Lecture 17:
Scattering in One Dimension
Part 2

Phy851 Fall 2009

Transfer Matrix

- For the purpose of propagating a wave through multiple elements, it is more convenient to relate the amplitudes on the right-side of the boundary to those on the left-side via the 'Transfer Matrix':

Definition of transfer matrix, M:

$$\begin{pmatrix} c \\ d \end{pmatrix} = M \begin{pmatrix} a \\ b \end{pmatrix}$$

We don't use T, because the T-matrix is something else

Boundary condition equations

$$a + b = c + d$$

$$k_1(a - b) = k_2(c - d)$$

b.c. eqs in matrix form

$$\begin{pmatrix} 1 & 1 \\ k_2 & -k_2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ k_1 & -k_1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

Solve for the right amplitudes in terms of the left

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ k_2 & -k_2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ k_1 & -k_1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

Calculate the inverse

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{2k_2}\begin{pmatrix} k_2 & 1 \\ k_2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ k_1 & -k_1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

Multiply matrices to get:

$$M(k_1, k_2) = \frac{1}{2k_2}\begin{pmatrix} k_2 + k_1 & k_2 - k_1 \\ k_2 - k_1 & k_2 + k_1 \end{pmatrix}$$

Extracting *r* and *t* from *M*

- Consider case $a=1,\ b=r,\ c=t,$ and $d=0$:

$$\begin{pmatrix} t \\ 0 \end{pmatrix} = M \begin{pmatrix} 1 \\ r \end{pmatrix}$$

$$M_{11} + M_{12} r = t$$

$$M_{21} + M_{22} r = 0$$

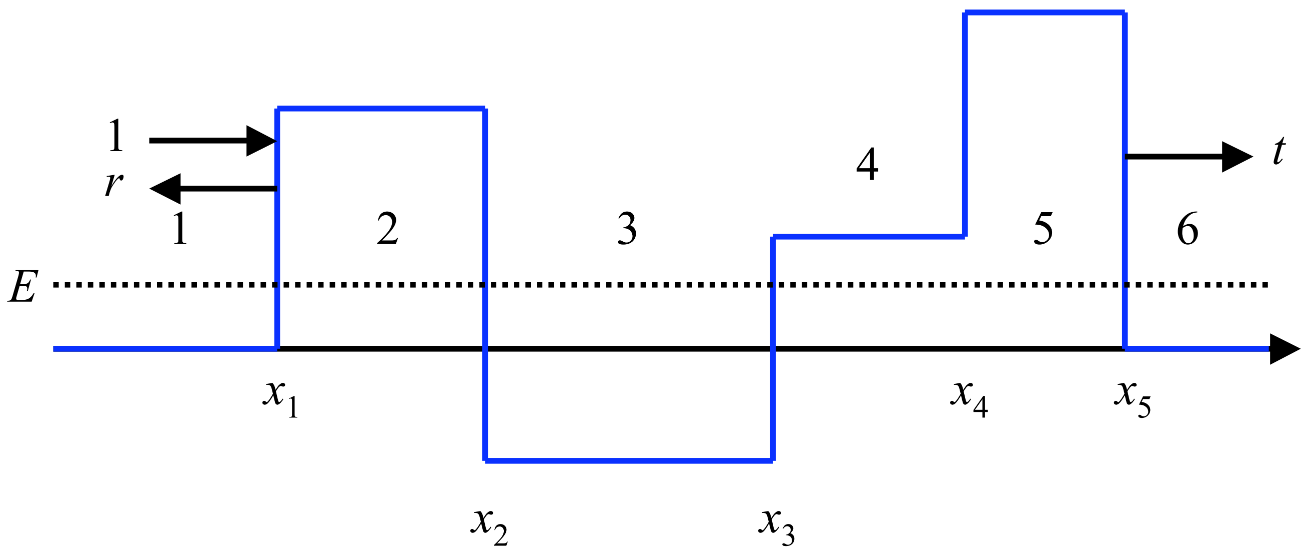
$$r = -\frac{M_{21}}{M_{22}}$$

$$t = M_{11} - \frac{M_{12} M_{21}}{M_{22}}$$

$$t = \frac{\det[M]}{M_{22}}$$

A Generalized Transfer Matrix approach to complex scattering potentials

- How would you go about computing *r* and *t* for a complicated structure?



Systematic approach:

- compute the Transfer Matrix for each element, then multiply them all together to get a full Transfer Matrix for the object
  - For a problem with only one boundary, it will always save time to plug and chug.
  - For 2 or more boundaries, the Transfer Matrix approach should save time
- Then compute probabilities via:

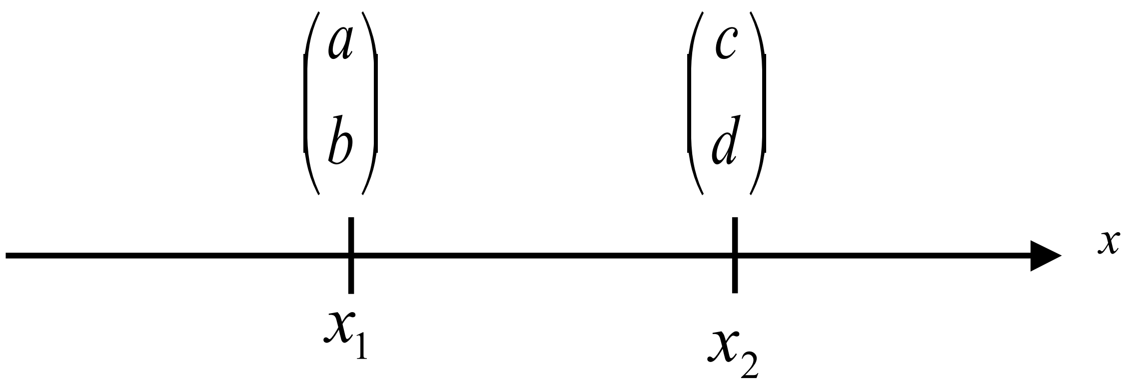
$$R = \left| \frac{M_{12}}{M_{22}} \right|^2 \qquad T = 1 - R$$

Definition of the 2-vector representation of the wavefunction

- At any point, *x*, along the wave, the state of the system, $\psi(x)$, will be represented by a two vector:

$$\psi(x) = \psi_R(x) + \psi_L(x) \rightarrow \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix}$$

- Example: consider a free particle:



$$\psi(x_1) = a + b \qquad \psi(x_2) = c + d$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

$$a = Ae^{ikx_1} \qquad c = Ae^{ikx_2}$$

$$b = Be^{-ikx_1} \qquad d = Be^{-ikx_2}$$

$$c = ae^{ik(x_2 - x_1)}$$

$$d = be^{-ik(x_2 - x_1)}$$

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} e^{ikL} & 0 \\ 0 & e^{-ikL} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$L = x_2 - x_1$$

The Transfer-matrix for free propagation

- Thus to propagate a two-vector over a distance L, with wavevector k, we have:

$$M_{free}(kL) = \begin{pmatrix} e^{ikL} & 0 \\ 0 & e^{-ikL} \end{pmatrix}$$

- A potential step can be characterized by k_1 and k_2. We already determined that the *M*-matrix is:

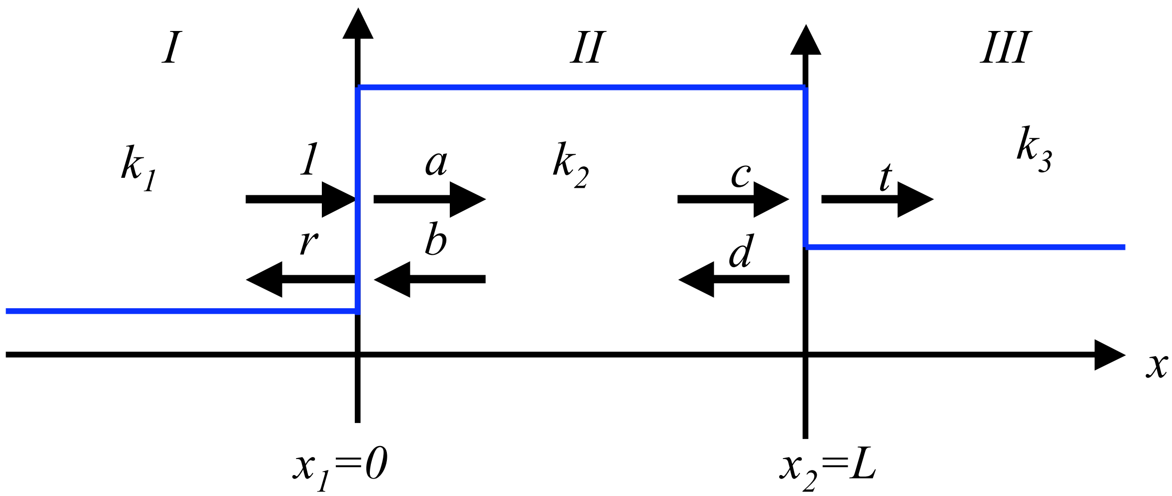
$$M_{step}(k_2, k_1) = \frac{1}{2k_2} \begin{pmatrix} k_2 + k_1 & k_2 - k_1 \\ k_2 - k_1 & k_2 + k_1 \end{pmatrix}$$

R L

- Now we are ready to go to work on some scattering problems

Transfer-Matrix Approach to Scattering through Multiple Elements

- Example: Scattering through two step potentials:



$$\begin{pmatrix} a \\ b \end{pmatrix} = M_{step}(k_2, k_1)\begin{pmatrix} 1 \\ r \end{pmatrix}$$

$$\begin{pmatrix} c \\ d \end{pmatrix} = M_{free}(k_2 L)\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} c \\ d \end{pmatrix} = M_{free}(k_2 L) M_{step}(k_2, k_1)\begin{pmatrix} 1 \\ r \end{pmatrix}$$

$$\begin{pmatrix} t \\ 0 \end{pmatrix} = M_{step}(k_3, k_2)\begin{pmatrix} c \\ d \end{pmatrix} \Rightarrow \begin{pmatrix} t \\ 0 \end{pmatrix} = M_{step}(k_3, k_2) M_{free}(k_2 L) M_{step}(k_2, k_1)\begin{pmatrix} 1 \\ r \end{pmatrix}$$

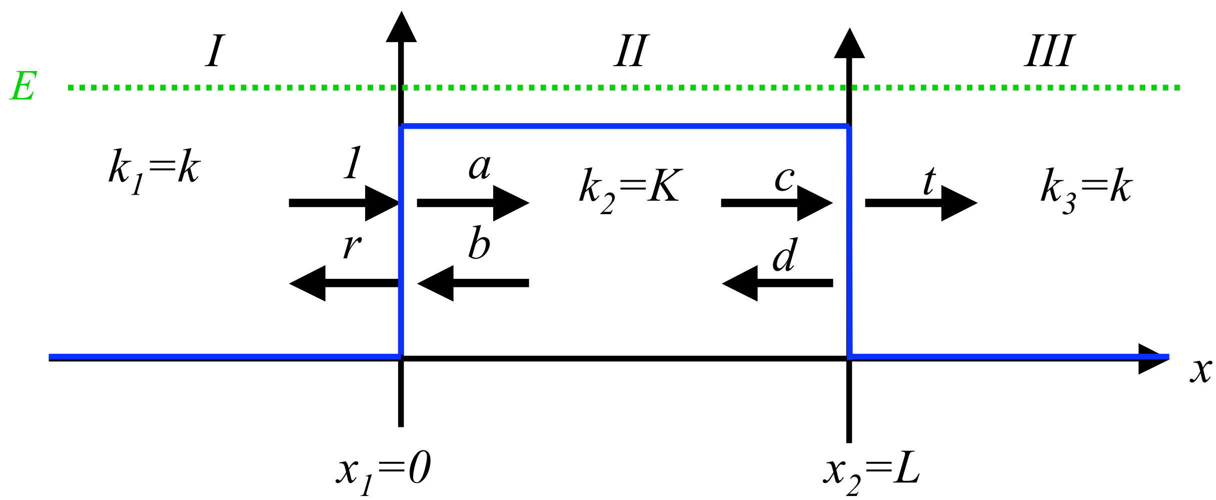
- Thus the full *M*-matrix for this scatterer is:

$$M = M_{step}(k_3, k_2) M_{free}(k_2 L) M_{step}(k_2, k_1)$$

$$R = \left|\frac{M_{12}}{M_{22}}\right|^2 \qquad T = 1 - R$$

Step potential result

- Example: Scattering through square barrier



- The full *M*-matrix for this scatterer is:

$$M = M_{step}(k,K)\,M_{free}(KL)\,M_{step}(K,k)$$

$$M_{step}(K,k) = \frac{1}{2K}\begin{pmatrix} K+k & K-k \\ K-k & K+k \end{pmatrix}$$

$$M_{free}(KL) = \begin{pmatrix} e^{iKL} & 0 \\ 0 & e^{-iKL} \end{pmatrix}$$

$$M_{step}(k,K) = \frac{1}{2k}\begin{pmatrix} k+K & k-K \\ k-K & k+K \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} \cos[KL] + i\dfrac{(k^2+K^2)}{2kK}\sin[KL] & -\dfrac{i(k^2-K^2)}{2kK}\sin[KL] \\ +\dfrac{i(k^2-K^2)}{2kK}\sin[KL] & \cos[KL] - i\dfrac{(k^2+K^2)}{2kK}\sin[KL] \end{pmatrix}$$

Example Continued:

$$\mathbf{M} = \begin{pmatrix} \cos[KL] + i\dfrac{(k^2+K^2)}{2kK}\sin[KL] & -\dfrac{i(k^2-K^2)}{2kK}\sin[KL] \\ +\dfrac{i(k^2-K^2)}{2kK}\sin[KL] & \cos[KL] - i\dfrac{(k^2+K^2)}{2kK}\sin[KL] \end{pmatrix}$$

- The reflection Amplitude is then:

$$r = -\frac{M_{21}}{M_{22}} = -\frac{(k^2-K^2)\sin[KL]}{(K^2+k^2)\sin[KL] + 2ikK\cos[KL]}$$

- So the reflection probability is

$$R = |r|^2 = \frac{(k^2-K^2)^2}{(k^2+K^2)^2 + 4k^2K^2\cot^2[KL]}$$

- This has a minimum of $R=0$ whenever $KL=n\pi$

$$\cot(n\pi) = \frac{\cos(n\pi)}{\sin(n\pi)} = \frac{(-1)^n}{0} = \infty \qquad T = 1 - R \rightarrow 1$$

  - These are the Transmission Resonances
  - This describes a frequency filter
  - Choose $L = (n/2)\lambda$ to transmit wavelength λ
  - For $KL=(n+1/2)\pi$, transmission is reduced to:

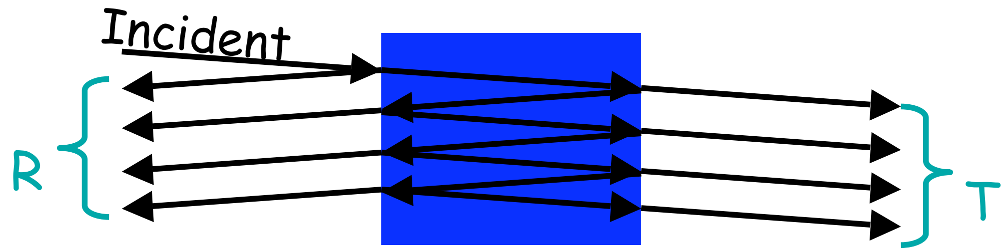
$$\cot\left(\left(n+\tfrac{1}{2}\right)\pi\right) = \frac{\cos\left(\left(n+\frac{1}{2}\right)\pi\right)}{\sin\left(\left(n+\frac{1}{2}\right)\pi\right)} = \frac{0}{(-1)^n} = 0$$

$$T = 1 - \frac{(k^2-K^2)^2}{(k^2+K^2)^2} = \frac{4K^2k^2}{(k^2+K^2)^2} \approx 4\frac{K^2}{k^2}$$

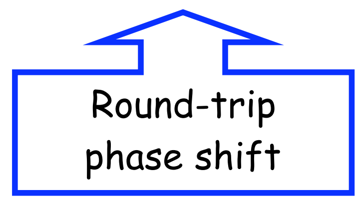
For $K << k$ can be very small

Transmission Resonance

- Analogy: light passing through a high-index medium



- Interference due to multiple Transmission Pathways
  - Constructive Interference when: $K(2L) = n(2\pi)$



$$T = 1 - R = 1 - \frac{\left((kL)^2 - (KL)^2\right)^2}{\left((kL)^2 + (KL)^2\right)^2 + 4(kL)^2(KL)^2\cot^2[KL]}$$

$$\frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 K^2}{2m} + V_0$$

$$k = \sqrt{K^2 + k_0^2}$$

$$kL = \sqrt{(KL)^2 + \alpha^2}$$

$$\alpha = k_0 L = \sqrt{\frac{2mV_0}{\hbar^2}}L$$

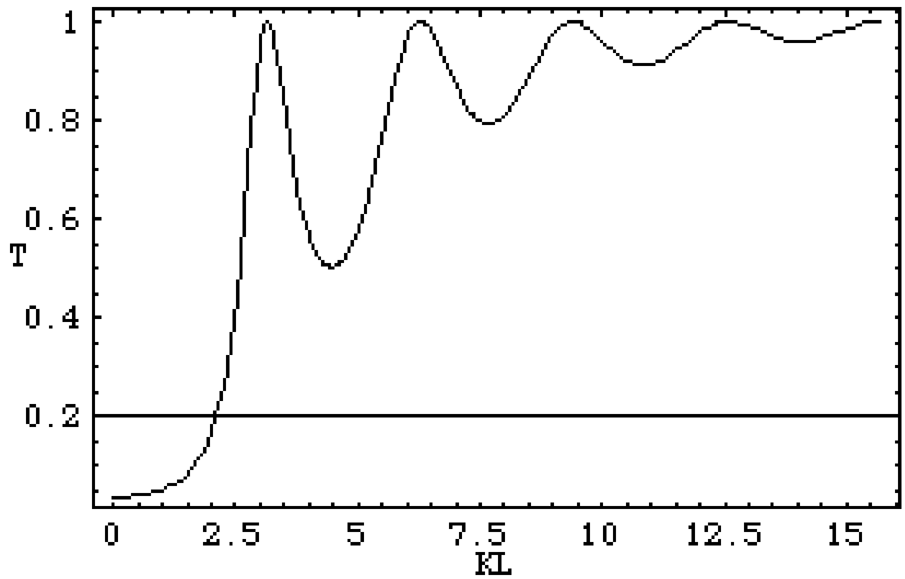
Depends only on KL and α

K is wavevector inside material

α is a measure of barrier `area'

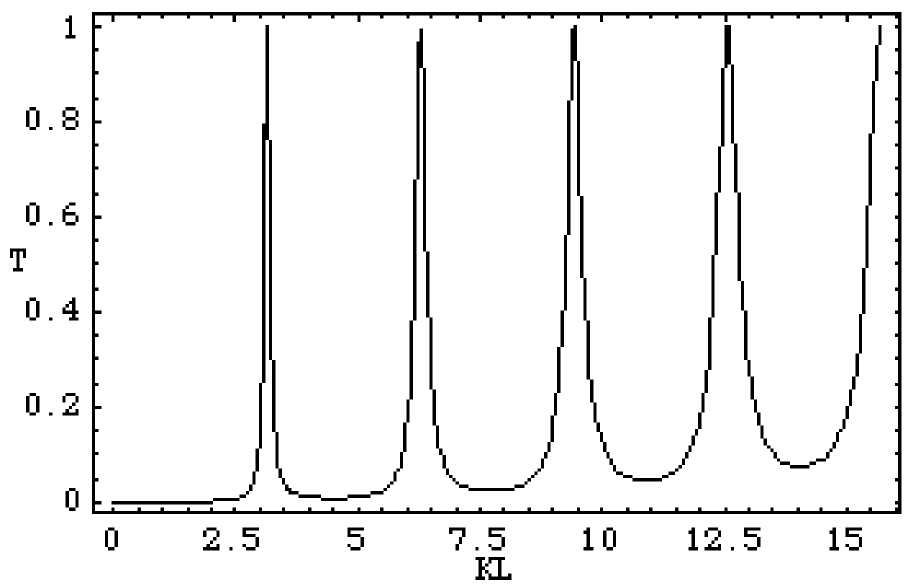
Resonance Transmission Profiles

$\alpha = 10$

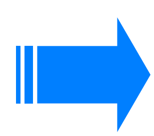


$\alpha = 100$

Plots made with Mathematica

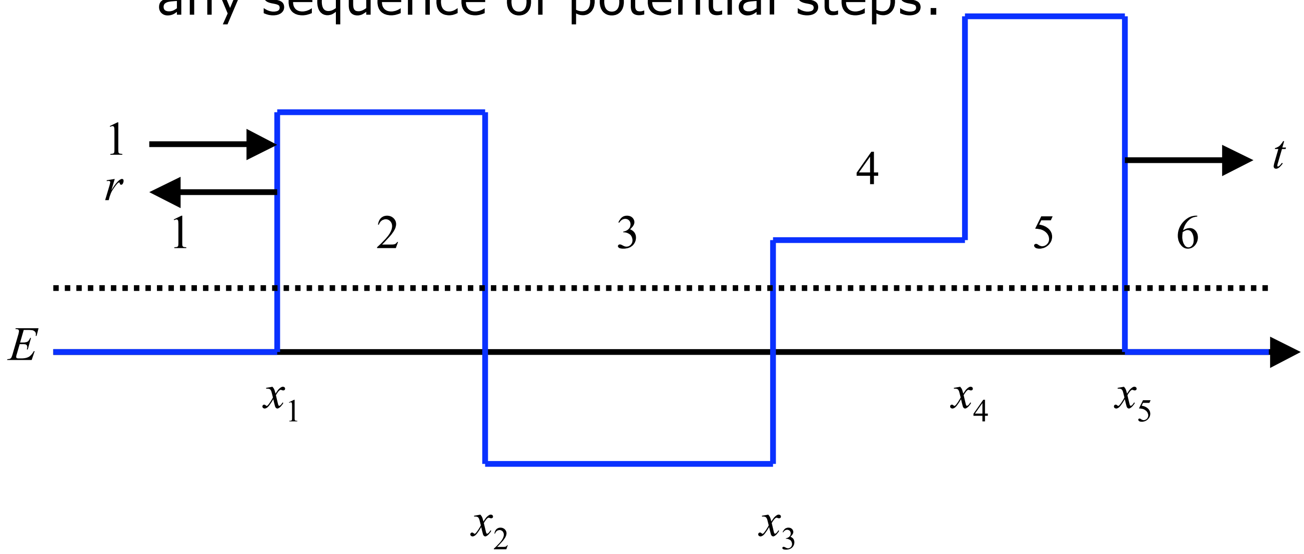


- Common Approach:
  - To increase α, increase V_0
  - Then use L to tune wavelength



Generalized Approach

- With this Method we can calculate the reflection and transmission amplitudes for any sequence of potential steps:



$$M = M_s^{[6,5]} M_f^{[5]} M_s^{[5,4]} M_f^{[4]} M_s^{[4,3]} M_f^{[3]} M_s^{[3,2]} M_f^{[2]} M_s^{[2,1]}$$

$$M_s^{[n,m]} = \frac{1}{2k_n} \begin{pmatrix} k_n + k_m & k_n - k_m \\ k_n - k_m & k_n + k_m \end{pmatrix} \qquad k_n = \frac{\sqrt{2m(E - V_n)}}{\hbar}$$

$$M_f^{[m]} = \begin{pmatrix} e^{ik_m(x_m - x_{m-1})} & 0 \\ 0 & e^{-ik_m(x_m - x_{m-1})} \end{pmatrix}$$

$$r = -\frac{M_{21}}{M_{22}} \qquad R = |r|^2 \qquad T = 1 - R$$