HOMEWORK ASSIGNMENT 1: Due Monday, 9/14/09

PHYS851 Quantum Mechanics I, Fall 2009

1. What is the relationship between $\langle\psi|\phi\rangle$ and $\langle\phi|\psi\rangle$? What is the relationship between the matrix elements of $\hat{M}^\dagger$ and the matrix elements of $\hat{M}$? Assuming that $H^\dagger = H$, what is $\langle n|H^\dagger|m\rangle$ in terms of $\langle m|H|n\rangle$?

2. Use the matrix representation and summation notation to prove that $(AB)^\dagger = B^\dagger A^\dagger$, where A and B are both operators. Use summation notation to expand $\langle\phi|AB|\psi\rangle^\dagger$ in terms of the constituent matrix elements and vector components?

3. Consider the discrete orthonormal basis $\{|m\rangle\}$, $m = 1, 2, 3, \ldots, M$ that spans an M-dimensional Hilbert space, $\mathcal{H}_M$.

   (a) Show that the identity operator, $\hat{I} = \sum_m |m\rangle\langle m|$, satisfies $\hat{I}^2 = \hat{I}$.

   (b) Form a new projector, $\hat{P}$, by removing the state $|3\rangle$, i.e. $\hat{P} = \sum_{m\neq 3} |m\rangle\langle m|$. Does $\hat{P}^2 = \hat{P}$? Is P also the identity operator?

   (c) Compute the trace of $\hat{P}$ and compare it to the trace of $\hat{I}$.

   (d) Based on the previous results, formulate the two necessary and sufficient conditions for an operator to be the identity operator in the space $\mathcal{H}_M$.

   (e) Now consider the continuous basis $\{|x\rangle\}$, whose elements are orthogonal, but delta-normalized, i.e. $\langle x|x'\rangle = \delta(x - x')$. Show that in this case, the projector $\hat{I} = \int dx\, |x\rangle\langle x|$ also satisfies $\hat{I}^2 = \hat{I}$.

4. Start from the equation $\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$. Hit it from the left with $\langle n|$ and insert the projector $\hat{I} = \sum_n |n\rangle\langle n|$ to derive the equation of motion for $c_n = \langle n|\psi(t)\rangle$? (Hint: be careful not to use the same symbol for two different things)

5. Suppose you have a set of d linearly independent vectors; $|e_1\rangle, |e_2\rangle, \ldots, |e_d\rangle$; in a d-dimensional space, but they are not orthonormal. The **Gram-Schmidt procedure** is a systematic algorithm for generating from it an orthonormal basis: $|e_1'\rangle, |e_2'\rangle \ldots, |e_d'\rangle$. The procedure is as follows:

   i Normalize the first vector by dividing by its norm: $|e_1'\rangle = \frac{|e_1\rangle}{||e_1||}$

   ii Find the projection of the second operator along the first and subtract it off: $|e_2''\rangle = |e_2\rangle - |e_1'\rangle\langle e_1'|e_2\rangle$ then normalize the resulting vector: $|e_2'\rangle = \frac{|e_2''\rangle}{||e_2''||}$

   iii Then subtract from $|e_3\rangle$ its projections along $|e_1'\rangle$ and $|e_2'\rangle$ and form $|e_3'\rangle$ by normalizing this vector, and so on for the remaining vectors

   (a) For the case $d = 3$ explicitly verify that the new unit vectors are orthogonal

   (b) Use the Gram-Schmidt procedure to generate an orthonormal basis from the vectors $|e_1\rangle = (1+i)|1\rangle + |2\rangle + i|3\rangle$, $|e_2\rangle = i|1\rangle + 3|2\rangle + |3\rangle$, and $|e_3\rangle = 28|2\rangle$. Please follow the procedure exactly as described above, i.e. first compute $|e_1'\rangle$ using $|e_1\rangle$ and so on.

   (c) Use the Gram-Schmidt procedure to generate an orthonormal basis from the vectors $|f_1\rangle$, $|f_2\rangle$, and $|f_3\rangle$, defined via $\langle x|f_1\rangle = e^{-x^2/2}$, $\langle x|f_2\rangle = xe^{-x^2/2}$, and $\langle x|f_3\rangle = x^2e^{-x^2/2}$. As your answer, give expressions for $\langle x|f_,'\rangle$, $\langle x|f_2'\rangle$, and $\langle x|f_3'\rangle$.

6. Two operators, A and B, can be represented by the matrices $A = \begin{pmatrix} -1 & i \\ 2i & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -i \\ i & 2 \end{pmatrix}$. Use matrix algebra to compute the elements of (a) $C = A + B$, (b) $C = AB$, (c) $C = [A, B] := AB - BA$, (d) $A^\dagger$ and $B^\dagger$, (e) B^{-1}. Verify that $BB^{-1} = 1$. (f) Does A have an inverse?

7. Let the set of states $|k\rangle$ be an alternate basis for the space spanned by the states $|x\rangle$ where x and k are continuous indices on the interval $(-\infty, \infty)$. This implies that $\langle k|k'\rangle = \delta(k - k')$ and $\int_{-\infty}^{\infty} dk\, |k\rangle\langle k| = \hat{I}$. Let the inner product be defined as $\langle x|k\rangle := \frac{1}{\sqrt{2\pi}} e^{ikx}$.

   (a) What is $\langle k|x\rangle$?

   (b) use the relation $\langle x|x'\rangle = \delta(x - x')$ to compute the integral $\int_{-\infty}^{\infty} dk\, e^{ik(x-x')}$.

   (c) Consider the vector $|f\rangle$, defined via $\langle x|f\rangle = \frac{1}{\sqrt[4]{\pi\sigma^2}} e^{-(x-x_0)^2/2\sigma^2}$. What is $||f||$?

   (d) What is the representation of $|f\rangle$ in the k-basis, i.e. $\langle k|f\rangle$? Hint: start with the thing you want, in this case $\langle k|f\rangle$, and insert the projector onto the basis in which $|f\rangle$ *is* known.