Lecture 1: Demystifying '$\hbar$' and 'i'

•We are often told that the presence of $\hbar$ distinguishes quantum from classical theories.

•One of the striking features of Schrödinger's equation is the fact that the variable, Ψ, is complex, whereas classical theories deal with real variables

QM:

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t)$$

CM:

$$\frac{d}{dt}x(t) = \frac{\partial}{\partial p}H(x,p)$$

$$\frac{d}{dt}p(t) = -\frac{\partial}{\partial x}H(x,p)$$

CE&M:

$$\frac{d}{dt}\vec{E}(\vec{r},t) = \frac{1}{c^2}\vec{\nabla}\times\vec{B}(\vec{r},t)$$

$$\frac{d}{dt}\vec{B}(\vec{r},t) = -\vec{\nabla}\times\vec{E}(\vec{r},t)$$

•Q: Is $\hbar$ necessary at all?

•By changing units we can of course make $\hbar$ disappear from QM

•But if it is truly fundamental, shouldn't this same choice of units make $\hbar$ appear then in CM?

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V(x)\psi$$

let $V(x) = V_0\, u\left(\frac{x}{x_0}\right)$

let $t = \frac{\hbar}{V_0}\tau \qquad x = x_0\rho$

$$i\frac{\cancel{\hbar} V_0}{\cancel{\hbar}}\frac{\partial}{\partial \tau}\psi = -\frac{\hbar^2}{2m x_0^2}\frac{\partial^2}{\partial \rho^2}\psi + V_0\, u(\rho)\psi$$

$$i\frac{\partial}{\partial \tau}\psi = -\frac{\hbar^2}{2m x_0^2 V_0}\frac{\partial^2}{\partial \rho^2}\psi + u(\rho)\psi$$

let $\boxed{m_0 = \frac{\hbar^2}{x_0^2 V_0}}$

$$i\frac{\partial}{\partial t}\psi(\rho,t) = -\frac{1}{2}\left(\frac{m_0}{m}\right)\frac{\partial^2}{\partial \rho^2}\psi(\rho,t) + u(\rho)\psi(\rho,t)$$

•If system has natural length scale and energy scale, then $\hbar$ is needed to relate then to the natural mass scale.

•What happens to CM in these units?

$$\frac{\partial}{\partial t}x = \frac{p}{m} \qquad \frac{\partial}{\partial t}p = -\frac{\partial}{\partial x}V$$

$$V \to V_0 u \qquad m \to m_0 \mu$$

$$x \to x_0 \rho \qquad p = p_0 \pi \qquad p_0 = \frac{m_0 x_0 V_0}{\hbar}$$

$$t \to \frac{\hbar}{V_0}\tau$$

$$\boxed{\frac{\partial}{\partial \tau}\rho = \frac{\pi}{\mu} \qquad \frac{\partial}{\partial \tau}\pi = -\frac{\partial u(\rho)}{\partial \rho}}$$

•Same mass scale makes CM dimensionless as well !

•Q: Are Maxwell's Eq's 'classical' or 'quantum'?

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0$$

•Apply De Broglie hypothesis to Einstein's equation:

$$E^2 - c^2 p^2 = m^2 c^4$$

$$E \to i\hbar\frac{\partial}{\partial t} \qquad p \to -i\hbar\frac{\partial}{\partial x}$$

$$\left[-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} + \hbar^2\nabla^2\right] = -m^2c^2$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi = \cancel{-\frac{m^2c^2}{\hbar^2}}\psi$$

let $m=0$

•The wavefunction of a massless particle obeys the Maxwell wave equation !

•So is *E* just the photon wavefunction?

•'Classical' E&M would be 'quantum' if the photon had mass

My opinion:

•Maxwell's equation is just as 'quantum' as Schrödinger's equation

•'Classical' EM is ray-optics

•Now, let's look at the 'i' issue:

$$i\frac{\partial}{\partial\tau}\psi = -\frac{1}{2\mu}\frac{\partial}{\partial\rho^2}\psi + U(\rho)\psi$$

•Separate ψ into real and imaginary parts:

let $\psi(\rho) = u(\rho) + i\,v(\rho)$

$$i\frac{\partial}{\partial t}u - \frac{\partial}{\partial t}v = -\frac{1}{2\mu}u'' - \frac{i}{2\mu}v'' + Uu + iVu$$

•No more 'i'

$$\frac{d}{dt}u(\vec{r},t) = H(u,v)\,v(\vec{r},t)$$
$$\frac{d}{dt}v(\vec{r},t) = -H(u,v)\,u(\vec{r},t)$$

•Structure looks familiar:

- •Two conjugate variables
- •Symmetric equations

CM:

$$\frac{d}{dt}x(t) = \frac{\partial}{\partial p}H(x,p)$$
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CE&M:

$$\frac{d}{dt}\vec{E}(\vec{r},t) = \frac{1}{c^2}\vec{\nabla}\times\vec{B}(\vec{r},t)$$
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•Can we put an 'i' in CM and make it look more like QM?

let $z = \frac{1}{\sqrt{2}}(x + ip)$ $\longleftrightarrow$ $x = z + z^*$

$z^* = \frac{1}{\sqrt{2}}(x - ip)$ $\qquad p = z - z^*$

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial z^*}{\partial x}\frac{\partial}{\partial z^*} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}\right)$$

$$\frac{\partial}{\partial p} = \frac{\partial z}{\partial p}\frac{\partial}{\partial z} + \frac{\partial z^*}{\partial p}\frac{\partial}{\partial z^*} = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right)$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial p}H(x,p) \qquad \Longrightarrow \qquad i\frac{d}{dt}z(t) = \frac{\partial}{\partial z^*}H(z,z^*)$$

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}H(x,p)$$

Newtons second Law

in Q.M.

$$H = \psi^*(x)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)\right]\psi(x) \quad \Leftarrow \text{Energy density}$$

$$i\frac{d}{dt}\psi(\vec{r},t) = \frac{\partial}{\partial\psi^*(\vec{r},t)}H(\psi,\psi^*)$$

- So what is going on?
  - The point is that QM is the correct theory
  - CM and CE&M are just approximations derived from QM
  - Thus they get their structures from QM