Lecture 16: Probability Current II and 1D Scattering

Phy851 Fall 2009

Continuity Equation

$$j(x-\varepsilon)-j(x+\varepsilon)=\frac{dP(x,t)}{dt}$$

$$j(x-\varepsilon)-j(x+\varepsilon)=\frac{d\rho(x,t)2\varepsilon}{dt}$$

$$\frac{j(x-\varepsilon)-j(x+\varepsilon)}{2\varepsilon}=\frac{d\rho(x,t)}{dt}$$

$$-\frac{d}{dx}j(x,t)=\frac{d}{dt}\rho(x,t)$$

- This is the standard continuity equation, valid for any kind of fluid

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- For energy eigenstates (stationary states), we need:

$$\frac{d}{dt}\rho(x,t)=0\Rightarrow\rho(x,t)=\rho(x,0)$$

$$\frac{d}{dt}j(x,t)=0\Rightarrow j(x,t)=j(x,0)$$

- This gives: $\frac{d}{dx}j(x,t)=0 \quad \Rightarrow \quad j(x,t)=j_0$

- Must have ***spatially uniform*** current in steady state (of course j can be zero)

Derivation of the probability current:

Starting points:

- probability density: $\rho(x,t) = \psi^*(x,t)\,\psi(x,t)$

$$\rho = \psi^*\psi$$

- continuity equation: $j' = -\dot{\rho}$ ← given ρ, defines j

$$j' = -\dot{\rho}$$
$$= -\dot{\psi}^*\psi - \psi^*\dot{\psi}$$

$$\dot{\psi} = \frac{i\hbar}{2m}\psi'' - \frac{i}{\hbar}V\psi \xrightarrow[\text{c.c.}]{} \dot{\psi}^* = -\frac{i\hbar}{2m}\psi^{*\prime\prime} + \frac{i}{\hbar}V\psi^*$$

therefore: $j' = \frac{i\hbar}{2m}\psi^{*\prime\prime}\psi - \cancel{\frac{i}{\hbar}V\psi^*\psi} - \frac{i\hbar}{2m}\psi^*\psi'' + \cancel{\frac{i}{\hbar}V\psi^*\psi}$

$$j' = \frac{i\hbar}{2m}\left(\psi^{*\prime\prime}\psi - \psi^*\psi''\right)$$

Trick: $\frac{d}{dx}\left(\psi^{*\prime}\psi - \psi^*\psi'\right) = \psi^{*\prime\prime}\psi + \cancel{\psi^{*\prime}\psi'} - \cancel{\psi^{*\prime}\psi'} - \psi^*\psi''$

$$= \psi^{*\prime\prime}\psi - \psi^*\psi''$$

$$\frac{d}{dx}j(x,t) = \frac{i\hbar}{2m}\frac{d}{dx}\left(\psi^{*\prime}\psi - \psi^*\psi'\right)$$

up to const.

$$\boxed{j(x,t) = -\frac{i\hbar}{2m}\left(\psi^*(x,t)\frac{d}{dx}\psi(x,t) - \psi(x,t)\frac{d}{dx}\psi^*(x,t)\right)}$$

Current of a plane wave

- For a plane wave we have:

$$\psi(x) = ae^{ikx}$$

- The corresponding probability current is:

$$j = -i\frac{\hbar}{2m}\left(\psi^*\psi' - \psi^{*\,\prime}\;\psi\right)$$

$$= -i\frac{\hbar|a|^2}{2m}\left(ik - (-ik)\right) = \underbrace{|a|^2}_{\text{density}}\;\underbrace{\frac{\hbar k}{m}}_{\text{velocity}}$$

- So for a plane wave, we find:

$$\boxed{\vec{j}(x,t) = \rho_0\vec{v}_0}$$

This result is fairly intuitive

Quantum Interference terms

- Consider a superposition of plane waves:

$$\psi(x) = a_1 e^{ik_1 x} + a_2 e^{ik_2 x}$$

- The probability density is:

$$\rho(x) = |a_1|^2 + |a_2|^2 + \underbrace{\left(a_1^* a_2 e^{i(k_2 - k_1)x} + c.c.\right)}_{\text{Interference Term}}$$

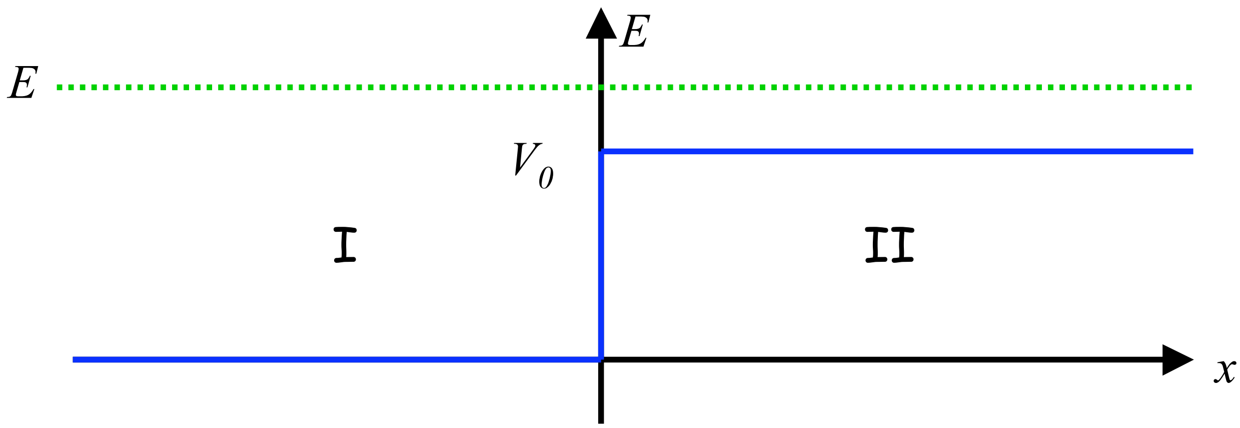
'Fringes'

- The probability current density is:

$$j(x) = |a_1|^2 \frac{\hbar k_1}{m} + |a_2|^2 \frac{\hbar k_2}{m} + \underbrace{\left(a_1^* a_2 e^{i(k_2 - k_1)x} + c.c\right)\frac{\hbar(k_1 + k_2)}{2m}}_{\text{Interference Term}}$$

- Note that the interference term in $j(x,t)$ vanishes for $k_2 = -k_1$
  - **This is always the case for *Energy Eigenstates* → Currents are then purely additive**
  - There is still interference in the *probability density* due to the presence of left and right currents, just not in the *probability current*.

Return to the Step Potential



$$\psi_I(x) = e^{ik_1x} + re^{-ik_1x} \qquad \psi_{II}(x) = te^{ik_2x}$$

$$\boxed{r = \frac{k_1 - k_2}{k_1 + k_2}} \qquad \boxed{t = \frac{2k_1}{k_1 + k_2}}$$

- The probability current density is:

$$j_I(x) = \frac{\hbar k_1}{m} - |r|^2 \frac{\hbar k_1}{m} = (1 - |r|^2)\frac{\hbar k_1}{m}$$

$$j_{II}(x) = |t|^2 \frac{\hbar k_2}{m}$$

- Spatially uniform current requires: $\boxed{j_I(x) = j_{II}(x)}$
  - So the probability conservation law is:

$$\boxed{(1 - |r|^2)\frac{\hbar k_1}{m} = |t|^2 \frac{\hbar k_2}{m}}$$

Rearrange terms to get more intuitive result

$$|r|^2 + |t|^2 \frac{k_2}{k_1} = 1 \qquad \boxed{j_{out} = j_{in}}$$

Continued

$$|r|^2 + |t|^2 \frac{k_2}{k_1} = 1$$

$$t = \frac{2k_1}{k_1 + k_2} \qquad r = \frac{k_1 - k_2}{k_1 + k_2}$$

- Transmission and reflection probabilities are derived from conservation law: $j_{in} = j_{out}$

$$T := \left| \frac{j_{out}(x > 0)}{j_{in}} \right|$$

$$R := \left| \frac{j_{out}(x < 0)}{j_{in}} \right|$$

Any constant we might have put in front of the incident wave would cancel out here

- For step potential, this gives:

$$T = \frac{|t|^2 k_2}{k_1} = \frac{4k_1 k_2}{|k_1 + k_2|^2} \qquad R = \frac{|r|^2 k_1}{k_1} = |r|^2$$

$$R + T = \frac{(k_1 - k_2)^2 + 4k_1 k_2}{(k_1 + k_2)^2} = \frac{k_1^2 + 2k_1 k_2 + k_2^2}{(k_1 + k_2)^2} = 1$$

Probability current: Summary/conclusions

- The proper way to compute probability in scattering is via probability current.

- The probability for a particle to scatter into a certain channel is the ratio of the outgoing current in that channel to the total incoming current.

- In 1D scattering at fixed energy, we can treat the left-traveling and right-traveling components of the current as independent
  - because there are no interference terms in the current density for +k and –k currents.
  - Allows us to group components into 'incoming' and 'outgoing' currents

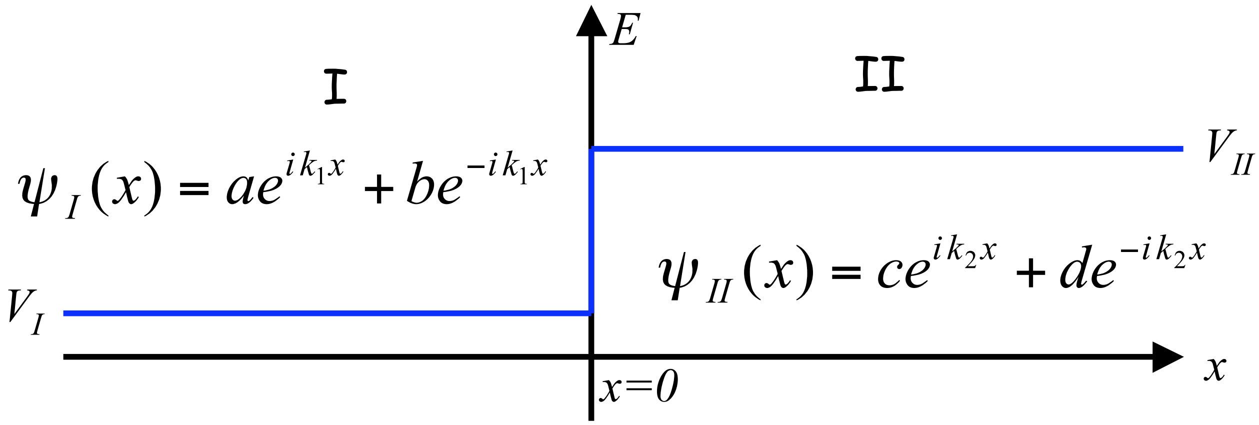
- For a plane-wave, the current is the amplitude squared times the velocity.

Important Shortcut:

- If you are asked to compute R and T, for a 1d scattering problem, you can:
  - Compute $R=|r|^2$
  - Then use $T=1-R$ (conservation law)
  - i.e. you don't need to compute t or worry about current unless specifically asked to do so.
  - **Wrong:** $T=|t|^2$ then $R = 1-T$ would not work

Scattering From a Potential Step revisited

- Lets solve the step potential again, but with the possibility for left *and* right travelling incoming waves:



$$k_1 = \left(U(E-V_I) + iU(V_I - E)\right)\frac{\sqrt{2m|E-V_I|}}{\hbar}$$

$$k_2 = \left(U(E-V_{II}) + iU(V_{II} - E)\right)\frac{\sqrt{2m|E-V_{II}|}}{\hbar}$$

- Boundary condition equations:

$$\psi_I(0) = \psi_{II}(0) \quad \Rightarrow \quad a + b = c + d$$

$$\psi_I'(0) = \psi_{II}'(0) \quad \Rightarrow \quad k_1\left(a - b\right) = k_2\left(c - d\right)$$

Scattering Matrix

- The scattering matrix gives the outgoing amplitudes in terms of the incoming amplitudes:

Definition of scattering matrix, S:

$$\begin{pmatrix} b \\ c \end{pmatrix} = S \begin{pmatrix} a \\ d \end{pmatrix}$$

$$a + b = c + d$$

$$k_1(a - b) = k_2(c - d)$$

Boundary condition equations

$$\Rightarrow$$

$$b - c = -a + d$$

$$k_1 b + k_2 c = k_1 a + k_2 d$$

Move outgoing amplitudes to l.h.s. and incoming to r.h.s.

$$\begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix}\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ k_1 & k_2 \end{pmatrix}\begin{pmatrix} a \\ d \end{pmatrix}$$

b.c. eqs in matrix form

Solve for the outgoing amplitudes

$$\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 1 \\ k_1 & k_2 \end{pmatrix}\begin{pmatrix} a \\ d \end{pmatrix}$$

Calculate the inverse

$$\begin{pmatrix} b \\ c \end{pmatrix} = \frac{1}{k_1 + k_2}\begin{pmatrix} k_2 & 1 \\ -k_1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ k_1 & k_2 \end{pmatrix}\begin{pmatrix} a \\ d \end{pmatrix}$$

Multiply matrices to get:

$$\begin{pmatrix} b \\ c \end{pmatrix} = \frac{1}{k_1 + k_2}\begin{pmatrix} k_1 - k_2 & 2k_2 \\ 2k_1 & k_2 - k_1 \end{pmatrix}\begin{pmatrix} a \\ d \end{pmatrix}$$

Result:

$$S = \frac{1}{k_1 + k_2}\begin{pmatrix} k_1 - k_2 & 2k_2 \\ 2k_1 & k_2 - k_1 \end{pmatrix}$$

Example: wave incident from left

- Consider case $a=1,\ b=r,\ c=t,$ and $d=0$:

$$\begin{pmatrix} b \\ c \end{pmatrix} = S \begin{pmatrix} a \\ d \end{pmatrix}$$

$$S = \frac{1}{k_1 + k_2}\begin{pmatrix} k_1 - k_2 & 2k_2 \\ 2k_1 & k_2 - k_1 \end{pmatrix}$$

$$\begin{pmatrix} r \\ t \end{pmatrix} = \frac{1}{k_1 + k_2}\begin{pmatrix} k_1 - k_2 & 2k_2 \\ 2k_1 & k_2 - k_1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- We find: $r = \dfrac{k_1 - k_2}{k_1 + k_2}$ $\qquad t = \dfrac{2k_1}{k_1 + k_2}$

- For left and right incoming waves:

$$\begin{pmatrix} r \\ t \end{pmatrix} = \frac{1}{k_1 + k_2}\begin{pmatrix} k_1 - k_2 & 2k_2 \\ 2k_1 & k_2 - k_1 \end{pmatrix}\begin{pmatrix} c_L \\ c_R \end{pmatrix}$$

$$r = \frac{(k_1 - k_2)c_L + 2k_2 c_R}{k_1 + k_2} \qquad t = \frac{2k_1 c_L + (k_2 - k_1)c_R}{k_1 + k_2}$$

$$j_{in} = |c_L|^2 k_1 + |c_R|^2 k_2$$

$$R = \frac{|r|^2 k_1}{j_{in}} = \frac{|k_1 - k_2|^2 |c_L|^2 + 4|k_2|^2 |c_R|^2 + 4\,\mathrm{Re}\left\{k_2^*(k_1 - k_2)c_R^* c_L\right\}}{|k_1 + k_2|^2\left(|c_L|^2 k_1 + |c_R|^2 k_2\right)}$$

$$T = 1 - R$$