PHYS851 Quantum Mechanics I, Fall 2009

HOMEWORK ASSIGNMENT 9: SOLUTIONS

1. **The Parity Operator:** [20 pts] Determine the matrix element $\langle x|\Pi|x'\rangle$ and use it to simplify the identity $\Pi = \int dx\, dx'\, |x\rangle\langle x|\Pi|x'\rangle\langle x'|$, then use this identity to compute Π^2, Π^3, and Π^n.

   From these results find an expression for $S(u) = \frac{\exp[\Pi u]}{\cosh u}$ in the form $f(u) + g(u)\Pi$.

   What is $\langle x|S(u)|\psi\rangle$? Express your answer in terms of $\psi_{even}(x) = \frac{1}{2}(\psi(x) + \psi(-x))$ and $\psi_{odd}(x) = \frac{1}{2}(\psi(x) - \psi(-x))$.

   Compute $\langle x|S(0)|\psi\rangle$, $\lim_{u\to\infty}\langle x|S(u)|\psi\rangle$, and $\lim_{u\to-\infty}\langle x|S(u)|\psi\rangle$.

   Answer:

$$\langle x|\Pi|x'\rangle = \langle x| - x'\rangle = \delta(x + x')$$

$$\Pi = \int dxdx'\, |x\rangle\delta(x + x')\langle x'| = \int dx|x\rangle\langle -x|$$

$$\Pi^2 = \int dxdx'\, |x\rangle\langle -x|x'\rangle\langle -x'| = \int dxdx'|x\rangle\delta(-x - x')\langle -x'| = \int dx|x\rangle\langle x| = 1$$

So $\Pi^3 = \Pi^2 \cdot \Pi = \Pi$, which generalizes to

$$\Pi^n = \begin{cases} 1; & n = \text{even} \\ \Pi; & n = \text{odd} \end{cases}$$

Now we have

$$\begin{aligned} S(u) &= \frac{e^{\Pi u}}{\cosh u} = \frac{I + \Pi u + \Pi^2\frac{u^2}{2} + \Pi^3\frac{u^3}{3!} + \dots}{\cosh u} \\ &= I\frac{1 + \frac{u^2}{2} + \frac{u^4}{4!} + \dots}{\cosh u} + \Pi\frac{u + \frac{u^2}{3!} + \frac{u^5}{5!} + \dots}{\cosh u} \\ &= \frac{\cosh u + \Pi\sinh u}{\cosh u} \\ &= I + \Pi\tanh u \end{aligned}$$

$$\begin{aligned} \langle x|S(u)|\psi\rangle &= \langle x|\psi\rangle + \tanh u\langle -x|\psi\rangle \\ &= \psi(x) + \tanh u\psi(-x) \\ &= \psi_{even}(x) + \psi_{odd}(x) + \tanh u\,(\psi_{even}(-x) + \psi_{odd}(-x)) \\ &= (1 + \tanh u)\psi_{even}(x) + (1 - \tanh u)\psi_{odd}(x) \end{aligned}$$

So that

$$\langle x|S(0)|\psi\rangle = \psi_{even}(x) + \psi_{odd}(x) = \psi(x)$$

$$\lim_{u\to\infty}\langle x|S(u)|\psi\rangle = 2\psi_{even}(x)$$

$$\lim_{u\to-\infty}\langle x|S(u)|\psi\rangle = 2\psi_{odd}(x)$$

2. [15 pts]The coherent state $|\alpha\rangle$ is defined by $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$, where the states $\{|n\rangle\}$ are the harmonic oscillator energy eigenstates.

   First, show that for $\alpha = 0$, the coherent state $|\alpha{=}0\rangle$ is exactly equal to the harmonic oscillator ground-state, $|n{=}0\rangle$.

   Then show that any other coherent state can be created by acting on the ground-state, $|0\rangle$, with the 'displacement operator' $D(\alpha)$, i.e. show that $|\alpha\rangle = D(\alpha)|0\rangle$, where

$$D(\alpha) := e^{\alpha A^\dagger - \alpha^* A} \tag{1}$$

   You may need the Zassenhaus formula $e^{B+C} = e^B e^C e^{-[B,C]/2}$, which is valid only when $[B,[B,C]] = [C,[B,C]] = 0$.

   What is $D(\alpha_2)|\alpha_1\rangle$?

   Answer:

   Part i) For $\alpha = 0$, we have

$$|\alpha\rangle\Big|_{\alpha=0} = e^{-0} \sum_{n=0}^{\infty} \frac{0^n}{\sqrt{n!}} |n\rangle = \sum_{n=0}^{\infty} \frac{\delta_{n,0}}{\sqrt{n!}} |n\rangle = |n\rangle\Big|_{n=0}$$

   Part ii) Let $B = \alpha A^\dagger$ and $C = -\alpha^* A$. Then $[B,C] = -|\alpha|^2 [A^\dagger, A] = |\alpha|^2$, which commutes with everything. We can therefore use the Zassenhaus formula, which gives

$$e^{\alpha A^\dagger - \alpha^* A} = e^{-|\alpha|^2/2} e^{\alpha A^\dagger} e^{-\alpha^* A}$$

   Now we have

$$e^{-\alpha^* A}|0\rangle = \sum_{n=0}^{\infty} \frac{(-\alpha^*)^n}{n!} A^n |0\rangle = \sum_{n=0}^{\infty} \frac{(-\alpha^*)^n}{n!} \delta_{n,0} |0\rangle = |0\rangle$$

   So we end up with

$$D(\alpha)|0\rangle = e^{-|\alpha|^2/2} e^{\alpha A^\dagger} |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (A^\dagger)^n |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = |\alpha\rangle$$

   Part iii) Using the Zassenhaus formula, we have

$$D(\alpha_1)D(\alpha_2) = e^{\alpha_1 A^\dagger - \alpha_1^* A + \alpha_2 A^\dagger - \alpha_2^* A} e^{\left[\alpha_1 A^\dagger - \alpha_1^* A, \alpha_2 A^\dagger - \alpha_2^* A\right]/2}$$

   now

$$\left[\alpha_1 A^\dagger - \alpha_1^* A, \alpha_2 A^\dagger - \alpha_2^* A\right] = -\alpha_1 \alpha_2^* [A^\dagger, A] - \alpha_1^* \alpha_2 [A, A^\dagger] = \alpha_1 \alpha_2^* - \alpha_2 \alpha_1^*$$

   and

$$e^{\alpha_1 A^\dagger - \alpha_1^* A + \alpha_2 A^\dagger - \alpha_2^* A} = D(\alpha_1 + \alpha_2)$$

   This gives

$$D(\alpha_1)D(\alpha_2) = e^{-\frac{(\alpha_1^* \alpha_2 - \alpha_2^* \alpha_1)}{2}} D(\alpha_1 + \alpha_2)$$

3. [15 pts] Consider a system described by the Hamiltonian $H = \hbar\kappa(A + A^\dagger)$. Use your results from the previous problem to determine $|\psi(t)\rangle$ for a system initially in the ground-state, $|\psi(0)\rangle = |0\rangle$.

   We know that $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$, so that

$$|\psi(t)\rangle = e^{(-i\kappa A^\dagger - i\kappa A)t}|0\rangle$$

   with $\alpha(t) = -i\kappa t$, we have $-\alpha^*(t) = -(i\kappa t) = -i\kappa t$, so we have

$$|\psi(t)\rangle = e^{\alpha A^\dagger - \alpha^* A}|0\rangle = |\alpha(t)\rangle$$

4. [10pts each] Cohen Tannoudji, pp341-350: problems 3.6, 3.7, 3.11

3.6 For these problems, the primary task is to set up the integral which gives the desired probability:

a.
$$N^2 \int_{-\infty}^{\infty} dxdydz\, e^{-|x|/a-|y|/b-|z|/c} = 1$$
$$\int_{-\infty}^{\infty} dx e^{-|x|/a} = 2\int_{0}^{\infty} dx e^{-x/a} = 2a$$

So that from symmetry we get
$$N^2 8abc = 1$$
which gives $N = 1/\sqrt{8abc}$.

b.
$$\begin{aligned} \mathcal{P} &= \int_0^a dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \frac{e^{-|x|/a-|y|/b-|z|/c}}{8abc} \\ &= \frac{1}{2a}\int_0^a dx\, e^{-|x|/a} \\ &= \frac{1}{2}\int_0^1 du\, e^{-u} \\ &= \frac{e-1}{2e} \end{aligned} \tag{2}$$

c. Based on the previous result and symmetry, we have
$$\mathcal{P} = \frac{(e-1)^2}{e^2}$$

d. The requested quantity is
$$\mathcal{P} = |\langle p_x = 0, p_y = 0, p_z = \hbar/c|\psi\rangle|^2 dp_x dp_y dp_z$$

$$\begin{aligned} \langle 0,0,\hbar/c|\psi\rangle &= \int dxdydz\, \langle 0,0,\hbar/c|xyz\rangle\langle xyz|\psi\rangle \\ &= \frac{1}{(2\pi\hbar)^{3/2}}\int dxdydz\, e^{-iz/c}\frac{e^{-|x|/2a-|y|/2b-|z|/2c}}{\sqrt{8abc}} \\ &= \frac{1}{(2\pi\hbar)^{3/2}\sqrt{8abc}}\int_{-\infty}^{\infty} dx\, e^{-|x|/2a}\int_{-\infty}^{\infty} dy e^{-|y|/2b}\int_{-\infty}^{\infty} dz\, e^{-|z|/2c-iz/c} \\ &= \frac{\sqrt{8}}{(2\pi\hbar)^{3/2}\sqrt{abc}}\int_0^{\infty} dx\, e^{-x/2a}\int_0^{\infty} dy e^{-y/2b}\int_0^{\infty} dz\, e^{-z/2c}\cos(z/c) \\ &= \frac{\sqrt{8}}{(2\pi\hbar)^{3/2}\sqrt{abc}}\frac{8abc}{5} \\ &= \frac{8\sqrt{8abc}}{5(2\pi\hbar)^{3/2}} \end{aligned} \tag{3}$$

So we have
$$\mathcal{P} = \frac{512abc}{25(2\pi\hbar)^3}dp_x dp_y dp_z = \frac{64abc}{25(\pi\hbar)^3}dp_x dp_y dp_z$$

3.7 a.

$$\mathcal{P} = \int_{x_1}^{x_2} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz |\psi(x,y,z)|^2$$

b.

$$\mathcal{P} = \int_{p_1}^{p_2} dp_x \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz |\psi(p_x,y,z)|^2$$

where $\psi(p_x,y,z) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx e^{-ip_x x/\hbar} \psi(x,y,z)$.

c.

$$\mathcal{P} = \int_{x_1}^{x_2} dx \int_{0}^{\infty} dy \int_{0}^{\infty} dp_z\, |\psi(x,y,p_z)|^2$$

where $\psi(x,y,p_z) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dz\, e^{-ip_z z/\hbar} \psi(x,y,z)$

d.

$$\mathcal{P} = \int_{p_1}^{p_2} dp_x \int_{p_3}^{p_4} dp_y \int_{p_5}^{p_6} dp_z\, |\psi(p_x,p_y,p_z)|^2$$

where $\psi(p_x,p_y,p_z) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, e^{-i(p_x x + p_y y + p_z z)/\hbar} \psi(x,y,z)$

If we extend the p_y and p_z limits to infinity we get

$$\begin{aligned} \mathcal{P} &= \int_{p_1}^{p_2} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z\, \langle\psi|p_x,p_y,p_z\rangle\langle p_x,p_y,p_z|\psi\rangle \\ &= \int_{p_1}^{p_2} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z\, \langle\psi|\Big(|p_x\rangle\langle p_x| \otimes |p_y,p_z\rangle\langle p_y,p_z|\Big)|\psi\rangle \end{aligned}$$

Now the identity operator can be written as

$$\begin{aligned} I &= I_x \otimes I_y \otimes I_z \\ &= I_x \otimes \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z\, |p_y,p_z\rangle\langle p_y,p_z| \\ &= I_x \otimes \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, |y,z\rangle\langle y,z| \end{aligned}$$

This shows that

$$\int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z\, |p_y,p_z\rangle\langle p_y,p_z| = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, |y,z\rangle\langle y,z|$$

which clearly makes sense.
Substituting this into the expression for $\mathcal{P}$ gives

$$\mathcal{P} = \int_{p_1}^{p_2} dp_x \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \langle\psi|p_x,y,z\rangle\langle p_x,y,z|\psi\rangle$$

which agrees with the answer to b.

e. **Method: Treat as probability problem.**
Standard probability theory tells us that if $u = f(x, y, z)$ then the probability density $p(u)$ is given by

$$\rho(u) = \int d^3r \, \rho(u|\vec{r})\rho(\vec{r})$$

where $\rho(u|\vec{r})$ is the probability density over u for fixed $\vec{r}$, and QM tells us that $\rho(\vec{r}) = |\psi(\vec{r})|^2$. Now we clearly must have $\rho(u|\vec{r}) = a\delta(u - f(\vec{r}))$, where the normalization constant is determined by requiring that

$$1 = \int_{-\infty}^{\infty} du \, \rho(u|\vec{r}) = a \int_{-\infty}^{\infty} du \, \delta(u - f(\vec{r})) = a$$

so that $a = 1$. This gives

$$\rho(u) = \int d^3r \; |\psi(\vec{r})|^2 \, \delta(u - f(\vec{r}))$$

and then

$$\mathcal{P} = \int_{u_1}^{u_2} du \, \rho(u) = \int_{u_1}^{u_2} du \int d^3r \, |\psi(\vec{r})|^2 \delta(u - f(\vec{r}))$$

3.11 a.

$$\mathcal{P} = dx_1 \int_\alpha^\beta dx_2 \, |\psi(x_1, x_2)|^2$$

b.

$$\mathcal{P} = dx_1 \int_{-\infty}^\infty dx_2 \, |\psi(x_1, x_2)|^2$$

c.

$$\mathcal{P} = \int_\alpha^\beta dx_1 \int_{-\infty}^\infty dx_2 \, |\psi(x_1, x_2)|^2 + \int_{-\infty}^\infty dx_1 \int_\alpha^\beta dx_2 |\psi(x_1, x_2)|^2 - \int_\alpha^\beta dx_1 \int_\alpha^\beta dx_2 |\psi(x_1, x_2)|^2$$

d.

$$\begin{aligned}\mathcal{P} \;&=\; \int_\alpha^\beta dx_1 \int_{-\infty}^\alpha dx_2 \, |\psi(x_1, x_2)|^2 + \int_\alpha^\beta dx_1 \int_\beta^\infty dx_2 \, |\psi(x_1, x_2)|^2 + \int_{-\infty}^\alpha dx_1 \int_\alpha^\beta dx_2 |\psi(x_1, x_2)|^2 \\ &\quad + \int_\beta^\infty dx_1 \int_\alpha^\beta dx_2 |\psi(x_1, x_2)|^2\end{aligned}$$

or equivalently

$$\mathcal{P} = \int_\alpha^\beta dx_1 \int_{-\infty}^\infty dx_2 \, |\psi(x_1, x_2)|^2 + \int_{-\infty}^\infty dx_1 \int_\alpha^\beta dx_2 |\psi(x_1, x_2)|^2 - 2 \int_\alpha^\beta dx_1 \int_\alpha^\beta dx_2 |\psi(x_1, x_2)|^2$$

e.

$$\mathcal{P} = \int_{p'}^{p''} dp_1 \int_\alpha^\beta dx_2 |\psi(p_1, x_2)|^2$$

where $\psi(p_1, x_2) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^\infty dx_1 \, e^{-ip_1 x_1/\hbar} \psi(x_1, x_2)$

f.

$$\mathcal{P} = \int_{p'}^{p''} dp_1 \int_{p'''}^{p''''} dp_2 \, |\psi(p_1, p_2)|^2$$

where $\psi(p_1, p_2) = \frac{1}{2\pi\hbar} \int_{-\infty}^\infty dx_1 \int_{-\infty}^\infty dx_2 \, e^{-i(p_1 x_1 + p_2 x_2)/\hbar} \psi(x_1, x_2)$

g. From the results of e., we find

$$\mathcal{P} = \int_{p'}^{p''} dp_1 \int_{-\infty}^\infty dx_2 |\psi(p_1, x_2)|^2$$

from the results of f., we find

$$\mathcal{P} = \int_{p'}^{p''} dp_1 \int_{-\infty}^\infty dp_2 \, |\psi(p_1, p_2)|^2$$

this shows that

$$\int_{-\infty}^\infty dx_2 \, |\psi(p_1, x_2)|^2 = \int_{-\infty}^\infty dp_2 \, |\psi(p_1, p_2)|^2$$

which follows because they are both equal to

$$\langle\psi| \Big(|p_1\rangle\langle p_1| \otimes I_2 \Big) |\psi\rangle$$

h.

$$\begin{aligned}\mathcal{P} &= \int_{-d}^{d} dx \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, \delta(x - x_1 + x_2)|\psi(x_1, x_2)|^2 \\ &= \int_{-d}^{d} dx \int_{-\infty}^{\infty} dx_1\, |\psi(x_1, x_1 - x)|^2\end{aligned}$$

$$\bar{x} = \langle X_1 - X_2 \rangle = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, (x_1 - x_2)|\psi(x_1, x_2)|^2$$