Lecture 10: Coordinate and Momentum Representations

- We will start by considering the quantum description of the motion of a particle in one dimension.

- In classical mechanics, the state of the particle is given by its position and momentum coordinates, x and p.

- In quantum mechanics, we will consider position and momentum as observables and therefore represent them by Hermitian operators, X and P, respectively.

- Based on experimental evidence, we have deduced that:

$$\left[X,P\right]=i\hbar I$$

Incompatible Observables

- If two operators do not commute, then an eigenstate of one cannot be an eigenstate of the other.

 Proof:

- Strategy: assume opposite and show contradiction

- Assume $|a,b\rangle$ such that

$$A|a,b\rangle = a|a,b\rangle$$

$$B|a,b\rangle = b|a,b\rangle$$

- Operate on $|a,b\rangle$ with $M := [A,B]$

$$M|a,b\rangle = AB|a,b\rangle - BA|a,b\rangle$$

$$= bA|a,b\rangle - aB|a,b\rangle$$

$$= (ba - ab)|a,b\rangle \quad \Rightarrow M|a,b\rangle = 0$$

- either $\underline{\det|M| = 0}$ or $|a,b\rangle = 0$

- if $\det|M| \neq 0$ then $|a,b\rangle$ does not exist

-Then A and B are compatible

- if $[A,B] = 0$, then $M = 0$, $\{|a,b\rangle\}$ do exist

-Then A and B are incompatible

Coordinate Eigenstates

- Clearly X and P are *incompatible*, thus a particle cannot simultaneously have a well-defined position *and* momentum

- Since X is a Hermitian operator, it follows that its eigenstates form a complete set of unit vectors:

– eigenvalue equation:

$\mathbb{R}$ = set of real numbers

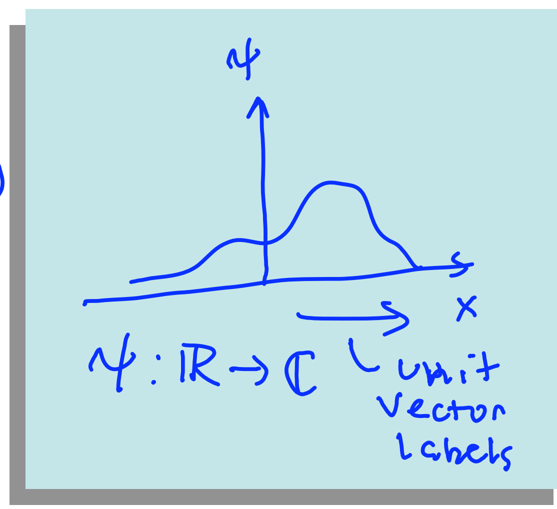
$$X|x\rangle = x|x\rangle \quad \forall x \in \mathbb{R}$$

– orthonormality:

$$\langle x|x'\rangle = \delta(x-x')$$

– closure:

$$\int_{-\infty}^{+\infty} dx\, |x\rangle\langle x| = I$$



– A state in Hilbert space is completely specified by its components in a 'physical' basis:

as $c_n = \langle n|\psi\rangle \Rightarrow \psi(x) = \langle x|\psi\rangle$

Momentum Representation

- The same logic applies also to momentum eigenstates:

$$P|p\rangle = p|p\rangle \quad \forall p \in \mathbb{R}$$

$$\langle p|p'\rangle = \delta(p-p')$$

$$\int dp\, |p\rangle\langle p| = I$$

$$\psi(p) := \langle p|\psi\rangle$$

- In order to move fluidly between coordinate and momentum basis, we need to know the transformation coefficients $\langle x|p\rangle$ and $\langle p|x\rangle$:

proof: $\psi(p) = \langle p|\psi\rangle$

$$= \int dx \langle p|x\rangle\langle x|\psi\rangle$$

$$= \int dx \underbrace{\langle p|x\rangle}_{?} \psi(x)$$

$$\psi(x) = \langle x|\psi\rangle$$

$$= \int dp \langle x|p\rangle\langle p|\psi\rangle$$

$$= \int dp \underbrace{\langle x|p\rangle}_{?} \psi(p)$$

Deriving $\langle x|p\rangle$

- This is the direct derivation
  - Most textbooks use a round-about approach to avoid mathematical subtleties
  - We will just tackle them head on

- start from $P|p\rangle = p|p\rangle$
- hit with $\langle x| \rightarrow$ $\langle x|P|p\rangle = p\langle x|p\rangle$
- insert I: $\int dx' \langle x|P|x'\rangle\langle x'|p\rangle = p\langle x|p\rangle$
  - thus we need to know $\langle x|P|x'\rangle$
    - i.e, the matrix elements of P in basis $\{|x\rangle\}$

Derivation of $\langle x|P|x'\rangle$

- All properties of X and P follow from: $[X, P] = i\hbar I$

- sandwich with $\langle x|$ → and ← $|x'\rangle$

$$\langle x|XP|x'\rangle - \langle x|PX|x'\rangle = i\hbar\langle x|x'\rangle$$

$$(x - x')\langle x|P|x'\rangle = i\hbar\langle x|x'\rangle$$

$$\langle x|P|x'\rangle = i\hbar\frac{\langle x|x'\rangle}{(x - x')}$$

$$\boxed{\langle x|P|x'\rangle = i\hbar\frac{\delta(x - x')}{(x - x')}}$$

- At first glance this looks like a monstrosity, it is zero for $x \neq x'$, but at $x = x'$, it is infinity divided by zero
  - By treating the delta-distribution correctly, we will see that we can easily understand the meaning of this result

Distribution Theory

- Q: is $\delta(x)$ a function?
  - A: no, technically it is a 'distribution'
  - A 'function' is a mapping from one space onto another:

$$y = f(x)$$

  - A 'distribution' is more general than a function
  - A distribution is defined only under integration

$$y = \int dx\, F(x) f(x)$$

    - Here, $F(x)$ is the distribution and $f(x)$ is an ordinary function
  - For example, the delta-distribution is defined by:

$$\int dx\, \delta(x - x_0) f(x) = f(x_0)$$

  - A distribution can also be defined as the limit of a sequence of functions. All properties of the distribution are *by definition*, the limiting properties of the sequence
    - Example:

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{\pi\sigma^2}} e^{-x^2/\sigma^2}$$

:

∴ properties of $\delta(x)$ are well defined

– sequence is not unique

$\langle x|P|x'\rangle$ is a Distribution

- Our previous result can be written as:

$$\langle x|P|x'\rangle = i\hbar \lim_{\sigma\to 0} \frac{1}{\sqrt{\pi}\,\sigma} \frac{e^{-(x-x')^2/\sigma^2}}{(x-x')}$$

  - This is well-behaved for any finite σ
  - This proves that it has a clear meaning.

- Insert result into original equation:

(For $\langle x|p\rangle$)

$$\int dx' \, i\hbar \frac{\delta(x-x')}{(x-x')} \langle x'|p\rangle = p\langle x|p\rangle$$

- Expand $\langle x\_|p\rangle$ around $x\_=x$:

$$\langle x'|p\rangle = \langle x|p\rangle + (x'-x)\frac{d}{dx}\langle x|p\rangle + \frac{(x'-x)^2}{2}\frac{d^2}{dx^2}\langle x|p\rangle + \dots$$

Continued

$$\int dx' \frac{\delta(x-x')}{(x-x')}\left[\langle x|p\rangle + (x'-x)\frac{d}{dx}\langle x|p\rangle + \frac{(x'-x)^2}{2}\frac{d^2}{dx^2}\langle x|p\rangle + \dots\right]$$

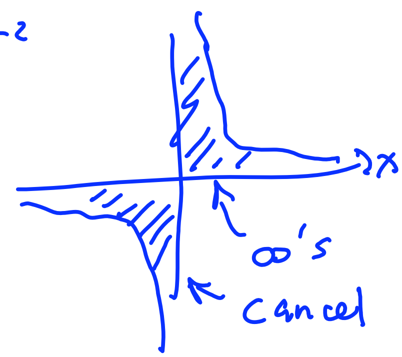
$$= -\frac{i}{\hbar}p\langle x|p\rangle$$

$$\langle x|p\rangle \overbrace{\int dx' \frac{\delta(x-x')}{(x-x')}}^{=\,?} - \frac{d}{dx}\langle x|p\rangle \overbrace{\int dx'\,\delta(x-x')}^{=\,1}$$

$$+ \frac{d^2}{dx^2}\langle x|p\rangle \underbrace{\int dx'\,(x-x')\,\delta(x-x')}_{=\,0} + \underbrace{\dots}_{=\,0} = -\frac{i}{\hbar}p\langle x|p\rangle$$

$$\int dx' \frac{\delta(x-x')}{(x-x')} = \lim_{\sigma\to 0}\frac{1}{\sqrt{\pi\sigma^2}}\int_{-\infty}^{+\infty} dx' \frac{e^{-(x-x')^2/\sigma^2}}{(x-x')}$$

$$= \lim_{\sigma\to 0}\frac{1}{\sqrt{\pi\sigma^2}}\int_{-\infty}^{+\infty} dx\,\frac{e^{-x^2/\sigma^2}}{x}$$



$$\therefore \int dx' \frac{\delta(x-x')}{(x-x')} = 0 \quad \text{by parity (odd function)}$$