Lecture 20:
Quantum SHO: Part 2

Phy851 Fall 2009

Recap

- Introduced dimensionless variables:

$$\overline{X} = \frac{X}{\lambda} \qquad \overline{P} = \frac{\lambda}{\hbar} P \qquad \overline{H} = \frac{H}{\hbar\omega} \qquad \lambda = \sqrt{\frac{\hbar}{m\omega}}$$

$$\overline{H} = \frac{1}{2}\overline{P}^2 + \frac{1}{2}\overline{X}^2$$

- Introduce 'normal variables':

$$A = \frac{1}{\sqrt{2}}\left(\overline{X} + i\overline{P}\right) \qquad A^\dagger = \frac{1}{\sqrt{2}}\left(\overline{X} - i\overline{P}\right) \qquad \left[A, A^\dagger\right] = 1$$

$$\overline{H} = A^\dagger A + \frac{1}{2}$$

- Energy eigenvalues:

$$\overline{H}\left|n\right\rangle = \left(n + 1/2\right)\left|n\right\rangle \qquad n = 0,1,2,3,\ldots$$

$$\left\langle n \middle| n\right\rangle = 1 \qquad \left\langle n \middle| n'\right\rangle = 0$$

- Raising and lowering operators:

$$A\left|n\right\rangle = c_n\left|n-1\right\rangle \qquad A^\dagger\left|n\right\rangle = d_n\left|n+1\right\rangle$$

Coefficients c_n and d_n

- Using n instead of ε, we have

$$A|n\rangle = c_n|n-1\rangle \qquad A^\dagger|n\rangle = d_n|n+1\rangle$$

$$\langle n|\overline{H}|n\rangle = n + 1/2$$

$$\langle n|A^\dagger A + 1/2|n\rangle = n + 1/2$$

$A^\dagger A|n\rangle = n|n\rangle$

$N = A^\dagger A$ (number operator)

$$\langle n|A^\dagger A|n\rangle = n$$

$$\langle n|A^\dagger A|n\rangle = n \qquad \langle n|AA^\dagger - 1|n\rangle = n$$

$$|c_n|^2\langle n-1|n-1\rangle = n \qquad \langle n|AA^\dagger|n\rangle = n+1$$

$$c_n = \sqrt{n} \qquad |d_n|^2\langle n+1|n+1\rangle = n+1$$

$$d_n = \sqrt{n+1}$$

$$A|n\rangle = \sqrt{n}|n-1\rangle \qquad A^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

Must memorize

How to Find Wavefunctions?

- Let us define:

$$\psi_n(x) = \langle x | n \rangle$$

- Let's start simple and try to find the ground state wavefunction:

$$\psi_0(x) = \langle x | 0 \rangle$$

- An equation involving only $|0\rangle$ is:

$$A|0\rangle = 0$$

- We can try to use this somehow:

$$\langle x | A | 0 \rangle = 0$$

- We can write A in terms of X and P:

$$A = \frac{1}{\sqrt{2}}\left(\frac{X}{\lambda} + i\frac{\lambda}{\hbar}P\right)$$

- Which gives:

$$\frac{1}{\sqrt{2}}\langle x | \left(\frac{X}{\lambda} + i\frac{\lambda}{\hbar}P\right) | 0 \rangle = 0$$

$$\frac{x}{\lambda}\psi_0(x) + \lambda\frac{d}{dx}\psi_0(x) = 0$$

Ground State Wavefunction

$$\frac{x}{\lambda}\psi_0(x) + \lambda\frac{d}{dx}\psi_0(x) = 0$$

- We can integrate this equation:

$$\frac{d}{dx}\psi_0(x) = -\frac{x}{\lambda^2}\psi_0(x)$$

$$\frac{1}{\psi_0(x)}d\psi_0(x) = -\frac{x}{\lambda^2}dx$$

$$\ln\psi_0(x) = -\frac{x^2}{2\lambda^2} + C$$

$$\psi_0(x) = N_0 e^{-\frac{x^2}{2\lambda^2}}$$

Ground state is a Gaussian of width λ

- Since we are familiar with Gaussians, we know that

$$N_0 = \left[\sqrt{\pi}\lambda\right]^{-1/2}$$

$$\psi_0(x) = \left[\sqrt{\pi}\lambda\right]^{-1/2} e^{-\frac{x^2}{2\lambda^2}}$$

First excited state:

- An equation relating $|1\rangle$ to $|0\rangle$ is: $|1\rangle = A^\dagger |0\rangle$

$$A^\dagger = \frac{1}{\sqrt{2}}\left(\frac{X}{\lambda} - i\frac{\lambda}{\hbar}P\right)$$

$$|1\rangle = \frac{1}{\sqrt{2}}\left(\frac{X}{\lambda} - i\frac{\lambda}{\hbar}P\right)|0\rangle$$

$$\langle x|1\rangle = \frac{1}{\sqrt{2}}\langle x|\left(\frac{X}{\lambda} - i\frac{\lambda}{\hbar}P\right)|0\rangle$$

$$\psi_1(x) = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - \lambda\frac{d}{dx}\right)\psi_0(x)$$

$$\psi_1(x) = \frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - \lambda\frac{d}{dx}\right)\left[\sqrt{\pi}\lambda\right]^{-1/2} e^{-\frac{x^2}{2\lambda^2}}$$

$$\psi_1(x) = \left[2\sqrt{\pi}\lambda\right]^{-1/2}\left(\frac{x}{\lambda} + \lambda\frac{2x}{2\lambda^2}\right)e^{-\frac{x^2}{2\lambda^2}}$$

$$\psi_1(x) = \left[2\sqrt{\pi}\lambda\right]^{-1/2} 2\frac{x}{\lambda} e^{-\frac{x^2}{2\lambda^2}}$$

Already properly normalized!

Creating multiple excitations

- We can always write $|n\rangle$ in terms of $|0\rangle$:

$$A^\dagger|n-1\rangle = \sqrt{n}|n\rangle$$

$$|n\rangle = \frac{A^\dagger}{\sqrt{n}}|n-1\rangle$$

$$|n\rangle = \frac{A^\dagger}{\sqrt{n}}\frac{A^\dagger}{\sqrt{n-1}}|n-2\rangle = \frac{\left(A^\dagger\right)^2}{\sqrt{n(n-1)}}|n-2\rangle$$

$$|n\rangle = \frac{A^\dagger}{\sqrt{n}}\frac{A^\dagger}{\sqrt{n-1}}\frac{A^\dagger}{\sqrt{n-2}}|n-3\rangle = \frac{\left(A^\dagger\right)^3}{\sqrt{n(n-1)(n-2)}}|n-3\rangle$$

$$\vdots$$

$$|n\rangle = \frac{\left(A^\dagger\right)^n}{\sqrt{n!}}|0\rangle$$

- Each time we act with $A^\dagger$ we increase the energy by $\hbar\omega$

- We call $A^\dagger$ the ‘creation operator’ because it creates a ‘quanta’ of energy

- Similarly, we call A an ‘annihilation operator’ because it removes a ‘quanta’ of energy

Wavefunction of n^{th} level

- Starting from: $$|n\rangle = \frac{\left(A^\dagger\right)^n}{\sqrt{n!}}|0\rangle$$

- We can hit with $\langle x|$ to get:

$$\langle x|n\rangle = \langle x|\frac{\left(A^\dagger\right)^n}{\sqrt{n!}}|0\rangle$$

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}}\left(\frac{x}{\lambda} - \lambda\frac{d}{dx}\right)\right)^n \psi_0(x)$$

$$\psi_n(x) = \left[\sqrt{\pi}\,2^n n!\lambda\right]^{-1/2}\left(\frac{x}{\lambda} - \lambda\frac{d}{dx}\right)^n e^{-\frac{x^2}{2\lambda^2}}$$

- The Hermite Polynomials are defined via:

$$H_n(x) = e^{\frac{x^2}{2}}\left(x - \frac{d}{dx}\right)^n e^{-\frac{x^2}{2}}$$

$$\psi_n(x) = \left[\sqrt{\pi}\,2^n n!\lambda\right]^{-1/2} H_n(x/\lambda)\, e^{-\frac{x^2}{2\lambda^2}}$$

Recursion Relation

- In practice, it is not practical for a computer to compute high *n* wavefunctions by differentiation

- Instead, an algorithm which relies only on multiplication is preferred

- To eliminate differentiation, we need to find an equation which does not contain P

- We can use the defining equation for X:

$$\langle x | X = x \langle x |$$

- Hit with $|n\rangle$ from right to get:

$$\langle x | X | n \rangle = x \psi_n(x)$$

- Express X in terms of A and $A^\dagger$:

$$X = \frac{\lambda}{\sqrt{2}}\left(A + A^\dagger\right)$$

$$\frac{\lambda}{\sqrt{2}} \langle x | A + A^\dagger | n \rangle = x \psi_n(x)$$

Recursion relation cont.

$$\frac{\lambda}{\sqrt{2}}\langle x|A + A^{\dagger}|n\rangle = x\psi_n(x)$$

- Use the relations:

$$A|n\rangle = \sqrt{n}|n-1\rangle$$

$$A^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$\frac{\lambda}{\sqrt{2}}\left(\sqrt{n}\psi_{n-1}(x) + \sqrt{n+1}\psi_{n+1}(x)\right) = x\psi_n(x)$$

- Let $n \rightarrow n\text{-}1$:

$$\frac{\lambda}{\sqrt{2}}\left(\sqrt{n-1}\psi_{n-2}(x) + \sqrt{n}\psi_n(x)\right) = x\psi_{n-1}(x)$$

- Solve for $\psi_n(x)$:

$$\psi_n(x) = \sqrt{\frac{2}{n}}\frac{x}{\lambda}\psi_{n-1}(x) - \sqrt{\frac{n-1}{n}}\psi_{n-2}(x)$$

- Iterate, starting from:

$$\psi_0(x) = \left[\sqrt{\pi}\lambda\right]^{1/2} e^{-\frac{x^2}{2\lambda^2}} \qquad \psi_1(x) = \left[2\sqrt{\pi}\lambda\right]^{1/2} 2\frac{x}{\lambda} e^{-\frac{x^2}{2\lambda^2}}$$

Summary

- Todays main results:

$$A|n\rangle = \sqrt{n}|n-1\rangle$$

$$A^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$|n\rangle = \frac{\left(A^\dagger\right)^n}{\sqrt{n!}}|0\rangle$$

$$\psi_n(x) = \left[\sqrt{\pi}\,2^n\,n!\,\lambda\right]^{-1/2} H_n\left(x/\lambda\right) e^{-\frac{x^2}{2\lambda^2}}$$

$$A|0\rangle = 0 \quad \Rightarrow \quad \psi_0(x) = \left[\sqrt{\pi}\lambda\right]^{-1/2} e^{-\frac{x^2}{2\lambda^2}}$$

$$A^\dagger|0\rangle = |1\rangle \quad \Rightarrow \quad \psi_1(x) = \left[2\sqrt{\pi}\lambda\right]^{-1/2} 2\frac{x}{\lambda} e^{-\frac{x^2}{2\lambda^2}}$$

$$\langle x|X|n-1\rangle = x\psi_{n-1}(x)$$

$$\psi_n(x) = \sqrt{\frac{2}{n}}\frac{x}{\lambda}\psi_{n-1}(x) - \sqrt{\frac{n-1}{n}}\psi_{n-2}(x)$$

Example Problem:

- For the n^{th} harmonic oscillator energy eigenstate $|n\rangle$, what is the position uncertainty ΔX?

$$\Delta X^2 = \langle X^2\rangle - \langle X\rangle^2$$

$$\langle X\rangle = \langle n|X|n\rangle \qquad X = \frac{\lambda}{\sqrt{2}}(A + A^\dagger)$$

$$= \frac{\lambda}{\sqrt{2}}\left(\langle n|A|n\rangle + \langle n|A^\dagger|n\rangle\right)$$

$$= \frac{\lambda}{\sqrt{2}}\left(\sqrt{n}\langle n|n-1\rangle + \sqrt{n+1}\langle n|n+1\rangle\right)$$

$$= 0$$

$$\langle X^2\rangle = \langle n|X^2|n\rangle$$

$$= \frac{\lambda^2}{2}\Big(\langle n|AA|n\rangle + \langle n|AA^\dagger|n\rangle + \langle n|A^\dagger A|n\rangle + \langle n|A^\dagger A^\dagger|n\rangle\Big) \qquad AA^\dagger = A^\dagger A + 1$$

$$= \frac{\lambda^2}{2}\left(2\langle n|A^\dagger A|n\rangle + 1\right)$$

$$= \frac{\lambda^2}{2}(2n+1) = \lambda^2\left(n+\tfrac{1}{2}\right)$$

$$\boxed{\Delta X = \lambda\sqrt{n+1/2}}$$