PHYS851 Quantum Mechanics I, Fall 2009

HOMEWORK ASSIGNMENT 4: Solutions

1. **The 2-Level Rabi Model:** The standard Rabi Model consists of a bare Hamiltonian $H_0 = \frac{\Delta}{2}\left(|2\rangle\langle 2| - |1\rangle\langle 1|\right)$ and a coupling term $V = \frac{\Omega^*}{2}|1\rangle\langle 2| + \frac{\Omega}{2}|2\rangle\langle 1|$.

   (a) What is the energy, degeneracy, and state vector of the bare ground state for $\Delta > 0$, $\Delta = 0$, and $\Delta < 0$?

   For $\Delta > 0$, the energy of the ground state is $-\Delta/2$, the degeneracy is 1, and the state vector is $|1\rangle$.
   For $\Delta = 0$, the energy of the ground state is 0, the degeneracy is 2 and the degenerate subspace is $\{|1\rangle, |2\rangle\}$.
   For $\Delta < 0$, the energy of the ground state si $\Delta/2 = -|\Delta|/2$, the degeneracy is 1, and the state vector is $|2\rangle$.

   (b) Let the full Hamiltonian be $H = H_0 + V$. Write down the 2x2 Hamiltonian matrix in the $\{|1\rangle, |2\rangle\}$ basis and then compute the 'dressed-state' energy levels for the case $\Omega \neq 0$. Use ω_g for the lowest eigenvalue, and ω_e for the highest (in energy).

   The matrix representation of H in the $\{|1\rangle, |2\rangle\}$ basis is:

$$H = \begin{pmatrix} -\frac{\Delta}{2} & \frac{\Omega^*}{2} \\ \frac{\Omega}{2} & \frac{\Delta}{2} \end{pmatrix} \tag{1}$$

   The characteristic equation is then:

$$\det|H - \omega I| = -\left(\frac{\Delta}{2} + \omega\right)\left(\frac{\Delta}{2} - \omega\right) - \frac{|\Omega|^2}{4} = \omega^2 - \frac{1}{4}\left(\Delta^2 + |\Omega|^2\right) = 0 \tag{2}$$

   the solutions are then

$$\omega_g = -\frac{1}{2}\sqrt{\Delta^2 + |\Omega|^2} \tag{3}$$

$$\omega_e = \frac{1}{2}\sqrt{\Delta^2 + |\Omega|^2} \tag{4}$$

   (c) Following the method shown in lecture (i.e. treating positive and negative detunings separately, and matching the limiting values of the dressed and bare eigenstates in the limits $|\Delta| \to \infty$), determine the normalized dressed-state eigenvectors. Label the state corresponding to ω_g as $|g\rangle$ and the other state as $|e\rangle$. Using Dirac notation, express the Full Hamiltonian as an operator in terms of the kets $|g\rangle$ and $|e\rangle$ and the corresponding bras, and then again using the kets $|1\rangle$ and $|2\rangle$ and the corresponding bras.

   The eigenvalue equation is $(H - \omega I)|\omega\rangle = 0$.
   Hitting this with $\langle 1|$ and inserting the projector $I = |1\rangle\langle 1| + |2\rangle\langle 2|$, then doing the same for $\langle 2|$, gives

$$(\langle 1|H|1\rangle - \omega)\langle 1|\omega\rangle + \langle 1|H|2\rangle\langle 2|\omega\rangle = 0 \tag{5}$$

$$\langle 2|H|1\rangle\langle 1|\omega\rangle + (\langle 2|H|2\rangle - \omega)\langle 2|\omega\rangle = 0 \tag{6}$$

   putting in the matrix elements and multiplying by 2 gives

$$-(\Delta + 2\omega)\langle 1|\omega\rangle + \Omega^*\langle 2|\omega\rangle = 0 \tag{7}$$

$$\Omega\langle 1|\omega\rangle + (\Delta - 2\omega)\langle 2|\omega\rangle = 0 \tag{8}$$

   The first equation gives, before normalization,

$$|\omega\rangle = \Omega^*|1\rangle + (\Delta + 2\omega)|2\rangle. \tag{9}$$

   The second gives,

$$|\omega\rangle = (\Delta - 2\omega)|1\rangle - \Omega|2\rangle. \tag{10}$$

For positive detuning, $\Delta > 0$, according to our answer for (a), we want $\lim_{\Omega\to 0}\langle 2|g\rangle = 0$ and $\lim_{\Omega\to 0}\langle 1|e\rangle \to 0$, thus we should use (10) for $|\omega_g\rangle$ and (9) for $|\omega_e\rangle$. This gives

$$|g\rangle = \frac{(\Delta + \sqrt{\Delta^2 + |\Omega|^2})|1\rangle - \Omega|2\rangle}{\sqrt{(\Delta + \sqrt{\Delta^2 + |\Omega|^2})^2 + |\Omega|^2}} \tag{11}$$

$$|e\rangle = \frac{\Omega^*|1\rangle + (\Delta + \sqrt{\Delta^2 + |\Omega|^2})|2\rangle}{\sqrt{(\Delta + \sqrt{\Delta^2 + |\Omega|^2})^2 + |\Omega|^2}} \tag{12}$$

For negative detuning, $\Delta < 0$, the limits are reversed, so we need to use (9) for $|g\rangle$ and (10) for $|e\rangle$, and multiply each by -1 (a nonphysical global phase factor), giving

$$|g\rangle = \frac{-\Omega^*|1\rangle + (|\Delta| + \sqrt{\Delta^2 + |\Omega|^2})|2\rangle}{\sqrt{(|\Delta| + \sqrt{\Delta^2 + |\Omega|^2})^2 + |\Omega|^2}} \tag{13}$$

$$|e\rangle = \frac{(|\Delta| + \sqrt{\Delta^2 + |\Omega|^2})|1\rangle + \Omega||2\rangle}{\sqrt{(|\Delta| + \sqrt{\Delta^2 + |\Omega|^2})^2 + |\Omega|^2}} \tag{14}$$

In the $\{|g\rangle, |e\rangle\}$ basis, the Hamiltonian is

$$H = \omega_g|g\rangle\langle g| + \omega_e|e\rangle\langle e| \tag{15}$$

while in the $\{|1\rangle, |2\rangle\}$ basis, this becomes

$$H = \frac{\Delta}{2}\left(|2\rangle\langle 2| - |1\rangle\langle 1|\right) + \frac{\Omega}{2}|2\rangle\langle 1| + \frac{\Omega^*}{2}|1\rangle\langle 2| \tag{16}$$

(d) Sketch the energy spectrum versus Ω for the case of fixed $\Delta > 0$. What are ω_g and ω_e at $\Omega = 0$? What are the corresponding dressed states. What are the limiting values of ω_g and ω_e, and their corresponding eigenvectors, in the limits $\Omega \to \infty$ and $\Omega \to -\infty$. What do you expect to be different for the case $\Delta < 0$?

For $\Delta > 0$ and $\Omega = 0$, we have $\omega_g = -\Delta/2$ and $\omega_e = \Delta/2$.
The corresponding dressed states are $|g\rangle = |1\rangle$ and $|e\rangle = |2\rangle$.
In the limit $|\Omega| \to \infty$, we apply the limit to (11) and (12), giving

$$|g\rangle \to \frac{1}{\sqrt{2}}\left(|1\rangle - \frac{\Omega}{|\Omega|}|2\rangle\right), \qquad |e\rangle \to \frac{1}{\sqrt{2}}\left(\frac{\Omega^*}{|\Omega|}|1\rangle + |2\rangle\right) \tag{17}$$

For $\Omega \to \infty$, we assume that Ω is real and positive, so this simplifies to

$$|g\rangle \to \frac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right), \qquad |e\rangle \to \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) \tag{18}$$

while for the limit $\Omega \to -\infty$, we assume Ω is real and negative, to get

$$|g\rangle \to \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right), \qquad |e\rangle \to \frac{1}{\sqrt{2}}\left(|2\rangle - |1\rangle\right) \tag{19}$$

Based on the symmetry of the Hamiltonian, for real Ω, taking $\Delta \to -\Delta$ is equivalent to swapping $|1\rangle \leftrightarrow |2\rangle$, so that the symmetric (+) states would be the same, while the antisymmetric (-) states would acquire a π phase shift.

2. **Adiabatic and Sudden Approximations** A 2-level quantum system is prepared initially in the ground-state of H_0 with a large, negative detuning, $\Delta(0) = \Delta_0 < 0$, and the coupling strength is initially zero, $\Omega(0) = 0$.

In the following, when a state $|\psi(t)\rangle$ is requested, give two expressions for $|\psi(t)\rangle$, one using the $\{|1\rangle, |2\rangle\}$ basis and the other using $\{|g\rangle, |e\rangle\}$, where the later always refers to the instantaneous values of the system parameters at the specified time.

(a) The coupling strength, $\Omega(t)$, is slowly increased over a duration T_1, to the value $\Omega(T_1) = \Omega_0$, with $|\Omega_0| \ll \Delta_0$, where $T_1 \gg \frac{1}{\Delta}$. What is the mean-energy, defined as $\langle H\rangle$ at time T_1? Give the state vector of the system $|\psi(T_1)\rangle$. Expand your results for the energy and the state to first-order in $\frac{\Omega_0}{\Delta_0}$.

The system begins in the ground state $|\psi(0)\rangle = |g(\Delta_0, 0)\rangle = |2\rangle$, with energy $\omega_g(\Delta_0, 0) = -|\Delta_0|/2$. As omega is increased, the smallest energy gap, $|\Delta_0|$, occurs at $t = 0$, thus $T \gg 1/|\Delta_0|$ is sufficient for adiabatic following.
At time $t = T_1$, the system is still in the ground-state, whose state is now given by

$$|\psi(T_1)\rangle = |g(\Delta_0, \Omega_0)\rangle = \frac{-\Omega_0^*|1\rangle + (|\Delta_0| + \sqrt{\Delta_0^2 + |\Omega_0|^2})|2\rangle}{\sqrt{(|\Delta_0| + \sqrt{\Delta_0^2 + |\Omega_0|^2})^2 + |\Omega_0|^2}} \tag{20}$$

because the system is an an energy eigenstate, its mean energy is just its energy eigenvalue,

$$\omega_g(\Delta_0, \Omega_0) = -\frac{1}{2}\sqrt{\Delta_0^2 + |\Omega_0|^2} \tag{21}$$

Thus the energy of the system as actually decreased during the adiabatic transition.
Expanding the energy and state at $t = T_1$ to first-order gives

$$|\psi(T_1)\rangle \approx |2\rangle - \frac{\Omega^*}{2|\Delta_0|}|1\rangle \tag{22}$$

$$\omega(\Delta_0, \Omega_0) \approx \Delta_0 \tag{23}$$

(b) The detuning is then increased to zero, over a very short duration T_2, while holding the coupling strength fixed, i.e. $\Omega(T_1 + t) = \Omega_0 \, \forall \, t \in (0, T_2)$. What condition on T_2 sufficient to permit one to use the Sudden Approximation (Hint: it is the opposite of the adiabatic condition)? Assuming that your condition is satisfied, and keeping only the zeroth-order term in your previous expression for $|\psi(T_1)\rangle$, what is $|\psi(T_1 + T_2)\rangle$?

As the magnitude of the detuning is decreased to zero, the smallest energy gap is $|\Omega_0|$, encountered at $t = T_1 + T_2$. Thus the validity condition for the sudden approximation is $T \gg 1/|\Omega_0|$.
Taking $|\psi(T_1)\rangle = |2\rangle$, in the sudden approximation, $|\psi(T_1 + T_2)\rangle = |\psi(T_1)\rangle$, which gives

$$|\psi(T_1 + T_2)\rangle = |2\rangle = \frac{1}{\sqrt{2}}\left(|g(0, \Omega_0)\rangle - \frac{\Omega_0}{|\Omega_0|}|e(0, \Omega_0)\rangle\right) \tag{24}$$

(c) The parameters are then held fixed for duration $T_3 = \frac{\pi}{|\Omega_0|}$. What is $|\psi(T_1 + T_2 + T_3)\rangle$? What is the mean energy as a function of time during this duration?

The parameters are held fixed at $\Delta = 0$, $\Omega = \Omega_0$, for an interval $T_3 = \pi/|\Omega_0|$, during which time the system freely evolves. The equations of motion are

$$\dot{c_1} = -i(\Omega_0^*/2)c_2 \tag{25}$$

$$\dot{c_2} = -i(\Omega_0/2)c_1 \tag{26}$$

These can be combined to give

$$\ddot{c_1} = -(|\Omega_0|^2/4)c_1 \tag{27}$$

which has the solution

$$c_1(t_0+t) = c_1(t_0)\cos(|\Omega_0|t/2) + \frac{2\dot{c}_1(t_0)}{|\Omega_0|}\sin(|\Omega_0|t/2) \tag{28}$$

$$= c_1(t_0)\cos(|\Omega_0|t/2) - i\frac{\Omega_0^*}{|\Omega_0|}c_2(t_0)\sin(|\Omega_0|t/2) \tag{29}$$

By symmetry, the solution for $c_2(t)$ is obtained from this by swapping the indices $1 \leftrightarrow 2$.
With $t_0 = T_1 + T_2$, and $t = T_3$, as well as the initial conditions $c_1(t_0) = 0$ and $c_2(t_0) = 1$, we find

$$c_1(T_1+T_2+T_3) = -i\frac{\Omega_0^*}{|\Omega_0|}\sin(|\Omega_0|T_3/2) \tag{30}$$

$$c_2(T_1+T_2+T_3) = \cos(|\Omega_0|T_3/2) \tag{31}$$

setting $T_3 = \pi/|\Omega_0|$ then gives

$$c_1(T_1+T_2+T_3) = -i\frac{\Omega_0^*}{|\Omega_0|} \tag{32}$$

$$c_2(T_1+T_2+T_3) = 0 \tag{33}$$

so that finally, we have

$$|\psi(T_1+T_2+T_3)\rangle = -i\frac{\Omega_0^*}{|\Omega_0|}|1\rangle = -i\frac{1}{\sqrt{2}}\left(\frac{\Omega_0^*}{|\Omega_0|}|g(0,\Omega_0)\rangle + |e(0,\Omega_0)\rangle\right) \tag{34}$$

The system is equally likely to be in the ground or excited states at $t = T_1 + T_2$, as can be deduced from (24). Under free evolution, these probabilities are conserved, so that during the evolution, we have $P_g(t) = 1/2$ and $P_e(t) = 1/2$. The average energy is $\langle H\rangle = \frac{1}{2}\hbar\omega_g + \frac{1}{2}\hbar\omega_e$. Since $\omega_g = -\omega_e$, the average energy is therefore zero throughout the evolution.

(d) Lastly, the detuning is adiabatically increased to $-\Delta_0$, over a duration, T_4. Give the adiabaticity condition on T_4, and give the state $|\psi(T_1+T_2+T_3+T_4)\rangle$.

Note that $-\Delta_0 = |\Delta_0| > 0$. The minimum gap energy for this leg of the trip is $|\Omega_0|$ which occurs at $t = T_1 + T_2 + T_3$. Thus the adiabaticity condition is $T_4 \gg 1/|\Omega_0|$.
Up to here, we haven't considered adiabatic following for a superposition. The adiabatic theorem predicts no transitions between eigenstates, but does allow for a phase to build up during the transition. For a single eigenstate, this is non-physical, but for a superposition, the difference between the two phase factors for the two states is physically observable. Thus we can only say that

$$|\psi(T_1+T_2+T_3+T_4) = \frac{1}{\sqrt{2}}\left(|g(|\Delta_0|,\Omega_0)\rangle + e^{i\phi}|e(|\Delta_0|,\Omega_0)\rangle\right) \tag{35}$$

where $\phi = \int_0^{T_4} dt\ (\omega_e(t) - \omega_g(t))$, is a relative phase which depends on the path taken. If we assume a linear rate of change, then we have $\phi = \int_0^{T_4} dt\ \sqrt{|\Delta_0|^2t^2 + |\Omega_0|^2} = \frac{T_4}{2}\sqrt{|\Omega_0|^2 + |\Delta_0|^2} + \frac{|\Omega_0|^2}{2|\Delta_0|}\arcsin\left(\frac{T_4|\Delta_0|}{|\Omega_0|}\right)$.

(e) Now we switch to a completely new system, whose Hamiltonian is also H_0. This system initially has the parameters $\Omega(0) = \Omega_0$, and $\Delta(0) = -\Delta_0$, where $\Delta_0 > 0$, $\Omega_0 > 0$, and $\Delta_0 \gg \Omega_0$. What is the initial state of this system, $|\psi(0)\rangle$? The detuning is then switched from $-\Delta_0$ to Δ_0, over a duration $\tau \ll 1/\Omega_0$. Use either the Sudden or Adiabatic approximation (whichever is appropriate) to determine the state $|\psi(\tau)\rangle$.

The initial state is not specified, however, as the minimum gap frequency is $|\Omega_0|$, we see that the sudden approximation would be appropriate, so $|\psi(\tau)\rangle = |\psi(0)\rangle$.

(f) Starting from the same initial state as in part (e), instead the switch from $-\Delta_0$ to Δ_0 is made over duration $\tau \gg 1/\Omega_0$. Use either the Sudden or Adiabatic approximation (whichever is appropriate) and give the state $|\psi(\tau)\rangle$ in this case.

In this case, the adiabatic approximation would be appropriate, so that for $|\psi(0)\rangle = |g(-\Delta_0, \Omega_0)\rangle \approx |2\rangle$, we would have $|\psi(\tau)\rangle = |g(\Delta_0, \Omega_0)\rangle \approx |1\rangle$. If the initial state where $|e(-\Delta_0, \Omega_0)\rangle \approx |1\rangle$, then the final state would be $|e(\Delta_0, \Omega_0)\rangle \approx |1\rangle$. This illustrates how that physical state changes when adiabatically traversing an avoided crossing.

3. **Prototypical Quantum Resonance:** Consider a two-level system described, in the $\{|1\rangle, |2\rangle\}$ basis, by the bare Hamiltonian,

$$H_0 = \begin{pmatrix} -\omega_0/2 & 0 \\ 0 & \omega_0/2 \end{pmatrix}$$

The system is then perturbed by a sinusoidal pertubation,

$$V(t) = \begin{pmatrix} 0 & \Omega\cos(\omega t) \\ \Omega\cos(\omega t) & 0 \end{pmatrix}$$

so that the total Hamiltonian is $H = H_0 + V(t)$.

(a) What is the resonance frequency of H_0?

The resonance frequency of H_0 is the difference between the bare energy levels, i.e. ω_0.

(b) Derive the equations of motion for $c_1(t) := \langle 1|\psi(t)\rangle$ and $c_2(t) = \langle 2|\psi(t)\rangle$.

We have $\dot{c}_j = \langle j|\frac{d}{dt}|\psi\rangle = -i\langle j|H|\psi\rangle$, which gives

$$\dot{c}_1 = i\frac{\omega_0}{2}c_1 - i\Omega\cos(\omega t)c_2 \tag{36}$$

$$\dot{c}_2 = -i\frac{\omega_0}{2}c_2 - i\Omega\cos(\omega t)c_1 \tag{37}$$

(c) Define a new set of variables via $c_1 = C_1 e^{i\omega t/2}$ and $c_2 = C_2 e^{-i\omega t/2}$. Define $\Delta := \omega_0 - \omega$, and re-express the equations of motion in terms of the new variables C_1 and C_2.

Differentiation gives $\dot{c_1} = \dot{C}_1 e^{i\omega t/2} + i\omega C_1 e^{i\omega t/2}$ and $\dot{c_2} = \dot{C}_2 e^{-i\omega t/2} - i\omega C_2 e^{-i\omega t/2}$, so that

$$\dot{C}_1 = i\frac{\Delta}{2}C_1 - i\Omega\cos(\omega t)e^{-i\omega t}C_2 \tag{38}$$

$$\dot{C}_2 = -i\frac{\Delta}{2}C_2 - i\Omega\cos(\omega t)e^{i\omega t}C_1 \tag{39}$$

(d) Group the constant terms together so that the new equations take the form (Be sure to expand the cosine onto exponentials):

$$\frac{d}{dt}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = -i\mathcal{H}_0\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \mathcal{V}(t)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

where $\mathcal{H}_0$ is a 2x2 matrix with time-independent coefficients, and $\mathcal{V}$ is a 2x2 matrix with time-varying coefficients.

with $\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$, we have

$$\frac{d}{dt}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = -i\begin{pmatrix} -\frac{\Delta}{2} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & \frac{\Delta}{2} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} - i\begin{pmatrix} 0 & \frac{\Omega}{2}e^{-i2\omega t} \\ \frac{\Omega}{2}e^{i2\omega t} & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \tag{40}$$

(e) What is the relation between $\mathcal{H}_0$ and the Rabi model? What is the condition on ω for $\mathcal{H}_0$ to generate Rabi oscillations of maximum amplitude?

H_0 is the exact Hamiltonian of the Rabi model. Maximum Rabi oscillations would occur at $\Delta = 0$, therefore the resonance condition is $\omega_0 = \omega$, which makes sense, as the drive frequency in $V(t)$ matches the resonance frequency of H_0.

(f) Find the eigenvalues of $\mathcal{H}_0$. What is the resonance frequency of a system governed by $\mathcal{H}_0$? Based on this, what is the condition on ω, so that the term $\mathcal{V}(t)$ can be safely ignored? This ignoring is called the 'Rotating Wave Approximation' or RWA for short.

The eigenvalues of $\mathcal{H}_0$ are $\pm\frac{1}{2}\sqrt{\Delta^2+\Omega^2}$. The term $\mathcal{V}(t)$ drives the new system $\mathcal{H}_0$ at frequencies $\pm 2\omega$. The condition to be far from resonance is therefore $\omega \gg \frac{1}{4}\sqrt{\Delta^2+\Omega^2}$.