PHYS851 Quantum Mechanics I, Fall 2009
HOMEWORK ASSIGNMENT 13

Topics Covered: Spin
Please note that the physics of spin-1/2 particles will figure heavily in both the final exam for 851, as well as the QM subject exam.

Spin-1/2: The Hilbert space of a spin-1/2 particle is the tensor product between the infinite dimensional 'motional' Hilbert space $\mathcal{H}^{(r)}$ and a two-dimensional 'spin' Hilbert space, $\mathcal{H}^{(s)}$. The spin Hilbert space is defined by three non-commuting observables, S_x, S_y, and S_z. These operators satisfy angular momentum commutation relations, so that simultaneous eigenstates of $S^2 = S_x^2 + S_y^2 + S_z^2$ and S_z exist. According to the general theory of angular momentum, these states can be designated by two quantum numbers, s, and m_s, where s must be either an integer or half integer, and $m_s \in \{s, s-1, \ldots, -s\}$. The theory of spin says that for a given particle, the value of s is fixed. A spin-1/2 particle has $s = 1/2$, so that $m_s \in \{-1/2, 1/2\}$. Since s never changes, we can label the two eigenstates of S_z as $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$, where $|\uparrow_z\rangle = |s{=}\frac{1}{2}, m_s{=}\frac{1}{2}\rangle$ and $|\downarrow_z\rangle = |s{=}\frac{1}{2}, m_s{=}-\frac{1}{2}\rangle$, so that

$$\begin{aligned} S_z|\uparrow_z\rangle &= \frac{\hbar}{2}|\uparrow_z\rangle & (1)\\ S_z|\downarrow_z\rangle &= -\frac{\hbar}{2}|\downarrow_z\rangle & (2) \end{aligned}$$

and

$$\begin{aligned} S^2|\uparrow_z\rangle &= \frac{3\hbar^2}{4}|\uparrow_z\rangle & (3)\\ S^2|\downarrow_z\rangle &= \frac{3\hbar^2}{4}|\downarrow_z\rangle. & (4) \end{aligned}$$

As eigenstates of an observable, these states must satisfy the orthonormality conditions $\langle\uparrow_z|\uparrow_z\rangle = 1$, $\langle\downarrow_z|\downarrow_z\rangle = 1$, and $\langle\uparrow_z|\downarrow_z\rangle = \langle\downarrow_z|\uparrow_z\rangle = 0$. The two vectors $\{|\uparrow\rangle, |\downarrow\rangle\}$ must therefore form a complete basis that spans $\mathcal{H}^{(s)}$, so that

$$I = |\uparrow_z\rangle\langle\uparrow_z| + |\downarrow_z\rangle\langle\downarrow_z|. \qquad (5)$$

1. In this problem you will derive the 2×2 matrix representations of the three spin observables from first principles:

   (a) In the basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$, the matrix representation of S_z is of course

   $$S_z = \begin{pmatrix} \langle\uparrow_z|S_z|\uparrow_z\rangle & \langle\uparrow_z|S_z|\downarrow_z\rangle \\ \langle\downarrow_z|S_z|\uparrow_z\rangle & \langle\downarrow_z|S_z|\downarrow_z\rangle \end{pmatrix}. \tag{6}$$

   Use Eqs. (1) and (2) to find the four matrix elements of S_z in the basis of its own eigenstates.

   (b) Invert the definitions $S_+ = S_x + iS_y$ and $S_- = S_x - iS_y$, to express S_x and S_y in terms of S_+ and S_-.

   (c) Use the equation

   $$S_\pm|s, m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s\pm1)}|s, m_s\pm1\rangle \tag{7}$$

   to find the matrix elements of S_+ and S_- in the basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle$.

   (d) From your answers to 13.1.b and 13.1.c, derive the matrix representations of S_x and S_y for spin-1/2.

   (e) Explicitly verify that these operators satisfy the angular momentum commutation relations.

   (f) Show explicitly that $S^2 = \hbar^2 s(s+1)I$.

   (g) Based on symmetry, write the 2×2 matrix representations of S_x, S_y, and S_z in the basis of eigenstates of S_y.

2. **Pauli spin matrices:** The Pauli spin matrices, σ_x, σ_y, and σ_z are defined via

   $$\vec{S} = \hbar s\vec{\sigma} \tag{8}$$

   (a) Use this definition and your answers to problem 13.1 to derive the 2×2 matrix representations of the three Pauli matrices in the basis of eigenstates of S_z.

   (b) For each Pauli matrix, find its eigenvalues, and the components of its normalized eigenvectors in the basis of eigenstates of S_z.

   (c) Use your answer to 13.2.b to obtain the eigenvalues of S_x, S_y, and S_z, as well as the components of the corresponding normalized eigenvectors in the basis of eigenstates of S_z.

3. Repeat problems 13.1.(a-d) and 13.2.a for the case of a spin-1 particle.

4. Consider an electron whose position is held fixed, so that it can be described by a simple two-component spinor (i.e. no $\vec{r}$ dependence). Let the initial state of the electron be spin up relative to the z-axis, $|\uparrow_z\rangle$. At time $t = 0$, a uniform magnetic field is applied along the y-axis.
What is the state-vector of the electron at time $t > 0$?

   Hint: Start by writing the Hamiltonian, which should contain only the spin-contribution to the magnetic dipole energy. Then propagate the state using the energy eigenvalue representation of the propagator, $U(t) = \sum_n |\omega_n\rangle e^{-i\omega_n t}\langle\omega_n|$.

5. Consider the most general normalized spin-1/2 state $|\psi\rangle = c_\uparrow|\uparrow_z\rangle + c_\downarrow|\downarrow_z\rangle$.

   (a) Compute $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$, with respect to this state.

   (b) Compute the variances ΔS_x, ΔS_y, and ΔS_z.

   (c) Prove that $\Delta S_x = \frac{\hbar}{2}\left|c_\uparrow^2 - c_\downarrow^2\right|$, $\Delta S_y = \frac{\hbar}{2}\left|c_\uparrow^2 + c_\downarrow^2\right|$, and $\Delta S_z = \hbar|c_\uparrow||c_\downarrow|$.

6. **The Stern-Gerlach effect:** A Stern-Gerlach analyzer (SGA) spatially separates the m_s states, relative to the axis of alignment, of any particle with spin sent through the device.

   (a) Consider an SGA aligned along the z-axis. At the location of the beam center, the magnetic field inside the SGA can be written to good approximation as $\vec{B}(\vec{r}) = B_0(z)\vec{e}_z$, where $B_0(z)$ is a monotonically increasing function of z. The operator for the spin magnetic dipole energy is $V_B = -\vec{\mu}\cdot\vec{B}(\vec{r})$. Due to conservation of energy, the electron must experience a force in the direction of decreasing dipole energy. Show that this force is orthogonal to the Lorentz force if the particle has charge, and that it will deflect the spin up and spin down states in opposite directions.

   (b) A single electron in the $|\uparrow_z\rangle$ state, is directed into SGA1, which is aligned along the x-axis. Determine the probabilities for the electron to exit SGA1 in the $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$ channels.

   (c) The output beam from SGA1 corresponding to the $|\downarrow_x\rangle$ channel is then directed into SGA2, which is aligned along the z-axis. While the output beam from SGA1 corresponding to the $|\uparrow_x\rangle$ channel is directed into SGA3, which is aligned along the unit vector $\frac{1}{\sqrt{2}}\vec{e}_z + \frac{1}{\sqrt{2}}\vec{e}_y$. Determine the probabilities for the electron to exit in each of the four output channels (i.e. two for SGA2 and two for SGA3).

7. Work through problem 9.1 on page 990 in Cohen-Tannoudji, transcribed below:

   Consider a spin 1/2 particle. Call its spin $\vec{S}$, and its orbital angular momentum, $\vec{L}$, and its state vector $|\psi\rangle$. The two functions $\psi_+(\vec{r})$ and $\psi_-(\vec{r})$ are defined by

$$\psi_\pm(\vec{r}) = \langle\vec{r}, \pm|\psi\rangle, \tag{9}$$

   where + indicates spin up relative to the z-axis, and − indicates spin down.
Assume that:

$$\begin{aligned}
\psi_+(\vec{r}) &= R(r)\left[Y_0^0(\theta,\phi) + \frac{1}{\sqrt{3}}Y_1^0(\theta,\phi)\right] &(10)\\
\psi_-(\vec{r}) &= \frac{R(r)}{\sqrt{3}}\left[Y_1^1(\theta,\phi) - Y_1^0(\theta,\phi)\right] &(11)
\end{aligned}$$

   where r, θ, and ϕ are the coordinates of the particle and $R(r)$ is a given function of r.

(a) What condition must $R(r)$ satisfy for $|\psi\rangle$ to be normalized?

(b) S_z is measured with the particle in state $|\psi\rangle$. What results can be found, and with what probabilities? Same question for L_z, then for S_x.

(c) A measurement of L^2, with the particle in state $|\psi\rangle$, yielded zero. What state describes the particle just after this measurement? Same question if the measurement of L^2 had given $2\hbar^2$.