PHYS851 Quantum Mechanics I, Fall 2009
HOMEWORK ASSIGNMENT 11

Topics Covered: Orbital angular momentum, center-of-mass coordinates
Some Key Concepts: angular degrees of freedom, spherical harmonics

1. [20 pts] In order to derive the properties of the spherical harmonics, we need to determine the action of the angular momentum operator in spherical coordinates. Just as we have $\langle x|P_x|\psi\rangle = -i\hbar\frac{d}{dx}\langle x|\psi\rangle$, we should find a similar expression for $\langle r\theta\phi|\vec{L}|\psi\rangle$. From $\vec{L} = \vec{R}\times\vec{P}$ and our knowledge of momentum operators, it follows that

$$\langle r\theta\phi|\vec{L}|\psi\rangle = -\imath\hbar\left(\vec{e}_x\left(y\frac{d}{dz} - z\frac{d}{dy}\right) + \vec{e}_y\left(z\frac{d}{dx} - x\frac{d}{dz}\right) + \vec{e}_z\left(x\frac{d}{dy} - y\frac{d}{dx}\right)\right)\langle r\theta\phi|\psi\rangle.$$

Cartesian coordinates are related to spherical coordinates via the transformations

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

and the inverse transformations

$$r = \sqrt{x^2+y^2+z^2}$$

$$\theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\phi = \arctan\left(\frac{y}{x}\right).$$

Their derivatives can be related via expansions such as

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial\theta}{\partial x}\partial_\theta + \frac{\partial\phi}{\partial x}\partial_\phi.$$

Using these relations, and similar expressions for ∂_y and ∂_z, find expressions for $\langle r\theta\phi|L_x|\psi\rangle$, $\langle r\theta\phi|L_y|\psi\rangle$, and $\langle r\theta\phi|L_z|\psi\rangle$, involving only spherical coordinates and their derivatives.

$\partial_x r = \frac{x}{r} = \sin\theta\cos\phi$
$\partial_x\theta = \frac{z^2}{r^2}\frac{x}{z\sqrt{x^2+y^2}} = \frac{\cos\theta\cos\phi}{r}$
$\partial_x\phi = -\frac{x^2}{x^2+y^2}\frac{y}{x^2} = -\frac{\csc\theta\sin\phi}{r}$
So $\frac{d}{dx} = \sin\theta\cos\phi\,\partial_r + \frac{\cos\theta\cos\phi}{r}\partial_\theta - \frac{\csc\theta\sin\phi}{r}\partial_\phi$

$\partial_y r = \frac{y}{r} = \sin\theta\sin\phi$
$\partial_y\theta = \frac{z^2}{r^2}\frac{y}{z\sqrt{x^2+y^2}} = \frac{\cos\theta\sin\phi}{r}$
$\partial_y\phi = \frac{x^2}{x^2+y^2}\frac{1}{x} = \frac{\csc\theta\cos\phi}{r}$
So $\frac{d}{dy} = \sin\theta\sin\phi\partial_r + \frac{\cos\theta\sin\phi}{r}\partial_\theta + \frac{\csc\theta\cos\phi}{r}\partial_\phi$

$\partial_z r = \frac{z}{r} = \cos\theta$

$\partial_z \theta = -\frac{z^2}{r^2} \frac{\sqrt{x^2+y^2}}{z^2} = -\frac{\sin\theta}{r}$
$\partial_z \phi = 0$
So $\frac{d}{dz} = \cos\theta \partial_r - \frac{\sin\theta}{r} \partial_\theta$

Now

$$\langle r\theta\phi|L_x|\psi\rangle = -i\hbar(y\frac{d}{dz} - z\frac{d}{dy})\langle r\theta\phi|\psi\rangle$$

So we can say

$$\begin{aligned} L_x &= i\hbar\left(y\frac{d}{dz} - z\frac{d}{dy}\right) \\ &= -i\hbar\left(r\sin\theta\cos\theta\sin\phi\partial_r - \sin^2\theta\sin\phi\partial_\theta - r\sin\theta\cos\theta\sin\phi\partial_r - \cos^2\theta\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi\right) \end{aligned}$$

Which means

$$\langle r\theta\phi|L_x|\psi\rangle = -i\hbar\left(-\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi\right)\langle r\theta\phi|\psi\rangle$$

Similarly we can say

$$\begin{aligned} L_y &= -i\hbar(z\frac{d}{dx} - x\frac{d}{dz}) \\ &= -i\hbar\left(r\sin\theta\cos\theta\cos\phi\partial_r + \cos^2\theta\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi - r\sin\theta\cos\theta\cos\phi\partial_r + \sin^2\theta\cos\phi\partial_\theta\right) \end{aligned}$$

so that

$$\langle r\theta\phi|L_y|\psi\rangle = -i\hbar\left(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi\right)\langle r\theta\phi|\psi\rangle$$

Lastly, we have

$$\begin{aligned} L_z &= -i\hbar(x\frac{d}{dy} - y\frac{d}{dx}) \\ &= -i\hbar\left(r\sin^2\theta\sin\phi\cos\phi\partial_r + \sin\theta\cos\theta\sin\phi\cos\phi\partial_\theta + \cos^2\phi\partial_\phi\right. \\ &\quad \left. + r\sin^2\theta\sin\phi\cos\phi\partial_r - \sin\theta\cos\theta\sin\phi\cos\phi\partial_\theta + \sin^2\phi\partial_\phi\right) \end{aligned}$$

so that

$$\langle r\theta\phi|L_z|\psi\rangle = -i\hbar\partial_\phi\langle r\theta\phi|\psi\rangle$$

2. [15pts] From your previous answer and the definition $L^2 = L_x^2 + L_y^2 + L_z^2$, prove that

$$\langle r\theta\phi|L^2|\psi\rangle = -\hbar^2 \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi^2} \right) \langle r\theta\phi|\psi\rangle.$$

$$\begin{aligned} L^2 &= -\hbar^2 \left[(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi)(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi) \right. \\ &\quad \left. + (\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi)(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi) + \partial_\phi^2 \right] \\ &= -\hbar^2 \left[\sin^2\phi\partial_\theta^2 + \cot\theta\sin\phi\cos\phi\partial_\theta\partial_\phi - \csc^2\theta\sin\phi\cos\phi\partial_\phi + \cot\theta\sin\phi\cos\phi\partial_\phi\partial_\theta + \cot\theta\cos^2\phi\partial_\theta \right. \\ &\quad + \cot^2\theta\cos^2\phi\partial_\phi^2 - \cot^2\theta\cos\phi\sin\phi\partial_\phi + \cos^2\phi\partial_\theta^2 - \cot\theta\cos\phi\sin\phi\partial_\theta\partial_\phi + \csc^2\theta\sin\phi\cos\phi\partial_\phi \\ &\quad \left. - \cot\theta\sin\phi\cos\phi\partial_\phi\partial_\theta + \cot\theta\sin^2\phi\partial_\theta + \cot^2\theta\sin^2\theta\partial_\phi^2 + \cot^2\theta\sin\phi\cos\phi\partial_\phi + \partial_\phi^2 \right] \\ &= -\hbar^2 \left[\partial_\theta^2 + \cot\theta\partial_\theta + (1 + \cot^2\theta)\partial_\phi^2 \right] \end{aligned}$$

Noting that

$$\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta = \partial_\theta^2 + \cot\theta\partial_\theta$$

and

$$1 + \cot^2\theta = \frac{1}{\sin^2\theta}$$

the proof is complete.

3. [10 pts] We can factorize the Hilbert space of a 3-D particle into radial and angular Hilbert spaces, $\mathcal{H}^{(3)} = \mathcal{H}^{(r)} \otimes \mathcal{H}^{(\Omega)}$. Two alternate basis sets that both span $\mathcal{H}^{(\Omega)}$ are $\{|\theta\phi\rangle\}$ and $\{|\ell m\rangle\}$. As the angular momentum operator lives entirely in $\mathcal{H}^{(\Omega)}$, we can use our results from problem 11.1 to derive an expression for $\langle\theta\phi|L_z|\ell m\rangle$. Combine this with the formula $L_z|\ell m\rangle = \hbar m|\ell m\rangle$, to derive and then solve a differential equation for the ϕ-dependence of $\langle\theta\phi|\ell m\rangle$. Your solution should give $\langle\theta\phi|\ell m\rangle$ in terms of the as of yet unspecified initial condition $\langle\theta|\ell m\rangle \equiv \langle\theta,\phi|\ell m\rangle\Big|_{\phi=0}$. What restrictions does this solution impose on the quantum number m, which describes the z-component of the orbital angular momentum? Since $m_{max} = \ell$, what restrictions are then placed on the total angular momentum quantum number ℓ?

$$\langle\theta\phi|L_z|\ell m\rangle = -i\hbar\frac{\partial}{\partial\phi}\langle\theta\phi|\ell m\rangle$$

and

$$\langle\theta\phi|L_z|\ell m\rangle = \hbar m\langle\theta\phi|\ell m\rangle$$

Thus

$$-i\hbar\frac{\partial}{\partial\phi}\langle\theta\phi|\ell m\rangle = \hbar m\langle\theta\phi|\ell m\rangle$$

The solution to this simple first-order differential equation is

$$\langle\theta\phi|\ell m\rangle = \langle\theta 0|\ell m\rangle e^{im\phi}$$

Since the wave-function must be single valued, we require m to be a whole integer. As $m_{max} = \ell$, this implies that ℓ must be a whole integer also.

4. [10 pts] Using $L_\pm = L_x \pm iL_y$ we can use the relation $L_+|\ell,\ell\rangle = 0$ and the expressions from problem 11.1 to write a differential equation for $\langle\theta\phi|\ell\ell\rangle$. Plug in your solution from 11.3 for the ϕ-dependence, and show that the solution is $\langle\theta\phi|\ell\ell\rangle = c_\ell e^{i\ell\phi}\sin^\ell(\theta)$. Determine the value of the normalization coefficient c_ℓ by performing the necessary integral.

We have $\langle\theta\phi|L_+|\ell\ell\rangle = 0$
This implies $\langle\theta\phi|L_x|\ell\ell\rangle + i\langle\theta\phi|L_y|\ell\ell\rangle = 0$
Using the expressions from 11.1 gives $(-\sin\theta\partial_\theta - \cot\theta\cos\phi\partial_\phi + i\cos\phi\partial_\theta - i\cot\theta\cos\phi\partial_\phi)\langle\theta\phi|\ell\ell\rangle = 0$
This simplifies to $(i(\cos\phi + i\sin\phi)\partial_\theta - \cot\theta(\cos\phi + i\sin\phi)\partial_\phi)\langle\theta\phi|\ell\ell\rangle = 0$
Factoring out the $e^{i\phi}$ gives $(i\partial_\theta - \cot\theta\partial_\phi)\langle\theta\phi|\ell\ell\rangle = 0$
Plugging in the solution from 11.3 gives $(i\partial_\theta - i\ell\cot\theta)\langle\theta 0|\ell\ell\rangle e^{-i\ell\phi} = 0$
which reduces to

$$\partial_\theta\langle\theta 0|\ell\ell\rangle = \ell\cot\theta\langle\theta 0|\ell\ell\rangle$$

Now $\partial_\theta\sin^\ell\theta = \ell\sin^{\ell-1}\theta\cos\theta = \ell\cot\theta\sin^\ell\theta$
So the solution is

$$\langle\theta\phi|\ell\ell\rangle = c_\ell e^{i\ell\phi}\sin^\ell\theta$$

The normalization integral is $\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, |\langle\theta\phi|\ell\ell\rangle|^2 = 1$
With our solution this becomes $|c_\ell|^2 \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, \sin^{2\ell}\theta = 1$
Performing the phi integral gives $2\pi|c_\ell|^2 \int_0^\pi \sin\theta d\theta\, \sin^{2\ell}\theta = 1$
u-substitution with $u = \cos\theta$ gives $2\pi|c_\ell|^2 \int_{-1}^1 du\,(1-u^2)^\ell = 1$
Since the integrand is even, this reduces to $4\pi|c_\ell|^2 \int_0^1 du\,(1-u^2)^\ell = 1$
From Mathematic we get $2\pi|c_\ell|^2 \frac{\Gamma[1/2]\Gamma[\ell+1]}{\Gamma[\ell+3/2]} = 1$
which gives $c_\ell = \sqrt{\frac{\Gamma(\ell+3/2)}{2\pi\Gamma[1/2]\Gamma[\ell+1]}}$

Thus we have

$$\langle\theta\phi|\ell\ell\rangle = \sqrt{\frac{\Gamma(\ell+3/2)}{2\pi\Gamma[1/2]\Gamma[\ell+1]}}\sin^\ell\theta e^{i\ell\phi}$$

For the special case $\ell = 3$ this gives $\langle\theta\phi|33\rangle = \frac{1}{8}e^{3i\phi}\sqrt{\frac{35}{\pi}}\sin^3\theta$, which agrees with the spherical harmonic $Y_3^3(\theta,\phi)$ up to a non-physical phase factor.

5. [10 pts] Using $L_-|\ell m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m-1)}|\ell, m-1\rangle$ together with your previous answers to derive an expression for $\langle\theta\phi|\ell, m-1\rangle$ in terms of $\langle\theta\phi|\ell m\rangle$. Explain how in principle you can now recursively calculate the value of the spherical harmonic $Y_\ell^m(\theta\phi) \equiv \langle\theta\phi|\ell m\rangle$ for any θ and ϕ and for any ℓ and m. Follow your procedure to derive properly normalized expressions for spherical harmonics for the case $\ell = 1$, $m = -1, 0, 1$.

To construct the other m states for the same ℓ, we can begin from the expression

$$\langle\theta\phi|L_-|\ell m\rangle = \hbar\sqrt{(\ell+m)(\ell-m+1)}\langle\theta\phi|\ell, m-1\rangle$$

Now

$$\begin{aligned}
\langle\theta\phi|L_-|\ell m\rangle &= \langle\theta\phi|L_x|\ell m\rangle - i\langle\theta\phi|L_y|\ell m\rangle \\
&= -i\hbar(-\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi)\langle\theta\phi|\ell m\rangle - \hbar(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi)\langle\theta\phi|\ell m\rangle \\
&= \hbar e^{-i\phi}(-\partial_\theta + i\cot\theta\partial_\phi)\langle\theta\phi|\ell m\rangle
\end{aligned}$$

Putting the pieces together gives

$$\langle\theta\phi|\ell, m-1\rangle = \frac{e^{-i\phi}(-\partial_\theta + i\cot\theta\partial_\phi)}{\sqrt{(\ell+m)(\ell-m+1)}}\langle\theta\phi|\ell m\rangle$$

Starting from our expression for $\langle\theta\phi|\ell\ell\rangle$, we can find $\langle\theta\phi|\ell, \ell-1\rangle$ by applying the above differential formula. Successive iterations will then generate all the remaining $\langle\theta\phi|\ell m\rangle$ states.

6. [10 pts] A particle of mass M is constrained to move on a spherical surface of radius a.

Does the system live in $\mathcal{H}^{(3)}$, $\mathcal{H}^{(r)}$, or $\mathcal{H}^{(\Omega)}$? What is the Hamiltonian? What are the energy levels and degeneracies? What are the wavefunctions of the energy eigenstates?

Because the radial motion is constrained to a fixed value, it is only necessary to consider the dynamics in $\mathcal{H}^{(\Omega)}$.

The Hamiltonian is then

$$H = \frac{L^2}{2Ma^2}$$

Choosing simultaneous eigenstates of L^2 and L_z, we have

$$H|\ell, m\rangle = \frac{\hbar^2 \ell(\ell+1)}{2Ma^2}|\ell, m\rangle$$

so that

$$E_\ell = \frac{\hbar^2 \ell(\ell+1)}{2Ma^2}$$

and

$$d_\ell = 2\ell + 1$$

The wavefunctions are the spherical harmonics

$$\langle \theta\phi|\ell, m\rangle = Y_m^\ell(\theta, \phi)$$

7. [10 pts] Two particles of mass M_1 and M_2 are attached to a massless rigid rod of length d. The rod is attached to an axle at its center-of-mass, and is free to rotate without friction in the x-y plane.

Describe the Hilbert space of the system and then write the Hamiltonian. What are the energy levels and degeneracies? What are the wavefunctions of the energy eigenstates?

Only a single angle, ϕ is required to specify the state of the system, where ϕ is the azimuthal angle, thus the Hilbert space is $\mathcal{H}^{(\phi)}$.

The Hamiltonian is then

$$H = \frac{L_z^2}{2I}$$

where

$$I = 2M\left(\frac{d}{2}\right)^2 = \frac{Md^2}{2}$$

is the moment of inertia. This gives

$$H = \frac{L_z^2}{Md^2}$$

The energy levels are then

$$E_m = \frac{\hbar^2 m^2}{Md^2}$$

where $m = 0, \pm 1, \pm 2, \pm 3, \ldots$
The energy levels all have a degeneracy of 2, except for E_0, which is not degenerate.
The wavefunctions are given by

$$\langle \phi | m \rangle = \frac{e^{im\phi}}{\sqrt{2\pi}}$$

8. [10 pts] For a two-particle system, the transformation to relative and center-of-mass coordinates is defined by

$$\vec{R} = \vec{R}_1 - \vec{R}_2$$

$$\vec{R}_{CM} = \frac{m_1\vec{R}_1 + m_2\vec{R}_2}{m_1 + m_2}$$

The corresponding momenta are defined by

$$\vec{P} = \mu \frac{d}{dt}\vec{R}$$

$$\vec{P}_{CM} = M \frac{d}{dt}\vec{R}_{CM}$$

where $\mu = m_1 m_2/M$ is the reduced mass, and $M = m_1 + m_2$ is the total mass. Invert these expressions to write $\vec{R}_1$, $\vec{R}_2$, $\vec{P}_1$, and $\vec{P}_2$ in terms of $\vec{R}$, $\vec{R}_{CM}$, $\vec{P}$, and $\vec{P}_{CM}$.

The solutions are

$$\vec{R}_1 = \vec{R}_{CM} + \frac{m_2}{M}\vec{R}$$

$$\vec{R}_2 = \vec{R}_{CM} - \frac{m_1}{M}\vec{R}$$

Writing $\vec{P}$ and $\vec{P}_{CM}$ in terms of $\vec{P}_1$ and $\vec{P}_2$ gives

$$\vec{P} = \frac{m_2\vec{P}_1 - m_1\vec{P}_2}{m_1 + m_2}$$

$$\vec{P}_{CM} = \vec{P}_1 + \vec{P}_2$$

Inverting this gives

$$\vec{P}_1 = \frac{m_1}{M}\vec{P}_{CM} + \vec{P}$$

$$\vec{P}_2 = \frac{m_2}{M}\vec{P}_{CM} - \vec{P}$$