PHYS851 Quantum Mechanics I, Fall 2009

HOMEWORK ASSIGNMENT 2:
Postulates of Quantum Mechanics

1. [10 pts] Assume that $A|\phi_n\rangle = a_n|\phi_n\rangle$ but that $\langle\phi_n|\phi_n\rangle \neq 1$. Prove that $|a_n\rangle = c|\phi_n\rangle$ is also an eigenstate of A. What is its eigenvalue? What should c be so that $\langle a_n|a_n\rangle$=1?

   Solution:

$$A|a_n\rangle = Ac|\phi_n\rangle = cA|\phi_n\rangle = ca_n|\phi_n\rangle = a_n(c|\phi_n\rangle) = a_n|a_n\rangle.$$

$$\langle a_n|a_n\rangle = |c|^2\langle\phi_n|\phi_n\rangle.$$

   Setting equal to unity requires

$$c = \frac{1}{\sqrt{\langle\phi_n|\phi_n\rangle}}.$$

2. [10 pts] Assume that $|\phi_n\rangle$ and $|a_n\rangle$ are degenerate eigenstates of A with eigenvalue a_n. They satisfy $\langle\phi_n|\phi_n\rangle = \langle a_n|a_n\rangle = 1$ and $\langle\phi_n|a_n\rangle \neq 1$. Show that the states $|a_n,1\rangle = |a_n\rangle$ and $|a_n,2\rangle = \frac{|\phi_n\rangle - |a_n\rangle\langle a_n|\phi_n\rangle}{|||\phi_n\rangle - |a_n\rangle\langle a_n|\phi_n\rangle||}$ are both eigenstates of A with eigenvalue a_n, and are mutually orthogonal.

   Now assume that the system is in an arbitrary state $|\psi\rangle$ when A is measured. Write an expression for the probability to obtain the result a_n. Write down the state of the system immediately after the measurement, assuming that a_n was obtained by random chance.

   Solution:

$$A|a_n,1\rangle = A|a_n\rangle = a_n|a_n\rangle$$

   Let $\mathcal{N} = |||\phi_n\rangle - |a_n\rangle\langle a_n|\phi_n\rangle||$,

$$A|a_n,2\rangle = A\frac{|\phi_n\rangle - |a_n\rangle\langle a_n|\phi_n\rangle}{\mathcal{N}} = \frac{A|\phi_n\rangle - A|a_n\rangle\langle a_n|\phi_n\rangle}{\mathcal{N}} = \frac{a_n|\phi_n\rangle - a_n|a_n\rangle\langle a_n|\phi_n\rangle}{\mathcal{N}} = a_n\frac{|\phi_n\rangle - |a_n\rangle\langle a_n|\phi_n\rangle}{\mathcal{N}}$$

$$\langle a_n,1|a_n,2\rangle = \langle a_n|\frac{|\phi_n\rangle - |a_n\rangle\langle a_n|\phi_n\rangle}{\mathcal{N}} = \frac{\langle a_n|\phi_n\rangle - \langle a_n|a_n\rangle\langle a_n|\phi_n\rangle}{\mathcal{N}} = \frac{\langle a_n|\phi_n\rangle - \langle a_n|\phi_n\rangle}{\mathcal{N}} = 0.$$

$$P(a_n) = \langle\psi|\Big(|a_n,1\rangle\langle a_n,1| + |a_n,2\rangle\langle a_n,2|\Big)|\psi\rangle = |\langle a_n,1|\psi\rangle|^2 + |\langle a_n,2|\psi\rangle|^2$$

$$|\psi'\rangle = \frac{1}{\sqrt{P(a_n)}}\Big(|a_n,1\rangle\langle a_n,1| + |a_n,2\rangle\langle a_n,2|\Big)|\psi\rangle = \frac{1}{\sqrt{P(a_n)}}\Big(|a_n,1\rangle\langle a_n,1|\psi\rangle + |a_n,2\rangle\langle a_n,2|\psi\rangle\Big)$$

3. [20 pts] Consider the wavefunction $\phi(x) \equiv \langle x|\phi\rangle = \mathcal{N}e^{-\frac{(x-x_0)^2}{2\sigma^2}}$. What should the value of $\mathcal{N}$ be so that $\langle\phi|\phi\rangle = 1$?

   What is the probability to find the system between position x and $x + dx$?
   Calculate the expectation value $\langle X\rangle \equiv \langle\phi|X|\phi\rangle$ and the uncertainty $\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2}$.
   Note: X is the position operator, satisfying $X|x\rangle = x|x\rangle$. Does the interpretation of $\langle X\rangle$ as the average value of x and Δx as the 'spread' around the average make sense for this example?
   Solution:

$$1 = \langle\phi|\phi\rangle = \int dx\, \langle\phi|x\rangle\langle x|\phi\rangle = \int dx\, |\phi(x)|^2 = \mathcal{N}^2 \int dx\, e^{-\frac{(x-x_0)^2}{\sigma^2}} = \mathcal{N}^2\sqrt{\pi\sigma^2}.$$

   Normalization thus requires $\mathcal{N} = \frac{1}{\sqrt[4]{\pi\sigma^2}}$.

   The probability to find the system between x and $x + dx$ is

$$dP(x) = |\langle x|\phi\rangle|^2 dx = \frac{1}{\sqrt{\pi}\sigma}e^{-x^2/\sigma^2}dx.$$

$$\langle X\rangle = \langle\phi|X|\phi\rangle = \int dx\, \langle\phi|X|x\rangle\langle x|\phi\rangle = \int dx\, x\langle\phi|x\rangle\langle x|\phi\rangle = \int_{-\infty}^{\infty} dx\, x|\phi(x)|^2 = \int_{-\infty}^{\infty} dx\, x\frac{e^{-\frac{(x-x_0)^2}{\sigma^2}}}{\sqrt{\pi}\sigma} = x_0.$$

   Similarly

$$\langle X^2\rangle = \frac{1}{\sqrt{\pi}\sigma}\int_{-\infty}^{\infty} dx\, x^2 e^{-\frac{(x-x_0)^2}{\sigma^2}} = x_0^2 + \frac{\sigma^2}{2}.$$

   This gives:

$$\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2} = \sqrt{x_0^2 + \frac{\sigma^2}{2} - x_0^2} = \frac{\sigma}{\sqrt{2}}.$$

   Since the probability density is a Gaussian centered at x_0 with width σ, interpreting $\langle X\rangle$ as the average and ΔX as the 'spread' makes sense.

4. [10 pts] Consider a particle whose wavefunction is that used in the previous problem. The position of the particle is then measured with a resolution of Δ. What is the probability that the particle is found within $\pm\Delta/2$ of position x? What is the state vector of the system immediately after the measurement, assuming that the particle was indeed found within $\pm\Delta/2$ of position x?

   Solution:
   Let $P(x, \Delta)$ be the probability that the system is found within $\pm\Delta$ of x.

$$P(x, \Delta) = \int_{x-\Delta/2}^{x+\Delta/2} dx'\, |\psi(x')|^2 = \int_{x-\Delta/2}^{x+\Delta/2} dx'\, \frac{e^{-x'^2/\sigma^2}}{\sqrt{\pi}\sigma}$$

   This integral expression is sufficient to receive full credit, but if you like you can plug this into mathematica, to get

$$\frac{1}{2}\left[\text{Erf}\left(\frac{\Delta - 2x}{2\sigma}\right) - \text{Erf}\left(\frac{\Delta + 2x}{2\sigma}\right)\right]$$

   The state after the measurement is

$$|\psi'\rangle = \frac{1}{\sqrt{P(x, \Delta)}}\int_{x-\Delta/2}^{x+\Delta/2} dx'\, |x'\rangle\langle x'|\psi\rangle = \frac{1}{\sqrt{P(x, \Delta)}}\int_{x-\Delta/2}^{x+\Delta/2} dx'\, |x'\rangle\frac{e^{-x'^2/2\sigma^2}}{\sqrt[4]{\pi\sigma^2}}.$$

5. Consider a quantum system with three energy eigenstates, which we can call $|1\rangle$, $|2\rangle$, and $|3\rangle$. The Hamiltonian of the system is

$$H = \hbar\left[(4\times 10^9\text{s}^{-1})|1\rangle\langle 1| + (1\times 10^9\text{s}^{-1})|2\rangle\langle 2| + (3\times 10^9\text{s}^{-1})|3\rangle\langle 3|\right].$$

Given that these states satisfy $H|n\rangle = E_n|n\rangle$, what are E_1, E_2, and E_3 in Joules?
Assume that the observable $\mathcal{F}$ is described by the Hermitian operator

$$F = 1|f_1\rangle\langle f_1| + 5|f_2\rangle\langle f_2| + 7|f_3\rangle\langle f_3|,$$

where $|f_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$, $|f_2\rangle = \frac{1}{\sqrt{3}}(|1\rangle - |2\rangle + |3\rangle)$, and $f_3\rangle = \frac{1}{\sqrt{6}}(|1\rangle - |2\rangle - 2|3\rangle)$.
Given that these states satisfy $F|f_n\rangle = f_n|f_n\rangle$, what are f_1, f_2, and f_3?
Do the eigenstates satisfy $\langle f_m|f_n\rangle = \delta_{mn}$?
At time $t = 0$, the system is in state $|\psi(0)\rangle = \frac{1}{\sqrt{6}}(|1\rangle + 2|2\rangle + |3\rangle)$.
If $\mathcal{F}$ is measured at time $t = 0$s, what are the possible results of the measurement?
For each possible result, what is the associated probability?
For each possible outcome, what is the state of the system immediately after the measurement?
Now assume that no measurement is made at $t = 0$, and instead the system evolves freely for $t = \pi$ ns at which time $\mathcal{F}$ is measured.
What are the possible outcomes and associated probabilities in this case?

Answer:
Using $H|n\rangle = E_n|n\rangle$, and taking $\hbar \approx 10^{-34}$J, we find $E_1 = 4\times 10^{-25}$J, $E_2 = 10^{-25}$J, and $E_3 = 3\times 10^{-25}$J.

From $F|f_n\rangle = f_n|f_n\rangle$, we see that $f_1 = 1$, $f_2 = 5$, and $f_3 = 7$.

Using $\langle m|n\rangle = \delta_{m,n}$, which follows from $H^\dagger = H$, we find $\langle f_1|f_1\rangle = \frac{1}{2}(1+1) = 1$, $\langle f_1|f_2\rangle = \frac{1}{\sqrt{6}}(1-1) = 0$, $\langle f_1|f_3\rangle = \frac{1}{sqrt12}(1-1) = 0$, $\langle f_2|f_2\rangle = \frac{1}{3}(1+1+1) = 1$, $\langle f_2|f_3\rangle = \frac{1}{\sqrt{18}}(1+1-2) = 0$, and $\langle f_3|f_3\rangle = \frac{1}{6}(1+1+4) = 1$. Thus we see that indeed $\langle f_m|f_n\rangle = \delta_{m,n}$.

The possible results of a measurement of $\mathcal{F}$, are the eigenvalues of F, i.e. 1, 5, or 7.
The corresponding probabilities are:
$P(1) = |\langle f_1|\psi(0)\rangle|^2 = |\frac{1}{\sqrt{12}}(1+2)|^2 = \frac{3}{4}$
$P(5) = |\langle f_2|\psi(0)\rangle|^2 = |\frac{1}{\sqrt{18}}(1-2+1)|^2 = 0$
$P(7) = |\langle f_3|\psi(0)\rangle|^2 = |\frac{1}{6}(1-2-2)|^2 = \frac{1}{4}$.

The state of the system at time $t = \pi\times 10^{-9}$s is:
$|\psi(t)\rangle = \frac{1}{\sqrt{6}}\left(e^{-iE_1t/\hbar}|1\rangle + e^{-iE_2t/\hbar}2|2\rangle + e^{-iE_3t/\hbar}|3\rangle\right) = \frac{1}{\sqrt{6}}\left(|1\rangle - 2|2\rangle - |3\rangle\right)$
Thus while the possible outcomes are still 1, 5, and 7, the corresponding probabilities are now:
$P(1) = |\langle f_1|\psi(t)\rangle|^2 = |\frac{1}{\sqrt{12}}(1-2)|^2 = \frac{1}{12}$
$P(5) = |\langle f_2|\psi(t)\rangle|^2 = |\frac{1}{\sqrt{18}}(1+2-1)|^2 = \frac{2}{9}$
$P(7) = |\langle f_3|\psi(t)\rangle|^2 = |\frac{1}{6}(1+2+2)|^2 = \frac{25}{36}$
Adding them together gives $P(1) + P(5) + P(7) = \frac{3+8+25}{36} = 1$, as it must.

6. [20 pts] A quantum system can be in one of four possible physical states, $\{|a_1\rangle, |a_2\rangle, |a_3\rangle, |a_4\rangle\}$, which are the eigenstates of the observable A. The Hamiltonian for this system is given in this basis by

$$H = \hbar\omega \sum_{m,n=1}^{4} \sqrt{mn}|a_m\rangle\langle a_n|. \tag{1}$$

The system is prepared initially in the state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle - |a_2\rangle)$. Numerically determine the state of the system $|\psi(t)\rangle$ at time $t = 5/\omega$.

Assume that the observable A is measured at $t = 5/\omega$. What are the probabilities $P(a_n)$ to obtain each possible result? For each possible outcome, give the state of the system immediately following the measurement.

This problem can be solved directly in Mathematica: The possible results are a_1, a_2, a_3, a_4, with

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In[5]:= H = Table[Sqrt[m n], {m, 1, 4}, {n, 1, 4}];

In[6]:= MatrixForm[H]

Out[6]//MatrixForm=
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$$\begin{pmatrix} 1 & \sqrt{2} & \sqrt{3} & 2 \\ \sqrt{2} & 2 & \sqrt{6} & 2\sqrt{2} \\ \sqrt{3} & \sqrt{6} & 3 & 2\sqrt{3} \\ 2 & 2\sqrt{2} & 2\sqrt{3} & 4 \end{pmatrix}$$

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In[7]:= ψ0 = {1 / Sqrt[2], -1 / Sqrt[2], 0, 0};

In[8]:= MatrixForm[ψ0]

Out[8]//MatrixForm=
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$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$$

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In[11]:= ψ = N[MatrixExp[-I H 5].ψ0];

In[12]:= MatrixForm[ψ]

Out[12]//MatrixForm=
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$$\begin{pmatrix} 0.708133 - 0.00768478\, i \\ -0.705656 - 0.0108679\, i \\ 0.00177729 - 0.0133104\, i \\ 0.00205224 - 0.0153696\, i \end{pmatrix}$$

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In[13]:= Pan = Table[Conjugate[ψ[[j]]] ψ[[j]], {j, 1, 4}];

In[14]:= MatrixForm[Pan]

Out[14]//MatrixForm=
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$$\begin{pmatrix} 0.501511 + 0.\, i \\ 0.498068 + 0.\, i \\ 0.000180326 + 0.\, i \\ 0.000240435 + 0.\, i \end{pmatrix}$$

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corresponding probabilities $P(a_1) = 0.501511$,$P(a_2) = 0.498068$, $P(a_3) = 0.000180326$, and $P(a_4) = 0.000240435$. If result a_j is obtained, the system collapses into state $|a_j\rangle$.