Parity inversion

- Symmetry under parity inversion is known as mirror symmetry

\[
P: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}
\]

- Formally, we say that \( f(x) \) is symmetric under parity inversion if \( f(-x) = f(x) \)

- We would say that \( f(x) \) is antisymmetric under parity inversion if \( f(-x) = -f(x) \)

- The universe is not symmetric under parity inversion (beta decay)
  - Unless there is mirror matter (and mirror photons)
  - Would interact only weakly with matter via gravity
Parity Operator

- Let us define the parity operator via:
  \[ \Pi |x\rangle = |-x\rangle \]

- Parity operator is Hermitian:
  \[ \langle x | \Pi | x' \rangle = \langle x | - x' \rangle = \delta(x + x') \]
  \[ \langle x' | \Pi | x \rangle^* = \langle x' | - x \rangle^* = \delta(x + x') \]
  \[ \Pi^\dagger = \Pi \]

- Parity operator is its own inverse
  \[ \Pi \Pi |x\rangle = \Pi |-x\rangle = |x\rangle \]
  \[ \Pi^2 = 1 \]

- Thus it must be Unitary as well
  \[ \Pi^\dagger = \Pi \]
  \[ \Pi = \Pi^{-1} \]
  \[ \Pi^\dagger = \Pi^{-1} \]
Properties of the Parity operator

• Parity acting to the left:

\[ \langle x | \Pi^\dagger = (\Pi | x)\rangle^\dagger = | -x \rangle^\dagger = \langle -x | \]

\[ \langle x | \Pi = \langle -x | \]

• What is the action of the parity operator on a generic quantum state?
  - Let: \[ |\psi'\rangle = \Pi |\psi\rangle \]

\[ \langle x | \psi' \rangle = \langle x | \Pi |\psi\rangle \]

\[ \langle x | \psi' \rangle = \langle -x |\psi \rangle \]

\[ \psi'(x) = \psi(-x) \]

\[ \psi'(-x) = \psi(x) \]

• Under parity inversion, we would say:

\[ \psi'(x') = \psi(x) \]
Eigenstates of Parity Operator

• What are the eigenstates of parity?
  – What states have well-defined parity?
  – Answer: even/odd states

• Proof:
  – Let: \[ \Pi |\pi\rangle = \pi |\pi\rangle \]
  – It follows that:
    \[ \Pi^2 |\pi\rangle = \pi^2 |\pi\rangle \]
  – But \( \Pi^2 = 1 \), which gives:
    \[ |\pi\rangle = \pi^2 |\pi\rangle \]
    \[ \pi^2 = 1 \]
    \[ \pi = \pm 1 \]

\[ \begin{align*}
\pi = +1 & \quad \Rightarrow \quad \langle x | \Pi | + \rangle = \langle x | + \rangle \\
\langle -x | + \rangle &= \langle x | + \rangle \\
\langle x | - \rangle &= -\langle x | - \rangle \\
\langle -x | - \rangle &= -\langle x | - \rangle
\end{align*} \]

Any Even function! Any Odd function!
Parity acting on Momentum states

\[ \Pi | p \rangle = \int dx \Pi | x \rangle \langle x | p \rangle \]
\[ = \int dx | -x \rangle \langle x | p \rangle \]
\[ = \int dx | x \rangle \langle -x | p \rangle \]

\[ \langle -x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}px} = \langle x | -p \rangle \]

\[ \Pi | p \rangle = \int dx | x \rangle \langle x | - p \rangle \]

\[ \Pi | p \rangle = | - p \rangle \]
Commutator of $X$ with $\Pi$

- First we can compute $\Pi X \Pi$:

$$\langle x | \Pi X \Pi | \psi \rangle = \langle -x | X \Pi | \psi \rangle$$

$$= -x \langle -x | \Pi | \psi \rangle$$

$$= -x \langle x | \psi \rangle$$

$$= -\langle x | X | \psi \rangle$$

$$\Pi X \Pi = -X$$

$$\Pi X \Pi^2 = -X \Pi$$

$$\Pi X = -X \Pi$$

$$\Pi X - X \Pi = -2X \Pi$$

- So $\Pi$ and $X$ do not commute.
Commutator of X with \( \Pi \)

- Next we can compute \([x^2, \Pi]\):

\[
\langle x | \Pi x^2 \Pi | \psi \rangle = \langle -x | X^2 \Pi | \psi \rangle = x^2 \langle -x | \Pi | \psi \rangle = x^2 \langle x | \psi \rangle = \langle x | X^2 | \psi \rangle
\]

\[
\Pi x^2 \Pi = X^2
\]

\[
\Pi X^2 \Pi^2 = X^2 \Pi
\]

\[
\Pi X^2 = X^2 \Pi
\]

\[
\Pi X^2 - X^2 \Pi = 0
\]

- So \( \Pi \) and \( X^2 \) do commute!
Commutator with Hamiltonian

• Same results must apply for $P$ and $P^2$, as the relation between $\Pi$ and $P$ is the same as between $\Pi$ and $X$.

• Thus

$$\left[ \Pi, \frac{P^2}{2M} \right] = 0$$

• If $\Pi$ commutes with $X^2$, then $\Pi$ commutes with any even function of $X$

$$\left[ \Pi, V_{\text{even}}(X^2) \right] = 0$$

• Let

$$H = \frac{P^2}{2M} + V_{\text{even}}(X)$$

• Then

$$\left[ \Pi, H \right] = 0$$

• This means that simultaneous eigenstates of $H$ and $P$ exist
Consequences for a free particle

- The Hamiltonian of a free particle is:
  \[ H = \frac{P^2}{2M} \]

- Energy eigenstates are doubly-degenerate:
  \[ H\left| k \right\rangle = \frac{\hbar^2 k^2}{2M} \quad \text{and} \quad H\left| -k \right\rangle = \frac{\hbar^2 k^2}{2M} \]

\[ \left| E,1 \right\rangle := \left| k \right\rangle \bigg|_{k = \frac{\sqrt{2ME}}{\hbar}} \quad \left| E,2 \right\rangle := \left| k \right\rangle \bigg|_{k = -\frac{\sqrt{2ME}}{\hbar}} \]

- Note that plane waves, \( |k\rangle \), are eigenstates of momentum and energy, but NOT parity

- But \( [H,\Pi] = 0 \), so eigenstates of energy and parity must exist
  \[ \left| E,+,\right\rangle := \frac{1}{\sqrt{2}} \left( \left| E,1 \right\rangle + \left| E,2 \right\rangle \right) \]
  \[ \left| E,-\right\rangle := \frac{1}{\sqrt{2}} \left( \left| E,1 \right\rangle - \left| E,2 \right\rangle \right) \]
Consequences for the SHO

• For the SHO we have:

\[
H = \frac{P^2}{2M} + \frac{1}{2} M \omega^2 X^2
\]

• Therefore \([H, \Pi]=0\), so simultaneous eigenstates of Energy and Parity must exist

• The energy levels are not-degenerate, so there is no freedom to mix and match states

• Thus the only possibility is that each energy level must have definite parity

• The Hermite Polynomials have definite parity: \(H_n(-x)=(-1)^n H_n(x)\)

• Thus we have:

\[
\Pi |n\rangle = (-1)^n |n\rangle
\]