

Lecture 20:
Quantum SHO: Part 2

Phy851 Fall 2009

Recap

- Introduced dimensionless variables:

$$\bar{X} = \frac{X}{\lambda} \quad \bar{P} = \frac{\lambda}{\hbar} P \quad \bar{H} = \frac{H}{\hbar\omega} \quad \lambda = \sqrt{\frac{\hbar}{m\omega}}$$

$$\bar{H} = \frac{1}{2} \bar{P}^2 + \frac{1}{2} \bar{X}^2$$

- Introduce 'normal variables':

$$A = \frac{1}{\sqrt{2}} (\bar{X} + i\bar{P}) \quad A^\dagger = \frac{1}{\sqrt{2}} (\bar{X} - i\bar{P}) \quad [A, A^\dagger] = 1$$

$$\bar{H} = A^\dagger A + \frac{1}{2}$$

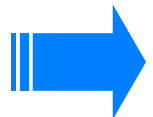
- Energy eigenvalues:

$$\bar{H}|n\rangle = (n + 1/2)\hbar\omega |n\rangle \quad n = 0, 1, 2, 3, \dots$$

$$\langle n|n\rangle = 1 \quad \langle n|n'\rangle = 0$$

- Raising and lowering operators:

$$A|n\rangle = c_n |n-1\rangle \quad A^\dagger|n\rangle = d_n |n+1\rangle$$



Coefficients c_n and d_n

- Using n instead of ε , we have

$$A|n\rangle = c_n|n-1\rangle \quad A^\dagger|n\rangle = d_n|n+1\rangle$$

$$\langle n|\bar{H}|n\rangle = n + 1/2$$

$$\langle n|A^\dagger A + 1/2|n\rangle = n + 1/2$$

$$A^\dagger A|n\rangle = n|n\rangle$$

$$N = A^\dagger A \text{ (number operator)}$$

$$\langle n|A^\dagger A|n\rangle = n$$

$$\langle n|A^\dagger A|n\rangle = n$$

$$\langle n|AA^\dagger - 1|n\rangle = n$$

$$|c_n|^2 \langle n-1|n-1\rangle = n$$

$$\langle n|AA^\dagger|n\rangle = n + 1$$

$$c_n = \sqrt{n}$$

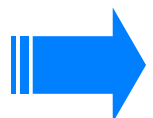
$$|d_n|^2 \langle n+1|n+1\rangle = n + 1$$

$$d_n = \sqrt{n+1}$$

$$A|n\rangle = \sqrt{n}|n-1\rangle$$

$$A^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

Must memorize



How to Find Wavefunctions?

- Let us define:

$$\psi_n(x) = \langle x | n \rangle$$

- Let's start simple and try to find the ground state wavefunction:

$$\psi_0(x) = \langle x | 0 \rangle$$

- An equation involving only $|0\rangle$ is:

$$A|0\rangle = 0$$

- We can try to use this somehow:

$$\langle x | A | 0 \rangle = 0$$

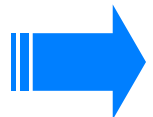
- We can write A in terms of X and P :

$$A = \frac{1}{\sqrt{2}} \left(\frac{X}{\lambda} + i \frac{\lambda}{\hbar} P \right)$$

- Which gives:

$$\frac{1}{\sqrt{2}} \langle x | \left(\frac{X}{\lambda} + i \frac{\lambda}{\hbar} P \right) | 0 \rangle = 0$$

$$\frac{x}{\lambda} \psi_0(x) + \lambda \frac{d}{dx} \psi_0(x) = 0$$



Ground State Wavefunction

$$\frac{x}{\lambda}\psi_0(x) + \lambda \frac{d}{dx}\psi_0(x) = 0$$

- We can integrate this equation:

$$\frac{d}{dx}\psi_0(x) = -\frac{x}{\lambda^2}\psi_0(x)$$

$$\frac{1}{\psi_0(x)} d\psi_0(x) = -\frac{x}{\lambda^2} dx$$

$$\ln\psi_0(x) = -\frac{x^2}{2\lambda^2} + C$$

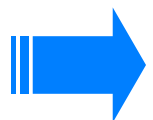
$$\psi_0(x) = N_0 e^{-\frac{x^2}{2\lambda^2}}$$

Ground state
is a Gaussian
of width λ

- Since we are familiar with Gaussians, we know that

$$N_0 = \left[\sqrt{\pi} \lambda \right]^{1/2}$$

$$\psi_0(x) = \left[\sqrt{\pi} \lambda \right]^{1/2} e^{-\frac{x^2}{2\lambda^2}}$$



First excited state:

- An equation relating $|1\rangle$ to $|0\rangle$ is: $|1\rangle = A^\dagger |0\rangle$

$$A^\dagger = \frac{1}{\sqrt{2}} \left(\frac{X}{\lambda} - i \frac{\lambda}{\hbar} P \right)$$

$$|1\rangle = \frac{1}{\sqrt{2}} \left(\frac{X}{\lambda} - i \frac{\lambda}{\hbar} P \right) |0\rangle$$

$$\langle x|1\rangle = \frac{1}{\sqrt{2}} \langle x| \left(\frac{X}{\lambda} - i \frac{\lambda}{\hbar} P \right) |0\rangle$$

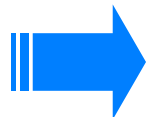
$$\psi_1(x) = \frac{1}{\sqrt{2}} \left(\frac{x}{\lambda} - \lambda \frac{d}{dx} \right) \psi_0(x)$$

$$\psi_1(x) = \frac{1}{\sqrt{2}} \left(\frac{x}{\lambda} - \lambda \frac{d}{dx} \right) [\sqrt{\pi\lambda}]^{1/2} e^{-\frac{x^2}{2\lambda^2}}$$

$$\psi_1(x) = [2\sqrt{\pi\lambda}]^{1/2} \left(\frac{x}{\lambda} + \lambda \frac{2x}{2\lambda^2} \right) e^{-\frac{x^2}{2\lambda^2}}$$

$$\psi_1(x) = [2\sqrt{\pi\lambda}]^{1/2} 2 \frac{x}{\lambda} e^{-\frac{x^2}{2\lambda^2}}$$

Already properly normalized!



Creating multiple excitations

- We can always write $|n\rangle$ in terms of $|0\rangle$:

$$A^\dagger |n-1\rangle = \sqrt{n} |n\rangle$$


$$|n\rangle = \frac{A^\dagger}{\sqrt{n}} |n-1\rangle$$

$$|n\rangle = \frac{A^\dagger}{\sqrt{n}} \frac{A^\dagger}{\sqrt{n-1}} |n-2\rangle = \frac{(A^\dagger)^2}{\sqrt{n(n-1)}} |n-2\rangle$$

$$|n\rangle = \frac{A^\dagger}{\sqrt{n}} \frac{A^\dagger}{\sqrt{n-1}} \frac{A^\dagger}{\sqrt{n-2}} |n-3\rangle = \frac{(A^\dagger)^3}{\sqrt{n(n-1)(n-2)}} |n-3\rangle$$

⋮

$$|n\rangle = \frac{(A^\dagger)^n}{\sqrt{n!}} |0\rangle$$

- Each time we act with A^\dagger we increase the energy by $\hbar\omega$
- We call A^\dagger the 'creation operator' because it creates a 'quanta' of energy
- Similarly, we call A an 'annihilation operator' because it removes a 'quanta' of energy 

Wavefunction of n^{th} level

- Starting from: $|n\rangle = \frac{(A^\dagger)^n}{\sqrt{n!}} |0\rangle$

- We can hit with $\langle x|$ to get:

$$\langle x|n\rangle = \langle x| \frac{(A^\dagger)^n}{\sqrt{n!}} |0\rangle$$

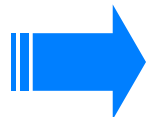
$$\psi_n(x) = \frac{1}{\sqrt{n!}} \left(\frac{1}{\sqrt{2}} \left(\frac{x}{\lambda} - \lambda \frac{d}{dx} \right) \right)^n \psi_0(x)$$

$$\psi_n(x) = \left[\sqrt{\pi} 2^n n! \lambda \right]^{1/2} \left(\frac{x}{\lambda} - \lambda \frac{d}{dx} \right)^n e^{-\frac{x^2}{2\lambda^2}}$$

- The **Hermite Polynomials** are defined via:

$$H_n(x) = e^{\frac{x^2}{2}} \left(x - \frac{d}{dx} \right)^n e^{-\frac{x^2}{2}}$$

$$\psi_n(x) = \left[\sqrt{\pi} 2^n n! \lambda \right]^{1/2} H_n(x/\lambda) e^{-\frac{x^2}{2\lambda^2}}$$



Recursion Relation

- In practice, it is not practical for a computer to compute high n wavefunctions by differentiation
- Instead, an algorithm which relies only on multiplication is preferred
- To eliminate differentiation, we need to find an equation which does not contain P
- We can use the defining equation for X :

$$\langle x|X = x\langle x|$$

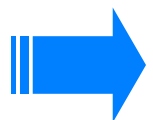
- Hit with $|n\rangle$ from right to get:

$$\langle x|X|n\rangle = x\psi_n(x)$$

- Express X in terms of A and A^\dagger :

$$X = \frac{\lambda}{\sqrt{2}}(A + A^\dagger)$$

$$\frac{\lambda}{\sqrt{2}}\langle x|A + A^\dagger|n\rangle = x\psi_n(x)$$



Recursion relation cont.

$$\frac{\lambda}{\sqrt{2}} \langle x | A + A^\dagger | n \rangle = x \psi_n(x)$$

- Use the relations:

$$A | n \rangle = \sqrt{n} | n-1 \rangle$$

$$A^\dagger | n \rangle = \sqrt{n+1} | n+1 \rangle$$

$$\frac{\lambda}{\sqrt{2}} \left(\sqrt{n} \psi_{n-1}(x) + \sqrt{n+1} \psi_{n+1}(x) \right) = x \psi_n(x)$$


- Let $n \rightarrow n-1$:

$$\frac{\lambda}{\sqrt{2}} \left(\sqrt{n-1} \psi_{n-2}(x) + \sqrt{n} \psi_n(x) \right) = x \psi_{n-1}(x)$$

- Solve for $\psi_n(x)$:

$$\psi_n(x) = \sqrt{\frac{2}{n}} \frac{x}{\lambda} \psi_{n-1}(x) - \sqrt{\frac{n-1}{n}} \psi_{n-2}(x)$$

- Iterate, starting from:

$$\psi_0(x) = \left[\sqrt{\pi \lambda} \right]^{1/2} e^{-\frac{x^2}{2\lambda^2}} \quad \psi_1(x) = \left[2\sqrt{\pi \lambda} \right]^{1/2} 2 \frac{x}{\lambda} e^{-\frac{x^2}{2\lambda^2}}$$


Summary

- Today's main results:

$$A|n\rangle = \sqrt{n}|n-1\rangle$$

$$A^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$|n\rangle = \frac{(A^\dagger)^n}{\sqrt{n!}}|0\rangle$$

$$\psi_n(x) = \left[\sqrt{\pi} 2^n n! \lambda \right]^{-1/2} H_n(x/\lambda) e^{-\frac{x^2}{2\lambda^2}}$$

$$A|0\rangle = 0 \quad \Rightarrow \quad \psi_0(x) = \left[\sqrt{\pi} \lambda \right]^{-1/2} e^{-\frac{x^2}{2\lambda^2}}$$

$$A^\dagger|0\rangle = |1\rangle \quad \Rightarrow \quad \psi_1(x) = \left[2\sqrt{\pi} \lambda \right]^{-1/2} 2 \frac{x}{\lambda} e^{-\frac{x^2}{2\lambda^2}}$$

$$\langle x|X|n-1\rangle = x\psi_{n-1}(x)$$

$$\psi_n(x) = \sqrt{\frac{2}{n}} \frac{x}{\lambda} \psi_{n-1}(x) - \sqrt{\frac{n-1}{n}} \psi_{n-2}(x)$$

Example Problem:

- For the n^{th} harmonic oscillator energy eigenstate $|n\rangle$, what is the position uncertainty ΔX ?

$$\Delta X^2 = \langle X^2 \rangle - \langle X \rangle^2$$

$$\langle X \rangle = \langle n | X | n \rangle \quad X = \frac{\lambda}{\sqrt{2}} (A + A^\dagger)$$

$$= \frac{\lambda}{\sqrt{2}} (\langle n | A | n \rangle + \langle n | A^\dagger | n \rangle)$$

$$= \frac{\lambda}{\sqrt{2}} (\sqrt{n} \langle n | n-1 \rangle + \sqrt{n+1} \langle n | n+1 \rangle)$$

$$= 0$$

$$\langle X^2 \rangle = \langle n | X^2 | n \rangle$$

$$= \frac{\lambda^2}{2} (\langle n | \cancel{A} A | n \rangle + \langle n | A A^\dagger | n \rangle + \langle n | A^\dagger A | n \rangle + \langle n | \cancel{A^\dagger} A^\dagger | n \rangle) \quad A A^\dagger = A^\dagger A + 1$$

$$= \frac{\lambda^2}{2} (2 \langle n | A^\dagger A | n \rangle + 1)$$

$$= \frac{\lambda^2}{2} (2n + 1) = \lambda^2 (n + \frac{1}{2})$$

$$\boxed{\Delta X = \lambda \sqrt{n + \frac{1}{2}}}$$

