PHYS851 Quantum Mechanics I, Fall 2009
HOMEWORK ASSIGNMENT 8: SOLUTIONS

Topics Covered: Algebraic approach to the quantized harmonic oscillator, coherent states.

Some Key Concepts: Oscillator length, creation and annihilation operators, the phonon number operator.

1. Start from the harmonic oscillator Hamiltonian $H = \frac{1}{2M}P^2 + \frac{1}{2}M\omega^2 X^2$. Make the change of variables $X \to \lambda\bar{X}$, $P \to \frac{\hbar}{\lambda}\bar{P}$, and $H \to \frac{\hbar^2}{M\lambda^2}\bar{H}$. Find the value of λ for which $\bar{H} = \frac{1}{2}\left(\bar{X}^2 + \bar{P}^2\right)$.

   We find
$$\frac{\hbar^2}{M\lambda^2}\bar{H} = \frac{\hbar^2}{2M\lambda^2}\bar{P}^2 + \frac{1}{2}M\omega^2\lambda^2\bar{X}^2 \tag{1}$$
   For all of the constants to cancel requires
$$M\omega^2\lambda^2 = \frac{\hbar^2}{M\lambda^2} \tag{2}$$
   solving for λ gives
$$\lambda = \sqrt{\frac{\hbar}{M\lambda}} \tag{3}$$

2. Write down the harmonic oscillator Hamiltonian in terms of ω, A, and $A^\dagger$, and then write the commutation relation between A and $A^\dagger$. Use these to derive the equation of motion for the expectation value $a(t) = \langle\psi(t)|A|\psi(t)\rangle$.
   Solve this equation for the general case $a(0) = a_0$.
   Prove that $a^*(t) := \langle A^\dagger\rangle = [a(t)]^*$.

   The Hamiltonian is
$$H = \hbar\omega(A^\dagger A + 1/2) \tag{4}$$
   and the commutator is
$$\left[A, A^\dagger\right] = 1 \tag{5}$$
   The equation of motion for the expectation value is
$$\begin{aligned} \frac{d}{dt}\langle A\rangle &= -\frac{i}{\hbar}\langle[A,H]\rangle \\ &= -i\omega\langle A\rangle \end{aligned} \tag{6}$$
   With $a(t) = \langle A\rangle$, the solution is
$$a(t) = a(0)e^{-i\omega t} \tag{7}$$
   For $A^\dagger$, we find
$$a^*(t) = \langle A^\dagger\rangle = \langle\psi(t)|A^\dagger|\psi(t)\rangle = \langle\psi(t)|A|\psi(t)\rangle^\dagger = \langle A\rangle^* = [a(t)]^* \tag{8}$$
   The point here is that we can say
$$a^*(t) = [a(0)]^* e^{i\omega t} \tag{9}$$
   without deriving a separate equation of motion for $A^\dagger$.

3. Starting from $\langle x|X|n{-}1\rangle = x\phi_{n-1}(x)$, express X in terms of A and $A^\dagger$, to derive a recursion relation of the form:

$$\phi_n(x) = f_n(x)\phi_{n-1}(x) + g_n(x)\phi_{n-2}. \tag{10}$$

Starting from $\phi_0(x) = [\sqrt{\pi}\lambda]^{-1/2}\, e^{-\frac{1}{2}(x/\lambda)^2}$, use your recursion relation to compute $\phi_2(x)$, $\phi_3(x)$, and $\phi_4(x)$.

$$\begin{aligned} x\phi_{n-1}(x) &= \langle x|X|n-1\rangle \\ &= \frac{\lambda}{\sqrt{2}}\left(\langle x|A|n{-}1\rangle + \langle x|A^\dagger|n{-}1\rangle\right) \\ &= \frac{\lambda}{\sqrt{2}}\left(\sqrt{n{-}1}\langle x|n-2\rangle + \sqrt{n}\langle x|n\rangle\right) \\ &= \frac{\lambda\sqrt{n{-}1}}{\sqrt{2}}\phi_{n-2}(x) + \frac{\lambda\sqrt{n}}{\sqrt{2}}\phi_n(x) \end{aligned} \tag{11}$$

solving for $\phi_n(x)$ gives

$$\phi_n(x) = \sqrt{\frac{2}{n}}\frac{x}{\lambda}\phi_{n-1}(x) - \sqrt{\frac{n{-}1}{n}}\phi_{n-2}(x) \tag{12}$$

with $n = 1, 2, 3, 4$ this gives

$$\begin{aligned} \phi_1(x) &= \sqrt{2}\frac{x}{\lambda}\phi_0(x) \\ &= [\sqrt{\pi}2\lambda]^{-1/2}\, 2\left(\frac{x}{\lambda}\right) e^{-\frac{1}{2}(x/\lambda)^2} \end{aligned} \tag{13}$$

$$\begin{aligned} \psi_2(x) &= \frac{x}{\lambda}\psi_1(x) - \sqrt{\frac{1}{2}}\psi_0(x) \\ &= \frac{\sqrt{2}}{[\sqrt{\pi}\lambda]^{1/2}}\frac{x^2}{\lambda^2}e^{-\frac{x^2}{2\lambda^2}} - \frac{1}{[2\sqrt{\pi}\lambda]^{1/2}}e^{-\frac{x^2}{2\lambda^2}} \\ &= \frac{1}{[2^2 2!\sqrt{\pi}\lambda]^{1/2}}\left(4\frac{x^2}{\lambda^2} - 2\right) e^{-\frac{x^2}{2\lambda^2}} \end{aligned}$$

$$\begin{aligned} \psi_3(x) &= \sqrt{\frac{2}{3}}\frac{x}{\lambda}\psi_2(x) - \sqrt{\frac{2}{3}}\psi_1(x) \\ &= \sqrt{\frac{2}{24\sqrt{\pi}\lambda}}\left(4\frac{x^2}{\lambda^2} - 2\right)\frac{x}{\lambda}e^{-\frac{x^2}{2\lambda^2}} - \sqrt{\frac{4}{3\sqrt{\pi}\lambda}}\frac{x}{\lambda}e^{-\frac{x^2}{2\lambda^2}} \\ &= \frac{1}{[2^3 3!\sqrt{\pi}\lambda]^{1/2}}\left(8\frac{x^3}{\lambda^3} - 12\frac{x}{\lambda}\right) e^{-\frac{x^2}{2\lambda^2}} \end{aligned}$$

$$\begin{aligned} \psi_4(x) &= \sqrt{\frac{2}{4}}\frac{x}{\lambda}\psi_3(x) - \sqrt{\frac{3}{4}}\psi_2(x) \\ &= \sqrt{\frac{1}{96\sqrt{\pi}\lambda}}\left(8\frac{x^4}{\lambda^4} - 12\frac{x^2}{\lambda^2}\right) e^{-\frac{x^2}{2\lambda^2}} - \sqrt{\frac{3}{32\sqrt{\pi}\lambda}}\left(4\frac{x^2}{\lambda^2} - 2\right) e^{-\frac{x^2}{2\lambda^2}} \\ &= \frac{1}{\sqrt{2^4 4!\sqrt{\pi}\lambda}}\left(16\frac{x^4}{\lambda^4} - 48\frac{x^2}{\lambda^2} + 12\right) e^{-\frac{x^2}{2\lambda^2}} \end{aligned}$$

4. Make the definition $\phi_n(p) = i^n\langle p|n\rangle$. Start from $\langle p|P|n{-}1\rangle = p\langle p|n{-}1\rangle$ and derive a recursion relation for $\phi_n(p)$ by making an analogy to your result from the last problem.

$$\begin{aligned}(-i)^{n-1}p\phi_{n-1}(P) &= \langle p|P|n{-}1\rangle \\ &= -i\frac{\hbar}{\sqrt{2}\lambda}\left(\langle p|A|n{-}1\rangle - \langle p|A^\dagger|n{-}1\rangle\right) \\ &= -i\frac{\hbar}{\sqrt{2}\lambda}\sqrt{n{-}1}(-i)^{n-2}\psi_{n-2}(p) + i\frac{\hbar}{\sqrt{2}\lambda}\sqrt{n}(-i)^n\psi_n(p)\end{aligned} \tag{14}$$

Solving for $\phi_n(p)$ gives

$$\phi_n(p) = \sqrt{\frac{2}{n}}\left(\frac{\lambda p}{\hbar}\right)\phi_{n-1}(p) - \sqrt{\frac{n-1}{n}}\phi_{n-2}(p) \tag{15}$$

This equation is exactly the same as the one for x, but with $\frac{x}{\lambda} \to \frac{\lambda p}{\hbar}$. Thus we must have

$$\begin{aligned}\langle p|n\rangle &= (-i)^n\phi_n(p) \\ &= (-i)^n\left[2^n n!\sqrt{\pi}\hbar/\lambda\right]^{-1/2} H_n\left(\frac{\lambda}{\hbar}p\right)e^{-\frac{1}{2}(\lambda p/\hbar)^2}\end{aligned} \tag{16}$$

5. Consider a particle in the potential

$$V(x) = \begin{cases} \frac{1}{2}M\omega^2x^2; & x > 0 \\ \infty; & x < 0 \end{cases}. \tag{17}$$

What boundary condition must the eigenstates satisfy at $x = 0$?
To find the eigenstates and eigenvalues, consider that the wave-function must also satisfy the harmonic oscillator wave equation for $x > 0$, as well be normalizable ($\lim_{x\to\infty}\psi(x) = 0$). Can you think of any states that you already know of that satisfy all three conditions?

The boundary condition at $x = 0$ is $\psi(0) = 0$, and of course $\psi(x) = 0$ for $x < 0$.
Because we require the states to satisfy the harmonic oscillator wave equation, that doesn't leave us much to choose from, other than the harmonic oscillator states,

$$\phi_n(x) = \left[\sqrt{\pi}2^n n!\lambda\right]^{-1/2} H_n(x/\lambda)e^{-\frac{1}{2}(x/\lambda)^2} \tag{18}$$

As we learned in lecture, in the case of an even potential, simultaneous eigenstates of energy and parity must exist. If there is no degeneracy, then the energy eigenstates will automatically be parity eigenstates. From problem 8.3, we see that $\phi_n(x)$ is even for n even, and odd for n odd. Since all odd functions must vanish at $x = 0$, we see that the eigenstates we are looking for are the odd n, harmonic oscillator eigenstates. Labeling them from 1 to ∞ in order of increasing energy, we let $n \to 2m-1$, and introduce the states $|m\rangle$, with $m = 1, 2, 3, \ldots$, where $E_n \to E_{2m-1}$ gives

$$E_m = \hbar\omega(2m - 1/2) \tag{19}$$

and $\phi_n(x) \to \sqrt{2}\phi_{2m-1}(x)$ gives

$$\phi_m(x) = \left[\sqrt{\pi}4^{m-1}(2m-1)!\lambda\right]^{-1/2}\phi_{2m-1}(x/\lambda)e^{-\frac{1}{2}(x/\lambda)^2} \tag{20}$$

Note that the extra factor $\sqrt{2}$ is because the normalization integral is now from $x = 0$ to $x = \infty$. The probability density must be doubled to compensate for the interval being halved.

6. Consider the potential $V(x) = a + bX + cX^2$. Let $H = \frac{1}{2M}P^2 + V(X)$, so that the energy eigenstates and eigenvalues are defined via $H|E_n\rangle = E_n|E_n\rangle$. Make a change of variables to complete the square and map the problem onto the harmonic oscillator problem and then determine the allowed energies, $\{E_n\}$ and corresponding eigenfunctions $\psi_n(x) := \langle x|E_n\rangle$.

We can rearrange $V(x)$ as

$$\begin{aligned} V(x) &= c\left[X^2 + \frac{b}{c}X\right] + a \\ &= c\left[X^2 + 2\frac{b}{2c}X + \frac{b^2}{4c^2}\right] + a - \frac{b^2}{4c} \\ &= c\left(X + \frac{b}{2c}\right)^2 + a - \frac{b^2}{4c} \end{aligned} \tag{21}$$

With $X' = X + \frac{b}{2c}$, and $P' = P$, we have $[X', P'] = [X, P] = i\hbar$, and

$$H = \frac{P'^2}{2M} + cX'^2 + a - \frac{b^2}{4c} \tag{22}$$

Thus by analogy with the Harmonic oscillator, we have

$$E_n = \hbar\omega(n + 1/2) + a - \frac{b^2}{4c} \tag{23}$$

with $c = \frac{1}{2}M\omega^2$, or

$$\omega = \sqrt{\frac{2c}{M}} \tag{24}$$

The eigenfunctions are

$$\phi_n(x') = \left[\sqrt{\pi}s^n n!\lambda\right]^{-1/2} H_n(x/\lambda)e^{-\frac{1}{2}(x'/\lambda)^2} \tag{25}$$

with

$$\lambda = \sqrt{\frac{\hbar}{M\omega}} = \left(\frac{\hbar^2}{2Mc}\right)^{1/4} \tag{26}$$

with $x' = x + \frac{b}{2c}$ this gives

$$\phi_n(x) = \left[\sqrt{\pi}s^n n!\lambda\right]^{-1/2} H_n\left(\frac{x}{\lambda} + \frac{b}{2c\lambda}\right) e^{-\frac{1}{2}(\frac{x}{\lambda} + \frac{b}{2c\lambda})^2} \tag{27}$$

7. Consider the potential

$$V(x) = \begin{cases} 0; & 0 < x < W < L \\ V_0 > 0; & W < x < L \\ \infty; & \text{otherwise} \end{cases} . \tag{28}$$

There are important boundary conditions at $x = 0$, $x = W$, and $x = L$, what are they? Assume that $E < V_0$, and make an ansatz for each of the two regions, which automatically satisfies the boundary conditions at $x = 0$ and $x = L$.

Show that the two boundary conditions at $x = W$ can only be satisfied for certain values of E, and give a transcendental equation whose solutions yield the allowed energies.

Let region 1 cover the interval $(0, W)$ and region 2 cover the interval (W, L).
The boundary condition at $x = 0$ are then

$$\psi_1(0) = 0 \tag{29}$$

The boundary conditions at $x = W$ are

$$\psi_1(W) = \psi_2(W) \tag{30}$$
$$\psi_1'(W) = \psi_2'(W) \tag{31}$$

and the boundary condition at $x = L$ is

$$\psi_2(L) = 0 \tag{32}$$

For $E < V_0$, we can use the ansatz

$$\psi_1(x) = A\sin(kx) \tag{33}$$
$$\psi_2(x) = B\sinh(\gamma(L - x)) \tag{34}$$

where $k = \sqrt{2ME}/\hbar$ and $\gamma = \sqrt{2M(V_0 - E)}/\hbar$. The nice thing about this ansatz is that it automatically satisfies the boundary conditions at $x = 0$ and $x = L$, so that we can find A and B from the boundary at $x = W$. Putting our ansatz into the Eqs. (30) and (31) gives

$$A\sin(kW) = B\sinh(\gamma(L - W)) \tag{35}$$
$$Ak\cos(kW) = -B\gamma\cosh(\gamma(L - W)) \tag{36}$$

solving (35) for B gives $B = A\sin(kW)/\sinh(\gamma(L - W))$. Putting this into (36) gives

$$Ak\cos(kW) = -A\gamma\tanh(\gamma(L - W))\sin(kW) \tag{37}$$

Thus we see that this equation has a solution only when

$$\cot(kW) = -\frac{\gamma}{k}\tanh(\gamma(L - W)) \tag{38}$$

in which case A will be set by normalization. As $k = k(E)$ and $\gamma = \gamma(E)$, the equation depends only on E,

$$\cot(\sqrt{2ME}W/\hbar) = -\sqrt{\frac{V_0}{E} - 1}\tanh(\sqrt{2M(V_0 - E)}(L - W)/\hbar) \tag{39}$$

Finding all values of E on the interval $0 < E < V_0$ will generate a list of the energy eigenstates with energies below V_0.

8. A three level system is described by the Hamiltonian $H = \hbar\Omega(t)\left(|1\rangle\langle 3| - |3\rangle\langle 1|\right) + \hbar\Delta(t)|2\rangle\langle 2|$.

   Determine the eigenvalues and eigenvectors of H.

   At time $t = 0$, the system is prepared in state $|1\rangle$, with $\Delta(0) = 0$ and $\Omega(0) = 0$. Then Ω is suddenly increased to a value of Ω_0, and held for a duration of T. What is the state of the system at time $t = T$?

   At time T, the operator $J = j_0\left(i|1\rangle\langle 2| - i|2\rangle\langle 1| + 3|3\rangle\langle 3|\right)$ is measured. What are the possible outcomes of the measurement and the associated probabilities?

   For each possible outcome, what is the state immediately after the measurement?

   In order for H to be Hermitian, we see that Ω must be purely imaginary. Writing $\Omega(t) = i\Omega_I(t)$ then gives

$$H = \hbar\Omega_I(t) i\left(|1\rangle\langle 3| - |3\rangle\langle 1|\right) + \hbar\Delta(t)|2\rangle\langle 2| \tag{40}$$

   We see that the Hilbert space defined by $\{|1\rangle, |2\rangle, |3\rangle\}$ separates into two uncoupled subspaces $\{|1\rangle, |3\rangle\}$ and $\{|2\rangle\}$. In the $\{|1\rangle, |3\rangle\}$ subspace we have

$$H = -\hbar\Omega_I(t)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{41}$$

   and in the $\{|2\rangle\}$ subspace we have

$$H = \hbar\Delta(t) \tag{42}$$

   The eigenvalues of the system are therefore $E_\pm = \pm i\hbar\Omega_I(t)$ and $E_0 = \hbar\Delta(t)$. For the case $\Omega_I > 0$, the corresponding eigenvectors are

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + i|3\rangle\right) \tag{43}$$

$$|0\rangle = |2\rangle \tag{44}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - i|3\rangle\right) \tag{45}$$

   We see that the initial state is

$$|1\rangle = \frac{1}{\sqrt{2}}\left(|-\rangle + |+\rangle\right) \tag{46}$$

   Thus at time T, the system is in state

$$|\psi(T)\rangle = \frac{1}{\sqrt{2}}\left(|-\rangle e^{-\Omega_0 T} + |+\rangle e^{\Omega_0 T}\right) \tag{47}$$

   with Ω_0 purely imaginary. We can write this in the original basis as

$$|\psi(T)\rangle = \cosh(\Omega_0 T)|1\rangle + i\sinh(\Omega_0 T)|3\rangle \tag{48}$$

   Generally speaking, the possible outcomes of a measurement of J are the eigenvalues of J. From inspection, we see that the possible outcomes are $-|j_0|$, $+|j_0|$, and $3j_0$. The corresponding eigenvectors of J are given by

$$
\begin{aligned}
|-j_0\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle + i|2\rangle\right) && (49)\\
|j_0\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle - i|2\rangle\right) && (50)\\
|3j_0\rangle &= |3\rangle && (51)
\end{aligned}
$$

for the state $|\psi(T)\rangle$, the corresponding probabilities are

$$
\begin{aligned}
P(-j_0) &= |\langle -j_0|\psi(T)\rangle|^2 \\
&= \frac{1}{2}\cosh(\Omega_0^* T)\cosh(\Omega_0 T) \\
&= \frac{1}{2}\cosh(-\Omega_0 T)\cosh(\Omega_0 T) \\
&= \frac{1}{2}\cosh(\Omega_0 T) \\
&= \frac{1}{2}\cos^2(|\Omega_0|T) && (52)
\end{aligned}
$$

$$
\begin{aligned}
P(j_0) &= |\langle j_0|\psi(T)\rangle|^2 \\
&= \frac{1}{2}\cos^2(|\Omega_0|T) && (53)
\end{aligned}
$$

and from conservation of probability, we must have

$$P(3j_0) = \sin^2(|\Omega_0|T) \tag{54}$$

If eigenvalue $-j_0$ is obtained, we have

$$|\psi(T^+)\rangle = |-j_0\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + i|2\rangle\right) \tag{55}$$

If eigenvalue j_0 is obtained, we have

$$|\psi(T^+)\rangle = |j_0\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - i|2\rangle\right) \tag{56}$$

Note that this leads to the well-known, but somewhat paradoxical result that before the measurement, the probability of the system to be in state $|2\rangle$ is zero, but after the measurement of J, it becomes non-zero.
Lastly, if the result $3j_0$ is obtained, we have

$$|\psi(T^+)\rangle = |3\rangle \tag{57}$$