PHYS851 Quantum Mechanics I, Fall 2009
HOMEWORK ASSIGNMENT 7

Topics Covered: 1D scattering problems with delta- and/or step-functions, transfer matrix approach to multi-boundary 1D scattering problems, finding bound-states for combinations of delta- and/or step-functions in 1D.

Some Key Concepts: Wave-vector, probability current, continuity equation, reflection and transmission amplitudes/probabilities, quantum tunneling, quantum reflection, continuity conditions, transfer matrix.

1. **The continuity equation:** The probability that a particle of mass m lies on the interval $[a, b]$ at time t is
$$P(t|a,b) = \int_a^b dx\, |\psi(x,t)|^2 \tag{1}$$
Differentiate (1) and use the definition of the probability current, $j = -\frac{i\hbar}{2m}\left(\psi^* \frac{d}{dx}\psi - \psi \frac{d}{dx}\psi^*\right)$, to show that
$$\frac{d}{dt}P(t|a,b) = j(a,t) - j(b,t). \tag{2}$$
Next, take the limit as $b-a \to 0$ of both (1) and (2), and combine the results to derive the continuity equation: $\frac{d}{dx}j(x,t) = -\frac{d}{dt}\rho(x,t)$.

2. **Bound-states of a delta-well:** The inverted delta-potential is given by
$$V(x) = -g\,\delta(x), \tag{3}$$
where $g > 0$. For a particle of mass m, this potential supports a single bound-state for $E = E_b < 0$.
   (a) Based on dimensional analysis, estimate the energy, E_b, using the only available parameters, $\hbar$, m, and g.
   (b) Assume a solution of the form:
$$\psi_b(x) = c\, e^{-\frac{|x|}{\lambda}}, \tag{4}$$
and use the delta-function boundary conditions at $x = 0$ to determine λ, as well as the the energy, E_b. You can then use normalization to determine c. What is $\langle X^2 \rangle$ for this bound-state?

3. **Inverted delta scattering:** Consider a particle of mass m, subject to the inverted delta-potential, $V(x) = -g\,\delta(x)$, with $g > 0$. Only this time, consider an incoming particle with energy $E > 0$. What are the transmission and reflection probabilities, T, and R?

4. **Combination of delta and step:** Consider a particle of mass m, whose potential energy is

$$V(x) = V_0 u(x) + g\delta(x), \tag{5}$$

where $u(x)$ is the unit step function and $V_0 > 0$.

(a) What are the two boundary conditions at $x = 0$ that $\psi(x)$ must satisfy?

(b) For an incident wave of the form e^{ikx}, use the 'plug and chug' approach to find the reflection and transmission amplitudes, r and t.

(c) Compute the reflection probability, R, and the transmission probability, T. What is the relationship between T and $|t|^2$?

(d) Lastly, compute the transfer matrix for this potential at the discontinuity point, $x = 0$.

(e) Compare your answer to the matrices

$$M_{\delta,step} = M_{step}(K,k)M_\delta(ka), \tag{6}$$

and

$$M_{step,\delta} = M_\delta(Ka)M_{step}(K,k), \tag{7}$$

where $K = \sqrt{k^2 - \frac{2mV_0}{\hbar^2}}$ and $a = \hbar^2/(Mg)$. Comment on your result.

5. **Delta function Fabry Perot Resonator:** Consider transmission of particles of mass m through two delta-function barriers, described by the potential

$$V(x) = g\delta(x) + g\delta(x - L), \tag{8}$$

where $g > 0$ and $L > 0$.

(a) First, compute the allowed k-values for an infinite square well of length L, where $k = \sqrt{2mE}/\hbar$.

(b) Next, use the transfer-matrix approach to compute the full transfer matrix of the resonator.

(c) Use the full transfer-matrix to compute the transmission probability, T, in terms of the dimensionless parameters $\theta = 2kL$ and $\Delta = 1/ka$, where $a = \hbar^2/(mg)$.

(d) Make plots of T versus θ for $\Delta = 1$, $\Delta = 2$, and $\Delta = 4$. Compare the location of the transmission resonances on each plot to the locations of the allowed k-values from part (a).

6. Consider a particle of mass m incident on a square potential barrier of height $V_0 > 0$, and width W. Consider the case where the incident energy, E, is smaller than V_0.

(a) Compute the probability to tunnel through the barrier, T, as function of the incident wave-vector, k.

(b) Write out the full form of the wavefunction of the particle in the tunneling region.

(c) Take limit as $W \to 0$ and $V_0 \to \infty$, while holding $V_0 W$ constant, and show that your answer agrees with the result for a delta-function potential, $V(x) = g\delta(x)$, with $g = V_0 W$.