PHYS851 Quantum Mechanics I, Fall 2009
HOMEWORK ASSIGNMENT 6

1. [10 points] The quantum state of a free-particle of mass, M, at time t is a wave-packet of the form

$$\psi(x,t) = \frac{1}{\sqrt{\Gamma(5/4)\sigma_0}} e^{-\frac{(x-x_0)^4}{2\sigma_0^4} + ip_0x/\hbar},$$

We can safely predict that the width of the wave packet will grow in time. Clearly, the spreading velocity v_s must be determined by the initial conditions x_0, p_0, and σ_0, as well as the mass, M, and Planck's constant, $\hbar$, as there are no other parameters around to use.

(a) [2 pts] If there are no external forces acting on the particle, which parameters can we rule out based on symmetry arguments ?

(b) [2 pts] Of the remaining parameters, how many unique ways are there to combine them to make an object with units of a velocity?

(c) [2 pts] Based on this result alone, give a units-based estimate for the velocity at which the wave-packet should spread.

(d) [2 pts] Again, by considering units alone, what energy scale, E_s, would you associate with a wave-packet of width σ_0?

(e) [2 pts] We can assign a temperature to the wave-packet by setting $E_s = k_B T$. Solve this equation for σ_0 as a function of temperature, T. This is known as the thermal de Broglie wavelength, or the thermal coherence length, usually denoted as λ_{dB}. It gives the length-scale on which a particle at temperature T exhibits spatial coherence (quantum superposition).

2. [10 points] The mass of a small virus is about 10^{-21}kg. What is the thermal coherence length of the virus at room temperature? What is it a liquid Helium temperature?

Imagine a nano-scale double-slit apparatus consisting of a metal plate with two slits, each 1nm accross, with a center-to-center separation of 4nm. What coherence length is required in order to observe a double-slit interference pattern when passing many copies of the virus through the double-slit apparatus? Explain your reasoning.

Assume that this coherence length can be obtained by velocity filtering. Based on what we have learned about wave-packet spreading, and assuming a longitudinal velocity of v_L, how far away from the apparatus would we need to put our detector in order to observe the interference fringes? Explain your reasoning

3. [30 pts] Derive the equations of motion for $x(t) := \langle\psi(t)|X|\psi(t)\rangle$ and $p(t) := \langle\psi(t)|P|\psi(t)\rangle$, for a particle of mass M, in

(a) free space, $V(X) = V_0$,

(b) a uniform gravitational field, $V(X) = MgX$, where $g = 9.8\,\mathrm{m\,s^{-1}}$,

(c) a harmonic oscillator potential $V(X) = \frac{1}{2}M\omega^2X^2$.

For each case, solve the equations, and give $x(t)$ and $p(t)$ in terms of the initial values $x(0)$ and $p(0)$.

4. [30 pts] For each of the three potentials in the above problem, derive a closed set of equations for

$$\begin{aligned} p^2(t) &:= \langle\psi(t)|P^2|\psi(t)\rangle, \\ xp(t) &:= \mathrm{Re}\langle\psi(t)|XP|\psi(t)\rangle = \frac{1}{2}\left[\langle\psi(t)|XP|\psi(t)\rangle + \langle\psi(t)|PX|\psi(t)\rangle\right], \\ x^2(t) &:= \langle\psi(t)|X^2|\psi(t)\rangle. \end{aligned}$$

For the initial conditions, $\langle X\rangle = 0$, $\langle P\rangle = 0$, $\langle X^2\rangle = \lambda_0^2/2$, $\langle XP\rangle = 0$ and $\langle P^2\rangle = \hbar^2/2\lambda_0^2$, solve the equations for the potentials in parts (a) and (b). (Hint: for (b) you will need to use your insert your results from the previous problem) For these initial conditions, we can interpret $x^2(t)$ as the square of the width of a wave-packet.

(a) How does the free-space result compare to our previous result for the spreading of a Gaussian wave-packet?

(b) Does the addition of a gravitational potential increase or decrease the spreading rate?

(c) For potential (c), find the condition on λ_0 such that no spreading occurs. (You don't need to solve the equations, look for the condition so that all of the time-derivatives are zero)

(d) **Extra credit:** Solve the equations for the potential in part (c), for the same initial conditions.

5. [5 pts] Consider the potential $V(X) = aX + bX^2 + cX^3$. Starting from the equation for $\langle X\rangle$, show that it is only possible to obtain a closed set of equations for the case $c = 0$.

6. [15 pts]For an incoming wave of energy E, find the wavefunction for all $x > 0$, for the potential

$$V(X) = \begin{cases} 0 & x > L \\ V_0 > 0 & 0 < x < L \\ \infty & x < 0 \end{cases}$$