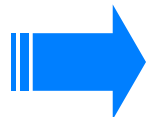


Lecture 24:
Tensor Product States

Phy851 Fall 2009



Basis sets for a particle in 3D

- Clearly the Hilbert space of a particle in three dimensions is not the same as the Hilbert space for a particle in one-dimension
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- In one dimension, X and P are *incompatible*
 - If you specify the wave function in coordinate-space, $\langle x|\psi\rangle$, its momentum-space state is completely specified as well:
 $\langle p|\psi\rangle = \int dx \langle p|x\rangle \langle x|\psi\rangle$
 - You thus specify a state by assigning an amplitude to every possible position **OR** by assigning an amplitude to every possible momentum

$$\psi(x) = \langle x|\psi\rangle \quad \text{or} \quad \psi(p) = \langle p|\psi\rangle$$

- In three dimensions, X , Y , and Z , are *compatible*.
 - Thus, to specify a state, you must assign an amplitude to each possible position in three dimensions.
 - This requires three quantum numbers

$$\psi(x, y, z) = \langle x, y, z|\psi\rangle$$

- So apparently, one basis set is:

$$\{|x, y, z\rangle\}$$

$$\exists |x, y, z\rangle \quad \forall (x, y, z) \in R^3$$



Definition of Tensor product

- Suppose you have a system with 10 possible states
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- Now you want to enlarge your system by adding ten more states to its Hilbert space.
 - The dimensionality of the Hilbert space increases from 10 to 20
 - The system can now be found in one of 20 possible states
 - This is a sum of two Hilbert sub-spaces
 - One quantum number is required to specify which state
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- Instead, suppose you want to combine your system with a second system, which has ten states of its own
 - The first system can be in 1 of its 10 states
 - The second system can be in 1 of *its* 10 states
 - *The state of the second system is independent of the state of the first system*
 - So two independent quantum numbers are required to specify the combined state
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- The dimensionality of the combined Hilbert space thus goes from 10 to $10 \times 10 = 100$
 - This combined Hilbert space is a *(Tensor) Product* of the two Hilbert sub-spaces



Formalism

- Let \mathcal{H}_1 and \mathcal{H}_2 be two Hilbert spaces
 - We will temporarily 'tag' states with a label to specify which space the state belongs to

$$|\psi\rangle^{(1)} \in \mathcal{H}_1 \quad |\phi\rangle^{(2)} \in \mathcal{H}_2$$

- Let the Hilbert space \mathcal{H}_{12} be the tensor-product of spaces \mathcal{H}_1 and \mathcal{H}_2 .

$$\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

- The Tensor product state $|\psi_{12}\rangle = |\psi_1\rangle^{(1)} \otimes |\psi_2\rangle^{(2)}$ belongs to \mathcal{H}_{12} .
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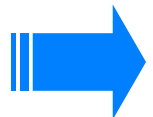
- **The KEY POINT TO 'GET' IS:**

- *Bras* and *kets* in the same Hilbert space **'attach'**.

$$\langle \psi_1 |^{(1)} \rightarrow \leftarrow | \psi_2 \rangle^{(1)} \Rightarrow \langle \psi_1 | \psi_2 \rangle$$

- BUT, *Bras* and *kets* in different Hilbert spaces **do not 'attach'**

$$\langle \psi_1 |^{(1)} \rightarrow \leftarrow | \varphi_1 \rangle^{(2)} \Rightarrow | \varphi_1 \rangle^{(2)} \langle \psi_1 |^{(1)}$$



Schmidt Basis

- The easiest way to find a good basis for a tensor product space is to use tensor products of basis states from each sub-space
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If: $\{|n_1\rangle^{(1)}\}$; $n_1=1,2,\dots,N_1$ is a basis in \mathcal{H}_1

$\{|n_2\rangle^{(2)}\}$; $n_2=1,2,\dots,N_2$ is a basis in \mathcal{H}_2

It follows that:

$\{|n_1, n_2\rangle^{(12)}\}$; $|n_1, n_2\rangle^{(12)} = |n_1\rangle^{(1)} \otimes |n_2\rangle^{(2)}$ is a basis in \mathcal{H}_{12} .

If System 1 is in state: $|\psi_1\rangle^{(1)} = \sum_{n_1=1}^{N_1} a_{n_1} |n_1\rangle^{(1)}$

and System 2 is in state: $|\psi_2\rangle^{(2)} = \sum_{n_2=1}^{N_2} b_{n_2} |n_2\rangle^{(2)}$

Then the combined system is in state:

$$|\psi_1, \psi_2\rangle^{(12)} = |\psi_1\rangle^{(1)} \otimes |\psi_2\rangle^{(2)} = \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} a_{n_1} b_{n_2} |n_1, n_2\rangle^{(12)}$$

- **Schmidt Decomposition Theorem:**

All states in a tensor-product space can be expressed as a linear combination of tensor product states

$$|\psi_{12}\rangle^{(12)} = \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} c_{n_1 n_2} |n_1, n_2\rangle^{(12)}$$



Entangled States

- The essence of quantum 'weirdness' lies in the fact that *there exist states* in the tensor-product space of *physically distinct systems* that are *not tensor product states*
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- A tensor-product state is of the form

$$|\psi\rangle^{(12)} = |\psi_1\rangle^{(1)} \otimes |\psi_2\rangle^{(2)}$$

- Tensor-product states are called 'factorizable'

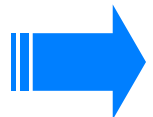
- The most general state is

$$|\psi_{12}\rangle^{(12)} = \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} c_{n_1 n_2} |n_1, n_2\rangle^{(12)}$$

- This may or may-not be 'factorizable'
-

- Non-factorizable states are called 'entangled'

- For an 'entangled state', each subsystem has *no independent objective reality*



Configuration Space

- The state of a quantum system of N particles in 3 dimensions lives in 'configuration space'
 - There are three quantum numbers associated with each particle
 - It takes $3N$ quantum numbers to specify a state of the full system
 - Coordinate Basis:

$$\{ |x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_N, y_N, z_N \rangle \}$$

or we could just write

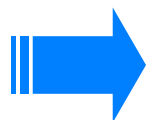
$$\{ | \vec{r}_1, \vec{r}_2, \dots, \vec{r}_N \rangle \}$$

This form is the
coordinate system
independent
representation

- Wavefunction:

$$\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = \langle \vec{r}_1, \vec{r}_2, \dots, \vec{r}_N | \psi \rangle$$

- To specify a state of N particles in d dimensions requires $d \cdot N$ quantum numbers



Tensor Products of Operators

- THEOREM:

Let $A^{(1)}$ act in \mathcal{H}_1 , and $B^{(2)}$ act in \mathcal{H}_2 ,

Then the tensor product operator $C^{(12)} = A^{(1)} \otimes B^{(2)}$ acts in \mathcal{H}_{12} .

- PROOF:

$$A^{(1)} = \sum_m a_m |a_m\rangle^{(1)} \langle a_m|^{(1)}$$

$$B^{(2)} = \sum_m b_m |b_m\rangle^{(2)} \langle b_m|^{(2)}$$

$$\begin{aligned} A^{(1)} \otimes B^{(2)} &= \sum_{m,n} a_m b_n |a_m\rangle^{(1)} \langle a_m|^{(1)} \otimes |b_n\rangle^{(2)} \langle b_n|^{(2)} \\ &= \sum_{m,n} a_m b_n \left(|a_m\rangle^{(1)} \otimes |b_n\rangle^{(2)} \right) \left(\langle a_m|^{(1)} \otimes \langle b_n|^{(2)} \right) \end{aligned}$$

$$C^{(12)} = \sum_{m,n} a_m b_n |a_m, b_n\rangle^{(12)} \langle a_m, b_n|^{(12)}$$

- The action of $C^{(12)}$ on a tensor-product state:

$$|\psi_1, \psi_2\rangle^{(12)} := |\psi_1\rangle^{(1)} \otimes |\psi_2\rangle^{(2)}$$

$$C^{(12)} |\psi_1, \psi_2\rangle^{(12)} = \sum_{m,n} a_m b_n |a_m, b_n\rangle^{(12)} \langle a_m | \psi_1 \rangle^{(1)} \langle b_n | \psi_2 \rangle^{(2)}$$



General form of Operators in Tensor-product spaces

- The most general form of an operator in \mathcal{H}_{12} is:

$$C^{(12)} = \sum_{m,n} c_{m,n;m',n'} |m,n\rangle^{(12)} \langle m',n'|^{(12)}$$

$$c_{m,n;m',n'} := \langle m,n|^{(12)} C^{(12)} |m',n'\rangle^{(12)}$$

- Here $|m,n\rangle$ may or may not be a tensor product state. The important thing is that it takes two quantum numbers to specify a basis state in \mathcal{H}_{12}
 - A basis that is not formed from tensor-product states is an 'entangled-state' basis
- In the beginning, you should always start with a tensor-product basis as your 'physical basis'
 - Then all operators are well-defined
 - Just expand states and operators onto tensor-product states
 - Then match up the bras and kets with their proper partners when taking inner products



Upgrading Subspace Operators

- Any operator in \mathbf{H}_1 can be upgraded to an operator in \mathbf{H}_{12} by taking the tensor product with the identity operator in \mathbf{H}_2 :

$$A^{(12)} = A^{(1)} \otimes I^{(2)}$$

- If A_1 is an observable in \mathbf{H}_1 , then it is also an observable in \mathbf{H}_{12} (since it remains Hermitian when upgraded).
- The spectrum of A_1 remains the same after upgrading

Proof:

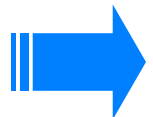
Let

$$A^{(1)} |a_m\rangle^{(1)} = a_m |a_m\rangle^{(1)}$$

Then:

$$\begin{aligned} A^{(1)} \otimes I^{(2)} \left(|a_m\rangle^{(1)} \otimes |\psi\rangle^{(2)} \right) &= \left(A^{(1)} |a_m\rangle^{(1)} \right) \otimes \left(I^{(2)} |\psi\rangle^{(2)} \right) \\ &= a_m |a_m\rangle^{(1)} \otimes |\psi\rangle^{(2)} \\ &= a_m \left(|a_m\rangle^{(1)} \otimes |\psi\rangle^{(2)} \right) \end{aligned}$$

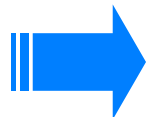
Note that $|\psi_2\rangle$ is completely arbitrary, but $|a_1\rangle \otimes |\psi_2\rangle$ is an eigenstate of $A^{(12)}$



Product of two Upgraded Operators

- Let $A^{(1)}$ and $B^{(2)}$ be observables in their respective Hilbert spaces
- Let $A^{(12)} = A^{(1)} \otimes I^{(2)}$ and $B^{(12)} = I^{(1)} \otimes B^{(2)}$.
- The product $A^{(12)}B^{(12)}$ is given by $A^{(1)} \otimes B^{(2)}$
 - Proof:

$$\begin{aligned}(A_1 \otimes I_2)(I_1 \otimes B_2) &= A_1 I_1 \otimes I_2 B_2 \\ &= A_1 \otimes B_2\end{aligned}$$



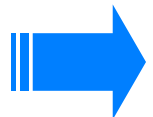
Compatible observables

- Let $A^{(1)}$ and $B^{(2)}$ be observables in their respective Hilbert spaces
- Let $A = A^{(1)} \otimes I^{(2)}$ and $B = I^{(1)} \otimes B^{(2)}$.
- **Theorem:** $[A, B] = 0$

Proof: $[A, B] = AB - BA$

$$\begin{aligned} &= (A_1 \otimes I_2)(I_1 \otimes B_2) - (I_1 \otimes B_2)(A_1 \otimes I_2) \\ &= (A_1 I_1) \otimes (I_2 B_2) - (I_1 A_1) \otimes (B_2 I_2) \\ &= A_1 \otimes B_2 - A_1 \otimes B_2 = 0 \end{aligned}$$

-
- Conclusion: any operator in \mathcal{H}_1 , is 'compatible' with any operator in \mathcal{H}_2 .
 - I.e. simultaneous eigenstates exist.
 - Let $A_1 |a_1\rangle = a_1 |a_1\rangle$ and $B_2 |b_2\rangle = b_2 |b_2\rangle$.
 - Let $a = a_1$ and $b = b_2$
 - Let $|ab\rangle = |a_1\rangle \otimes |b_2\rangle$.
 - Then $AB|ab\rangle = ab|ab\rangle$.



'And' versus 'Or'

- The tensor product correlates with a system having property A **and** property B
 - Dimension of combined Hilbert space is product of dimensions of subspaces associated with A and B
 - **Example:** start with a system having 4 energy levels. Let it interact with a 2 level system. The Hilbert space of the combined system has 8 possible states.
-
- Hilbert spaces are added when a system can have either property A **or** property B
 - Dimension of combined Hilbert space is sum of dimensions of subspaces associated with A and B
 - **Example:** start with a system having 4 energy levels. Add 2 more energy levels to your model, and the dimension goes from 4 to 6



Example #1 : Particle in Three Dimensions

- Let \mathcal{H}_1 be the Hilbert space of functions in one dimension

- The projector is: $I_1 = \int dx |x\rangle\langle x|$

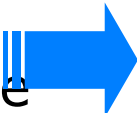
- So a basis is: $\{|x\rangle\}$

- Then $\mathcal{H}_3 = (\mathcal{H}_1)^3$ would then be the Hilbert space of square integrable functions in three dimensions.

- Proof:
$$\begin{aligned} I_3 &= I_1 \otimes I_1 \otimes I_1 \\ &= \int dx |x\rangle\langle x| \otimes \int dy |y\rangle\langle y| \otimes \int dz |z\rangle\langle z| \\ &= \int dx dy dz |x\rangle\langle x| \otimes |y\rangle\langle y| \otimes |z\rangle\langle z| \\ &= \int dx dy dz (|x\rangle \otimes |y\rangle \otimes |z\rangle) (\langle x| \otimes \langle y| \otimes \langle z|) \\ &= \int dx dy dz |x, y, z\rangle\langle x, y, z| \end{aligned}$$

$$|x, y, z\rangle \equiv |x\rangle \otimes |y\rangle \otimes |z\rangle \quad \psi(x, y, z) = \langle x, y, z | \psi \rangle$$

$$|\psi\rangle = \int dx dy dz |x, y, z\rangle \psi(x, y, z)$$

- **Note:** \mathcal{H}_3 is also the Hilbert space of three particles in one-dimensional space 

Three-dimensional Operators

- We can define the vector operators:

$$\vec{R} = X \hat{x} + Y \hat{y} + Z \hat{z}$$

$$\vec{P} = P_x \hat{x} + P_y \hat{y} + P_z \hat{z}$$

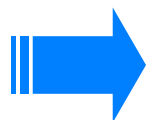
$$\vec{R}|x, y, z\rangle = (x \hat{x} + y \hat{y} + z \hat{z})|x, y, z\rangle$$

$$\text{or } \vec{R}|\vec{r}\rangle = \vec{r}|\vec{r}\rangle$$

- Note that: $X = X^{(1)} \otimes I^{(2)} \otimes I^{(3)}$ and $P_y = I^{(1)} \otimes P^{(2)} \otimes I^{(3)}$ so that $[X, P_y] = 0$.

- With $R_1 = X$ $P_1 = P_x$
 $R_2 = Y$ $P_2 = P_y$
 $R_3 = Z$ $P_3 = P_z$

- We can use: $[R_j, R_k] = [P_j, P_k] = 0$
 $[R_j, P_k] = i\hbar \delta_{jk}$



Example #2: Two particles in One Dimension

- For two particles in one-dimensional space, the Hilbert space is $(\mathcal{H}_1)^2$.

$$|x_1, x_2\rangle = |x_1\rangle^{(1)} \otimes |x_2\rangle^{(2)}$$

$$\psi(x_1, x_2) = \langle x_1, x_2 | \psi \rangle$$

$$I = \int dx_1 dx_2 |x_1, x_2\rangle \langle x_1, x_2| = 1$$

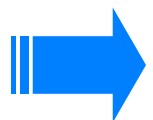
$$X_1 = X_1 \otimes I_2$$

$$P_2 = I_1 \otimes P_2$$

etc...

$$[X_j, X_k] = [P_j, P_k] = 0$$

$$[X_j, P_k] = i\hbar \delta_{jk}$$



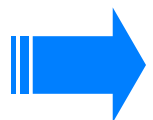
Hamiltonians

- One particle in three dimensions:
 - Each component of momentum contributes additively to the Kinetic Energy

$$H = \frac{1}{2m} (P_x^2 + P_y^2 + P_z^2) + V(X, Y, Z)$$
$$= \frac{1}{2m} \vec{P} \cdot \vec{P} + V(\vec{R})$$

- Two particles in one dimension:

$$H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + V(X_1, X_2)$$



Conclusions

- The 'take home' messages are:
 - The combined Hilbert space of two systems, dimensions d_1 and d_2 , has dimension $d_{1\otimes 2} = d_1 \cdot d_2$
 - A physical basis set for the combined Hilbert space, $\mathcal{H}_{1\otimes 2}$ can be formed by taking all possible products of one basis state from space \mathcal{H}_1 with one basis state from \mathcal{H}_2 .

$$\{|n_1\rangle^{(1)}\} \supseteq \mathcal{H}_1, \quad \{|n_2\rangle^{(2)}\} \supseteq \mathcal{H}_2$$

$$|n_1, n_2\rangle := |n_1\rangle^{(1)} \otimes |n_2\rangle^{(2)}$$

$$\therefore \{|n_1, n_2\rangle\} \supseteq \mathcal{H}_{12} := \mathcal{H}_1 \otimes \mathcal{H}_2$$

- In a tensor product space, a bra from one subspace can only attach to a ket from the same subspace:

$$\langle \psi_1 |^{(1)} \rightarrow \leftarrow | \psi_2 \rangle^{(1)} \Rightarrow \langle \psi_1 | \psi_2 \rangle$$

$$\langle \psi_1 |^{(1)} \rightarrow \leftarrow | \varphi_1 \rangle^{(2)} \Rightarrow | \varphi_1 \rangle^{(2)} \langle \psi_1 |^{(1)}$$

- For N particles (spin 0) in d dimensions, $d \cdot N$ quantum numbers are required to specify a unit-vector in any basis

$$|\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N\rangle$$

$$|n_{1x}, n_{1y}, n_{1z}, n_{2x}, n_{2y}, n_{2z}, \dots, n_{Nx}, n_{Ny}, n_{Nz}\rangle$$

