PHYS851 Quantum Mechanics I, Fall 2009

HOMEWORK ASSIGNMENT 4

1. **The 2-Level Rabi Model:** The standard Rabi Model consists of a bare Hamiltonian $H_0 = \frac{\Delta}{2}\left(|2\rangle\langle 2| - |1\rangle\langle 1|\right)$ and a coupling term $V = \frac{\Omega^*}{2}|1\rangle\langle 2| + \frac{\Omega}{2}|2\rangle\langle 1|$.

   (a) What is the energy, degeneracy, and state vector of the bare ground state for $\Delta > 0$, $\Delta = 0$, and $\Delta < 0$?

   (b) Let the full Hamiltonian be $H = H_0 + V$. Write down the 2x2 Hamiltonian matrix in the $\{|1\rangle, |2\rangle\}$ basis and then compute the 'dressed-state' energy levels for the case $\Omega \neq 0$. Use ω_g for the lowest eigenvalue, and ω_e for the highest (in energy).

   (c) Following the method shown in lecture (i.e. treating positive and negative detunings separately, and matching the limiting values of the dressed and bare eigenstates in the limits $|\Delta| \to \infty$), determine the normalized dressed-state eigenvectors. Label the state corresponding to ω_g as $|g\rangle$ and the other state as $|e\rangle$. Using Dirac notation, express the Full Hamiltonian as an operator in terms of the kets $|g\rangle$ and $|e\rangle$ and the corresponding bras, and then again using the kets $|1\rangle$ and $|2\rangle$ and the corresponding bras.

   (d) Sketch the energy spectrum versus Ω for the case of fixed $\Delta > 0$. What are ω_g and ω_e at $\Omega = 0$? What are the corresponding dressed states. What are the limiting values of ω_g and ω_e, and their corresponding eigenvectors, in the limits $\Omega \to \infty$ and $\Omega \to -\infty$. What do you expect to be different for the case $\Delta < 0$?

2. **Adiabatic and Sudden Approximations** A 2-level quantum system is prepared initially in the ground-state of H_0 with a large, negative detuning, $\Delta(0) = \Delta_0 < 0$, and the coupling strength is initially zero, $\Omega(0) = 0$.

   In the following, when a state $|\psi(t)\rangle$ is requested, give two expressions for $|\psi(t)\rangle$, one using the $\{|1\rangle, |2\rangle\}$ basis and the other using $\{|g\rangle, |e\rangle\}$, where the later always refers to the instantaneous values of the system parameters at the specified time.

   (a) The coupling strength, $\Omega(t)$, is slowly increased over a duration T_1, to the value $\Omega(T_1) = \Omega_0$, with $|\Omega_0| \ll \Delta_0$, where $T_1 \gg \frac{1}{\Delta}$. What is the mean-energy, defined as $\langle H\rangle$ at time T_1? Give the state vector of the system $|\psi(T_1)\rangle$. Expand your results for the energy and the state to first-order in $\frac{\Omega_0}{\Delta_0}$.

   (b) The detuning is then decreased to zero, over a very short duration T_2, while holding the coupling strength fixed, i.e. $\Omega(T_1 + t) = \Omega_0 \,\forall\, t \in (0, T_2)$. What condition on T_2 sufficient to permit one to use the Sudden Approximation (Hint: it is the opposite of the adiabatic condition)? Assuming that your condition is satisfied, and keeping only the zeroth-order term in your previous expression for $|\psi(T_1)\rangle$, what is $|\psi(T_1 + T_2)\rangle$?

   (c) The parameters are then held fixed for duration $T_3 = \frac{\pi}{|\Omega_0|}$. What is $|\psi(T_1 + T_2 + T_3)\rangle$? What is the mean energy as a function of time during this duration?

   (d) Lastly, the detuning is adiabatically decreased to $-\Delta_0$, over a duration, T_4. Give the adiabaticity condition on T_4, and give the state $|\psi(T_1 + T_2 + T_3 + T_4)\rangle$.

   (e) Now we switch to a completely new system, whose Hamiltonian is also H_0. This system initially has the parameters $\Omega(0) = \Omega_0$, and $\Delta(0) = -\Delta_0$, where $\Delta_0 > 0$, $\Omega_0 > 0$, and $\Delta_0 \gg \Omega_0$. What is the initial state of this system, $|\psi(0)\rangle$? The detuning is then switched from $-\Delta_0$ to Δ_0, over a duration $\tau \ll 1/\Omega_0$. Use either the Sudden or Adiabatic approximation (whichever is appropriate) to determine the state $|\psi(\tau)\rangle$.

   (f) Starting from the same initial state as in part (e), instead the switch from $-\Delta_0$ to Δ_0 is made over duration $\tau \gg 1/\Omega_0$. Use either the Sudden or Adiabatica approximation (whichever is appropriate) and give the state $|\psi(\tau)\rangle$ in this case.

3. **Prototypical Quantum Resonance:** Consider a two-level system described, in the $\{|1\rangle, |2\rangle\}$ basis, by the bare Hamiltonian,

$$H_0 = \begin{pmatrix} -\omega_0/2 & 0 \\ 0 & \omega_0/2 \end{pmatrix}$$

The system is then perturbed by a sinusoidal pertubation,

$$V(t) = \begin{pmatrix} 0 & \Omega\cos(\omega t) \\ \Omega\cos(\omega t) & 0 \end{pmatrix}$$

so that the total Hamiltonian is $H = H_0 + V(t)$.

(a) What is the resonance frequency of H_0?

(b) Derive the equations of motion for $c_1(t) := \langle 1|\psi(t)\rangle$ and $c_2(t) = \langle 2|\psi(t)\rangle$.

(c) Define a new set of variables via $c_1 = C_1 e^{i\omega t/2}$ and $c_2 = C_2 e^{-i\omega t/2}$. Define $\Delta := \omega_0 - \omega$, and re-express the equations of motion in terms of the new variables C_1 and C_2.

(d) Group the constant terms together so that the new equations take the form (Be sure to expand the cosine onto exponentials):

$$\frac{d}{dt}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = -i\mathcal{H}_0 \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \mathcal{V}(t) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

where $\mathcal{H}_0$ is a 2x2 matrix with time-independent coefficients, and $\mathcal{V}$ is a 2x2 matrix with time-varying coefficients.

(e) What is the relation between $\mathcal{H}_0$ and the Rabi model? What is the condition on ω for $\mathcal{H}_0$ to generate Rabi oscillations of maximum amplitude?

(f) Find the eigenvalues of $\mathcal{H}_0$. What is the resonance frequency of a system governed by $\mathcal{H}_0$? Based on this, what is the condition on ω, so that the term $\mathcal{V}(t)$ can be safely ignored? This ignoring is called the 'Rotating Wave Approximation' or RWA for short.