Momentum vs. Wavevector

- Instead of momentum, it is often convenient to use wavevector states:

Wavevector definition: $K = \frac{P}{\hbar}$

Important commutators: $[X, K] = i \qquad [P, K] = 0$

Eigenvalue equation: $K|k\rangle = k|k\rangle$

Relation to momentum eigenstates: $|k\rangle = c|p\rangle\big|_{p=\hbar k}$

$\langle k|k'\rangle = \delta(k-k')$ Normalization

$\int_{-\infty}^{+\infty} dk\, |k\rangle\langle k| = 1$ Closure

$\langle x|k\rangle = \frac{e^{ikx}}{\sqrt{2\pi}}$ Wavefunction

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$$\langle p|p'\rangle = \delta(p-p') = \delta(\hbar k - \hbar k') = \frac{1}{\hbar}\delta(k-k') = \frac{1}{\hbar}\langle k|k'\rangle$$

$$\therefore |k\rangle = \sqrt{\hbar}\,|p\rangle\big|_{p=\hbar k}$$

Wavefunction in K-space

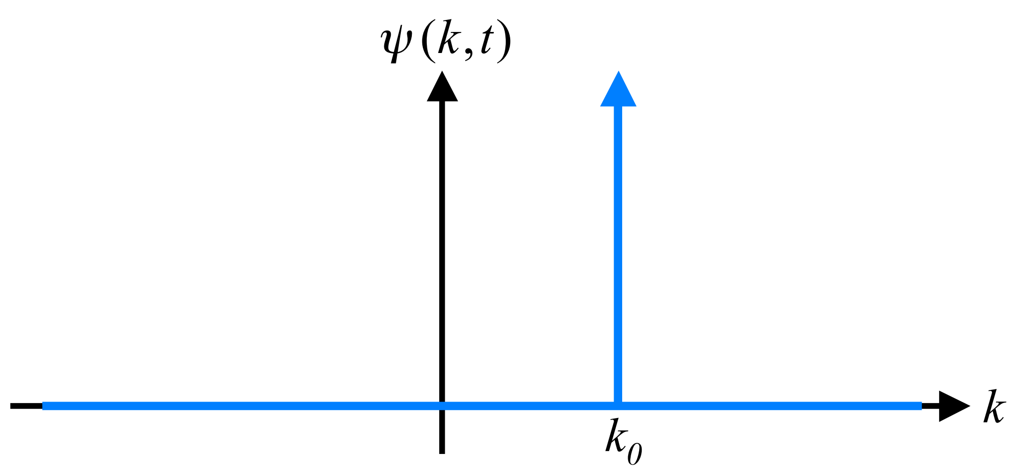
- What is the wavefunction in K-basis?

Let $\left|\psi(0)\right\rangle = \left|k_0\right\rangle$

Then: $\left|\psi(t)\right\rangle = e^{-i\frac{\hbar k_0^2}{2M}t}\left|k_0\right\rangle$

$$\psi(k,t) = \left\langle k \middle| \psi(t)\right\rangle = e^{-i\frac{\hbar k_0^2 t}{2m}}\left\langle k \middle| k_0\right\rangle$$

$$= e^{-i\frac{\hbar k_0^2 t}{2m}}\delta(k-k_0)$$



Wavefunction in X-space

- Quantum mechanical kinetic-energy eigenstate:

$$|\psi(t)\rangle = e^{-i\frac{p_0^2}{2M\hbar}t}|p_0\rangle$$

  - Only Global phase is changing in time
  - Global phase can't be observed
- Is anything actually moving?

$$\psi(x,t) = \frac{e^{i\frac{p}{\hbar}\left(x-\frac{p_0}{2M}t\right)}}{\sqrt{2\pi\hbar}} \qquad |\psi(x,t)|^2 = \frac{1}{2\pi\hbar}$$

  - The spatial phase-pattern is moving, but at:

$$v = p_0/2M$$

  - This is called the `phase velocity'
    - Not related to particle velocity in classical limit
- Classical free-particle motion:

$$x(t) = x_0 + v\,t$$

$$p(t) = p_0$$

- How can we have a `classical limit' of QM if nothing moves at v=p /M?

`Motion' in QM

- The answer is that in QM motion is in interference effect

- Consider a quantum-superposition of two plane-waves:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\frac{\hbar k_0^2}{2M}t}|k_0\rangle + e^{-i\frac{2\hbar k_0^2}{M}t}|2k_0\rangle\right]$$

$$\psi(x,t) = \frac{e^{ik_0\left(x-\frac{\hbar k_0}{2M}t\right)}}{2\sqrt{\pi}} + \frac{e^{i2k_0\left(x-\frac{\hbar k_0}{M}t\right)}}{2\sqrt{\pi}}$$

$$|\psi(x,t)|^2 = \frac{1}{2\pi}\left[1 + \frac{e^{i\left(k_0x-\frac{3}{2}\frac{\hbar k_0}{M}t\right)}}{2} + \frac{e^{-i\left(k_0x-\frac{3}{2}\frac{\hbar k_0}{M}t\right)}}{2}\right]$$

$$= \frac{1}{2\pi}\left[1 + \cos\left(k_0x - \frac{3}{2}\frac{\hbar k_0}{M}t\right)\right]$$

$$= \frac{1}{\pi}\cos^2\left[\frac{1}{2}\left(k_0x - \frac{3}{2}\frac{\hbar k_0}{M}t\right)\right]$$

This is a moving `standing wave'

- **The interference pattern can move!**

Wavepacket formation

- Measurement of position must produce a \`localized' state
  - How does this \`wavepacket' then move?

Physical versus non-physical states:

- States like $|x_0\rangle$ and $|k_0\rangle$ have $\langle\psi|\psi\rangle=\infty$
- All physical states must have $\langle\psi|\psi\rangle=1$

- CONCLUSION 1: states such as $|x_0\rangle$ and $|k_0\rangle$ are non-physical, and can therefore only be used as intermediate states in calculations

- CONCLUSION 2: Since a measurement of X produces the nonphysical state $|x_0\rangle$, such a measurement must be impossible

- However, real detectors have finite resolution
  - call the resolution σ
  - Result of measurement is therefore: $x = x_0 \pm \sigma$
  - After measurement, state vector is projected onto this subspace, as

$$|\psi'\rangle = \int_{x_0-\sigma}^{x_0+\sigma} dx\,|x\rangle\langle x|\psi\rangle$$

  - This state will be a \`wavepacket' with width σ

Gaussian Wavepackets

- A proto-typical wave packet is the Gaussian state $|x_0,\sigma\rangle$, defined via:

$$\langle x | x_0, \sigma \rangle = \frac{1}{\left[\pi\sigma^2\right]^{1/4}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}$$

  - Has width σ, centered at $x=x_0$
  - The width σ can be arbitrarily small, but we will always have $\langle\psi|\psi\rangle=1$

- Such wavepackets are the physical \`position' states

- A wavepacket centered at x_0 and moving with velocity $v=p_0/m$, has the wavefunction:

$$\langle x | x_0, p_0, \sigma \rangle = \frac{1}{\left[\pi\sigma^2\right]^{1/4}} e^{-\frac{(x-x_0)^2}{2\sigma^2} + i\frac{p_0 x}{\hbar}}$$

- How does the intial state $|x_0,p_0,\sigma\rangle$ evolve in time?

Analytic solution exists:

$$\psi(x,t) = \left[\pi\left(\sigma + \frac{i\hbar t}{m\sigma}\right)^2\right]^{-1/4} e^{-\frac{\left(x - \frac{p_0 t}{m} - x_0\right)^2}{2\sigma^2\left(1 + \frac{i\hbar t}{m\sigma^2}\right)} + i\frac{p_0\left(x - \frac{p_0 t}{2m}\right)}{\hbar}}$$

$$\psi(x,0) = \langle x | x_0, p_0, \sigma \rangle$$

Phase and Group Velocities

$$\psi(x,t)=\left[\pi\left(\sigma+\frac{i\hbar t}{m\sigma}\right)^2\right]^{-1/4} e^{-\frac{\left(x-\frac{p_0 t}{m}-x_0\right)^2}{2\sigma^2\left(1+\frac{i\hbar t}{m\sigma^2}\right)}+i\frac{p_0\left(x-\frac{p_0 t}{2m}\right)}{\hbar}}$$

- We can see that the phase velocity is

$$v_p=\frac{p_0}{2m}$$

- What does the *probability density* look like?

$$\left|\psi(x,t)\right|^2=\frac{1}{\sqrt{\pi}\sigma(t)}e^{-\frac{(x-x_0(t))^2}{\sigma^2(t)}}$$

$$x_0(t)=x_0+\frac{p_0}{m}t \qquad \sigma(t)=\sigma\sqrt{1+\left(\frac{\hbar t}{m\sigma^2}\right)^2}$$

- We see that the center of the wavepacket moves at the velocity $v_g=\dfrac{p_0}{m}$
  - We call this the `group velocity'
- We can see that the group velocity correlates with the velocity of a classical particle having the same momentum

Wavepacket Spreading

- The width of the Gaussian wavepacket is:

$$\sigma(t) = \sigma\sqrt{1+\left(\frac{\hbar t}{m\sigma^2}\right)^2}$$

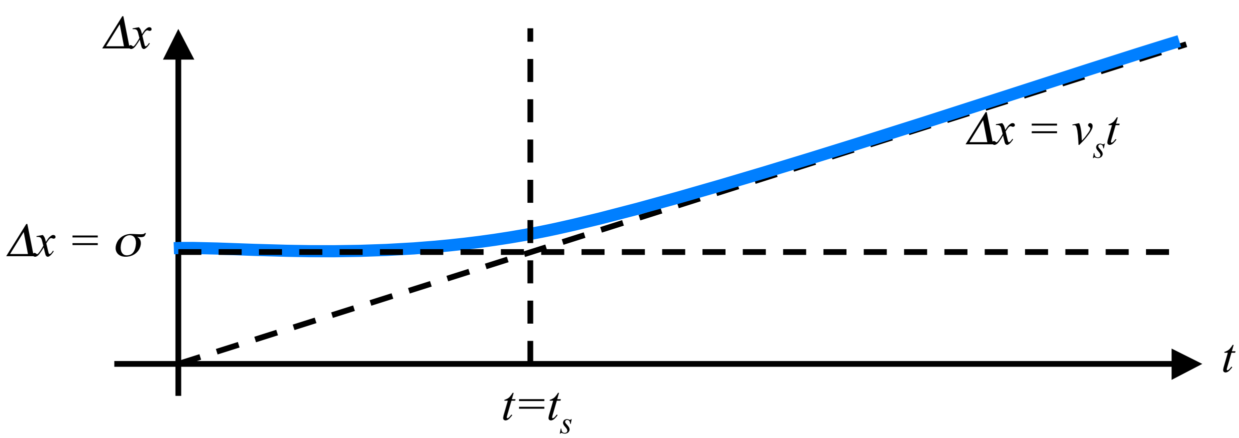
- Thus the timescale for wavepacket spreading is:

$$t_s = \frac{m\sigma^2}{\hbar} \qquad \sigma(t) = \sigma\sqrt{1+(t/t_s)^2}$$

  - For $t << t_s$ we can ignore spreading $\sigma(t) \approx \sigma$
  - For $t >> t_s$ the size of the wavepacket grows linearly in time:

$$\sigma(t) \approx \frac{\hbar}{m\sigma} t$$

  - Thus `spreading' at the velocity: $v_s = \frac{\hbar}{m\sigma}$
  - The smaller the wavepacket, the faster it will spread!



Electron Wavepacket

- Lets consider an electron:
  - m = 10^{-30} kg
  - If position was measured using light, then the electron will at most be localized to the size of the wavelength
  - Visible light $\lambda = 10^{-7}$ m

$$t_s = \frac{m\sigma^2}{\hbar} = \frac{10^{-30}\,kg \cdot 10^{-14}\,m^2}{10^{-34}\,kg\,m^2\,s^{-1}} = 10^{-10}\,s$$

- Electron wavepacket will start spreading 0.1 ns after being localized

$$v_s = \frac{\hbar}{m\sigma} = \frac{10^{-34}\,kg\,m^2\,s^{-1}}{10^{-30}\,kg \cdot 10^{-7}\,m} = 10^3\,\frac{m}{s} = 1\frac{km}{s}$$

- Spreading velocity, 1 km/s, is fairly large

- Hard to keep an electron pinned down to a deterministic position
  - → Classical Mechanics will not describe physics correctly
  - Repeated measurement of the electrons position could maintain localization, but random nature of measurements would introduce `quantum Brownian motion' so that CM will still not be correct

Wavepacket for a Baseball

- Here we have $m = 1\ \text{kg}$
  - Let us still consider the center-of-mass of the electron to be localized by neutron diffraction to .1 nanometer

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$$t_s = \frac{m\sigma^2}{\hbar} = \frac{1kg \cdot 10^{-20} m^2}{10^{-34} kg\, m^2\, s^{-1}} = 10^{14} s$$

- So the wavepacket will only start spreading after 10^{14} s = 30 million years

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$$v_s = \frac{\hbar}{m\sigma} = \frac{10^{-34} kg\, m^2\, s^{-1}}{1kg \cdot 10^{-10} m} = 10^{-24} \frac{m}{s}$$

$\frac{1\text{ nm}}{300\text{ million years}}$

- After which it will start to spread at a rate of 10^{-24} m/s.
  - In another 30 million years, it will have doubled in size

- Note: the age of the universe is 13 billion years
- In that time it will have reached 10 nm.

- So classical mechanics (i.e. well-defined/deterministic position and momentum) should do pretty well