Lecture 11:
X and P, part II

PHY851/fall 2009

Quick review:

$$P|p\rangle := p|p\rangle$$

$$\int dx' \langle x|P|x'\rangle\langle x'|p\rangle = p\langle x|p\rangle$$

$$\langle x|P|x'\rangle = i\hbar \frac{\delta(x-x')}{(x-x')}$$

$$i\hbar \int dx' \frac{\delta(x-x')}{(x-x')}\Big[\underbrace{\langle x|p\rangle}_{\text{"0"}} + (x'-x)\frac{d}{dx}\langle x|p\rangle + \underbrace{\frac{(x'-x)^2}{2}\frac{d^2}{dx^2}\langle x|p\rangle + \dots}_{\text{"0"}}\Big]$$

$$= p\langle x|p\rangle$$

$$\frac{d}{dx}\langle x|p\rangle \int dx' \delta(x-x') = \frac{i}{\hbar} p\langle x|p\rangle$$

$$\frac{d}{dx}\langle x|p\rangle = \frac{i}{\hbar} p\langle x|p\rangle$$

$$\int \frac{1}{\langle x|p\rangle} d\langle x|p\rangle = \int \frac{i}{\hbar} p\, dx$$

$$\ln\langle x|p\rangle + \ln C = \frac{i}{\hbar} px$$

$$\langle x|p\rangle = C e^{\frac{ipx}{\hbar}}$$

Normalization

- The constant C is found from normalization:

want: $\langle x | x' \rangle = \delta(x-x')$

$$\int dp \langle x|p\rangle\langle p|x'\rangle = \delta(x-x')$$

$$\int dp\, |C|^2 e^{ip(x-x')/\hbar} = \delta(x-x')$$

$$|C|^2 2\pi\, \delta\left(\frac{x-x'}{\hbar}\right) = \delta(x-x')$$

rule: $\delta(ax) = \frac{1}{|a|}\delta(x)$

$$|C|^2 2\pi\hbar\, \delta(x-x') = \delta(x-x')$$

- This leads to the results:

$$\boxed{\begin{aligned} \langle x | p \rangle &= \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \\ \langle p | x \rangle = \langle x | p \rangle^* &= \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \end{aligned}}$$

The momentum operator

- But we have really learned even more.
  - By the same logic we must have: i.e., let $|p\rangle \to |\psi\rangle$

$$\langle x|P|\psi\rangle = \int dx' \langle x|P|x'\rangle\langle x'|\psi\rangle$$

$$= -i\hbar\frac{d}{dx}\langle x|\psi\rangle$$

$\hookrightarrow = \langle x|\psi\rangle + (x'-x)\frac{d}{dx}\langle x|\psi\rangle + \text{etc.}\ldots$

  - So when we say:

$$P = -i\hbar\frac{d}{dx}$$

While you can argue that its meaning might be understandable, my point is just that it is not proper use of Dirac notation.

  - We really mean:

$$\boxed{\langle x|P|\psi\rangle = -i\hbar\frac{d}{dx}\langle x|\psi\rangle}$$

  - In fact, this is all you need to remember

$|\psi\rangle \Rightarrow |x'\rangle$

$\hookrightarrow \langle x|P|x'\rangle = -i\hbar\frac{d}{dx}\delta(x-x')$

equivalent to $i\hbar\frac{\delta(x-x')}{(x-x')}$

I think that a correct statement would be:

$$P = -i\hbar\int dx|x\rangle\frac{d}{dx}\langle x|$$

This is now clearly an operator in Hilbert space, whose meaning is precisely defined by the definition of the derivative.

It then seems reasonable to represent the r.h.s. as:

$$P = -i\hbar\frac{d}{dX}$$

In analogy to:

$$V(X) = -i\hbar\int dx|x\rangle V(x)\langle x|$$

`Coordinate Representation'

- If we make a decision to work in x-basis only, we can abandon Dirac notation, and resort to another common notation:
- The **`coordinate representation'** of the QM theory for a single particle in 1D is related to the Dirac-notation representation via:

$$X \rightarrow x$$

$$P \rightarrow -i\hbar \frac{d}{dx}$$

$$|\psi\rangle \rightarrow \psi(x)$$

$$\langle\psi| \rightarrow \psi^*(x)$$

$$\langle A(X,P)\rangle \rightarrow \int_{-\infty}^{+\infty} dx \psi^*(x) A\left(x, -i\hbar \frac{d}{dx}\right)\psi(x)$$

Gives correct QM description of motion of a particle

Deriving the wave-equation of a particle

- Start from Schrödinger's Equation:

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

- For a particle we have:

$$H = \frac{P^2}{2M} + V(X) \longleftarrow \text{Empirical result}$$

- Hit with $\langle x|$ from left:

$$i\hbar \frac{d}{dt}\langle x|\psi\rangle = \frac{\langle x|P^2|\psi\rangle}{2m} + \langle x|V(X)|\psi\rangle$$

- Use the properties:

from $$\langle x|P|\psi\rangle = -i\hbar \frac{d}{dx}\langle x|\psi\rangle$$

$$\langle x|V(X) = \langle x|V(x)$$

- To arrive at:

$$i\hbar \frac{d}{dt}\psi(x) = -\frac{\hbar^2}{2M}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x)$$

  - Where, as usual:

$$\psi(x) = \langle x|\psi\rangle$$

Time evolution of the wavefunction

- To find the time evolution of a wavefunction, we can start from the form of the propagator in energy-eigenstate basis:

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$$

$$U(t,t_0) = \sum_n |\omega_n\rangle e^{-i\omega_n(t-t_0)} \langle\omega_n|$$

$$|\psi(t)\rangle = \sum_n |\omega_n\rangle e^{-i\omega_n(t-t_0)} \langle\omega_n|\psi(t_0)\rangle$$

what if all we know is $\psi(x,t_0)$?

$$\langle x|\psi(t)\rangle = \sum_n \langle x|\omega_n\rangle e^{-i\omega_n(t-t_0)} \int dx' \langle\omega_n|x'\rangle\langle x'|\psi(t_0)\rangle$$

$$\langle x|\psi(t)\rangle = \psi(x,t)$$

let $\langle x|\omega_n\rangle = \phi_n(x)$ "mode function"

$$c_n(t_0) = \int dx\, \phi_n^*(x)\,\psi(x,t_0)$$

$$\psi(x,t) = \sum_n \phi_n(x)\, e^{-i\omega_n(t-t_0)}\, c_n(t_0)$$

let $\psi(x,t) := \sum_n \phi_n(x)\, c_n(t)$

$$\boxed{c_n(t) = e^{-i\omega_n(t-t_0)}\, c_n(t_0)} \rightarrow \frac{d}{dt}|c_n|^2 = 0$$

Example: a Free Particle

- Consider a free particle of mass M.
- The Hamiltonian is thus:

$$H = \frac{p^2}{2M}$$

- Since $H=H(P)$, it is clear that the momentum eigenstates are also eigenstates of H :

$$H|p\rangle = \frac{1}{2M} P^2 |p\rangle$$

$$= \frac{p^2}{2m} |p\rangle$$

$$E(p) = \frac{p^2}{2M}$$

- Assume that the initial wavefunction, $\psi(x,0)$ is known

- What is $\psi(x,t)$, the wavefunction at a later time?

$$\psi(x,t) = \int dp \int dx' e^{ip(x-x')/\hbar - i\frac{p^2}{2m\hbar}t} \psi(x',0)$$