PHYS851 Quantum Mechanics I, Fall 2009
HOMEWORK ASSIGNMENT 8

Topics Covered: Algebraic approach to the quantized harmonic oscillator, coherent states.

Some Key Concepts: Oscillator length, creation and annihilation operators, the phonon number operator.

1. Start from the harmonic oscillator Hamiltonian $H = \frac{1}{2M}P^2 + \frac{1}{2}M\omega^2 X^2$. Make the change of variables $X \to \lambda\bar{X}$, $P \to \frac{\hbar}{\lambda}\bar{P}$, and $H \to \frac{\hbar^2}{M\lambda^2}\bar{H}$. Find the value of λ for which $\bar{H} = \frac{1}{2}\left(\bar{X}^2 + \bar{P}^2\right)$.

2. Write down the harmonic oscillator Hamiltonian in terms of ω, A, and $A^\dagger$, and then write the commutation relation between A and $A^\dagger$. Use these to derive the equation of motion for the expectation value $a(t) = \langle\psi(t)|A|\psi(t)\rangle$.
   Solve this equation for the general case $a(0) = a_0$.
   Prove that $a^*(t) := \langle A^\dagger\rangle = [a(t)]^*$.

3. Starting from $\langle x|X|n{-}1\rangle = x\phi_{n-1}(x)$, express X in terms of A and $A^\dagger$, to derive a recursion relation of the form:
$$\phi_n(x) = f_n(x)\phi_{n-1}(x) + g_n(x)\phi_{n-2}. \tag{1}$$
   Starting from $\phi_0(x) = \left[\sqrt{\pi}\lambda\right]^{-1/2} e^{-\frac{1}{2}(x/\lambda)^2}$, use your recursion relation to compute $\phi_2(x)$, $\phi_3(x)$, and $\phi_4(x)$.

4. Make the definition $\phi_n(p) = i^n\langle p|n\rangle$. Start from $\langle p|P|n{-}1\rangle = p\langle p|n{-}1\rangle$ and derive a recursion relation for $\phi_n(p)$ by making an analogy to your result from the last problem.

5. Consider a particle in the potential
$$V(x) = \begin{cases} \frac{1}{2}M\omega^2x^2; & x > 0 \\ \infty; & x < 0 \end{cases}. \tag{2}$$
   What boundary condition must the eigenstates satisfy at $x = 0$?
   To find the eigenstates and eigenvalues, consider that the wave-function must also satisfy the harmonic oscillator wave equation for $x > 0$, as well be normalizable ($\lim_{x\to\infty}\psi(x) = 0$). Can you think of any states that you already know of that satisfy all three conditions?

6. Consider the potential $V(x) = a + bX + cX^2$. Let $H = \frac{1}{2M}P^2 + V(X)$, so that the energy eigenstates and eigenvalues are defined via $H|E_n\rangle = E_n|E_n\rangle$. Make a change of variables to complete the square and map the problem onto the harmonic oscillator problem and then determine the allowed energies, $\{E_n\}$ and corresponding eigenfunctions $\psi_n(x) := \langle x|E_n\rangle$.

7. Consider the potential

$$V(x) = \begin{cases} 0; & 0 < x < W < L \\ V_0 > 0; & W < x < L \\ \infty; & \text{otherwise} \end{cases} . \tag{3}$$

There are important boundary conditions at $x = 0$, $x = W$, and $x = L$, what are they? Assume that $E < V_0$, and make an ansatz for each of the two regions, which automatically satisfies the boundary conditions at $x = 0$ and $x = L$.

Show that the two boundary conditions at $x = W$ can only be satisfied for certain values of E, and give a transcendental equation whose solutions yield the allowed energies.

8. A three level system is described by the Hamiltonian $H = \hbar\Omega(t)\left(|1\rangle\langle 3| - |3\rangle\langle 1|\right) + \hbar\Delta(t)|2\rangle\langle 2|$.

Determine the eigenvalues and eigenvectors of H.

At time $t = 0$, the system is prepared in state $|1\rangle$, with $\Delta(0) = 0$ and $\Omega(0) = 0$. Then Ω is suddenly increased to a value of Ω_0, and held for a duration of T. What is the state of the system at time $t = T$?

At time T, the operator $J = j_0\left(i|1\rangle\langle 2| - i|2\rangle\langle 1| + 3|3\rangle\langle 3|\right)$ is measured. What are the possible outcomes of the measurement and the associated probabilities?

For each possible outcome, what is the state immediately after the measurement?