_z commutes with J^2 and J_z^2.

5. [20 pts] With $J_\pm = J_x \pm iJ_y$, express J_+J_- and J_-J_+ in terms of the operators J^2 and J_z, then compute the the following commutators:

$$\begin{aligned} J_+J_- &= (J_x + iJ_y)(J_x - iJ_y) \\ &= J_x^2 + J_y^2 + i[J_y, J_x] \\ &= J^2 - J_z^2 + \hbar J_z \end{aligned}$$

likewise

$$J_-J_+ = J^2 - J_z^2 - \hbar J_z$$

(a) $[J_+, J_-]$

$[J_+, J_-] = J_+J_- - J_-J_+ = 2\hbar J_z$

(b) $[J_\pm, J^2]$

$[J_\pm, J^2] = [J_x, J^2] \pm i[J_y, J^2] = 0$

(c) $[J_\pm, J_z]$

$[J_\pm, J_z] = [J_x, J_z] \pm i[J_y, J_z] = -i\hbar J_y \pm (-\hbar J_x) = -\hbar J_\pm$

(d) $[J_\pm, J_x]$

$[J_\pm, J_x] = \pm i[J_y, J_x] = \pm \hbar J_z$

(e) $[J_\pm, J_y]$

$[J_\pm, J_y] = [J_x, J_y] = i\hbar J_z$

6. [20 pts] Let $|j, m\rangle$ be the standard simultaneous eigenstate of J^2 and J_z. (a) What are $J^2|j, m\rangle$ and $J_z|j, m\rangle$ in terms of j and m? (b) What are the allowed values of j? (c) For a given j-value, what are the allowed values of m? First, re-write your answers to (a), (b), and (c) ten times, then compute the following matrix elements:

(a) $J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$ and $J_z|j, m\rangle = \hbar m|j, m\rangle$
(b) $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$
(c) $m \in \{-j, -j{+}1, \ldots, j\}$

Optional:

$J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$ $J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$ $J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$ $J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$ $J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$
$J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$ $J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$ $J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$ $J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$ $J^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$
$J_z|j, m\rangle = \hbar m|j, m\rangle$ $J_z|j, m\rangle = \hbar m|j, m\rangle$ $J_z|j, m\rangle = \hbar m|j, m\rangle$ $J_z|j, m\rangle = \hbar m|j, m\rangle$ $J_z|j, m\rangle = \hbar m|j, m\rangle$ $J_z|j, m\rangle = \hbar m|j, m\rangle$ $J_z|j, m\rangle = \hbar m|j, m\rangle$
$J_z|j, m\rangle = \hbar m|j, m\rangle$ $J_z|j, m\rangle = \hbar m|j, m\rangle$ $J_z|j, m\rangle = \hbar m|j, m\rangle$ $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$ $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$ $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$ $j \in$
$\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$ $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$ $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$ $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$ $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$ $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$
$j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$ $m \in \{-j, -j{+}1, \ldots, j\}$ $m \in \{-j, -j{+}1, \ldots, j\}$ $m \in \{-j, -j{+}1, \ldots, j\}$ $m \in \{-j, -j{+}1, \ldots, j\}$ $m \in \{-j, -j{+}1, \ldots, j\}$
$m \in \{-j, -j{+}1, \ldots, j\}$ $m \in \{-j, -j{+}1, \ldots, j\}$ $m \in \{-j, -j{+}1, \ldots, j\}$ $m \in \{-j, -j{+}1, \ldots, j\}$ $m \in \{-j, -j{+}1, \ldots, j\}$

(d) $\langle j, m|J^2|j', m'\rangle = \hbar^2 j(j+1)\, \delta_{j,j'}\delta_{m,m'}$

(e) $\langle j, m|J_z|j', m'\rangle = \hbar m\, \delta_{j,j'}\delta_{m,m'}$

(f) $\langle j, m|J_\pm|j'm'\rangle = \hbar\sqrt{j'(j'{+}1) - m'(m'{\pm}1)}\,\delta_{j,j'}\delta_{m,m'\pm 1}$

(g) $\langle j, m|J_x|j'm'\rangle = \frac{1}{2}\langle j, m|\,(J_+ + J_-)\,|j'm'\rangle$
$= \frac{\hbar}{2}\delta_{j,j'}\left(\sqrt{j'(j'{+}1) - m'(m'{+}1)}\,\delta_{m,m'+1} + \sqrt{j'(j'{+}1) - m'(m'{-}1)}\,\delta_{m,m'-1}\right)$

(h) $\langle j, m|J_y|j'm'\rangle = \frac{1}{2i}\langle j, m|\,(J_+ - J_-)\,|j'm'\rangle$
$= \frac{\hbar}{2i}\delta_{j,j'}\left(\sqrt{j'(j'{+}1) - m'(m'{+}1)}\,\delta_{m,m'+1} - \sqrt{j'(j'{+}1) - m'(m'{-}1)}\,\delta_{m,m'-1}\right)$

7. [10 pts] Consider a system described by the Hamiltonian

$$H = \Delta J_z + U(J_x^2 + J_y^2) \tag{1}$$

where J_x, J_y, and J_z are the three components of a generalized angular momentum operator, and Δ and U are constants. What are the energy levels of this system?

With $J_x^2 + J_y^2 = J^2 - J_z^2$, the Hamiltonian becomes

$$H = U(J^2 - J_z^2) + \Delta J_z$$

The eigenstates of this are the standard $|j, m\rangle$ states, with eigenvalues given by

$$H|j, m\rangle = \left(\hbar^2 U(j^2 + j - m^2) + \hbar\Delta m\right)|j, m\rangle$$