Lecture 24:
Tensor Product States

Phy851 Fall 2009

Basis sets for a particle in 3D

- Clearly the Hilbert space of a particle in three dimensions is not the same as the Hilbert space for a particle in one-dimension

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- In one dimension, X and P are *incompatible*
  - If you specify the wave function in coordinate-space, $\langle x|\psi\rangle$, its momentum-space state is completely specified as well: $\langle p|\psi\rangle = \int dx \langle p|x\rangle\langle x|\psi\rangle$
  - You thus specify a state by assigning an amplitude to every possible position ***OR*** by assigning and amplitude to every possible momentum

$$\psi(x) = \langle x|\psi\rangle \quad \text{or} \quad \psi(p) = \langle p|\psi\rangle$$

- In three dimensions, X, Y, and Z, are *compatible.*
  - Thus, to specify a state, you must assign an amplitude to each possible position in three dimensions.
  - This requires three quantum numbers

$$\psi(x,y,z) = \langle x,y,z|\psi\rangle$$

  - So apparently, one basis set is:

$$\{|x,y,z\rangle\}$$

$$\exists\, |x,y,z\rangle\ \forall\ (x,y,z) \in R^3$$

Definition of Tensor product

- Suppose you have a system with 10 possible states

- Now you want to enlarge your system by adding ten more states to its Hilbert space.
  - The dimensionality of the Hilbert space increases from 10 to 20
  - The system can now be found in one of 20 possible states
  - This is a sum of two Hilbert sub-spaces
  - One quantum number is required to specify which state

- Instead, suppose you want to combine your system with a second system, which has ten states of its own
  - The first system can be in 1 of its 10 states
  - The second system can be in 1 of *its* 10 states
    - *The state of the second system* is *independent of the state of the first system*
  - So two independent quantum numbers are required to specify the combined state

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- The dimensionality of the combined Hilbert space thus goes from 10 to 10x10=100
  - This combined Hilbert space is a (*Tensor*) *Product* of the two Hilbert sub-spaces

Formalism

- Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces
  - We will temporarily 'tag' states with a label to specify which space the state belongs to

$$|\psi\rangle^{(1)} \in \mathcal{H}_1 \qquad |\phi\rangle^{(2)} \in \mathcal{H}_2$$

- Let the Hilbert space $\mathcal{H}_{12}$ be the tensor-product of spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

$$\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}$$

- The Tensor product state $|\psi_{12}\rangle = |\psi_1\rangle^{(1)} \otimes |\psi_2\rangle^{(2)}$ belongs to $\mathcal{H}_{12}$.

- **The KEY POINT TO 'GET' IS:**
  - *Bras* and *kets* in the same Hilbert space **'attach'**.

$$\langle\psi_1|^{(1)} \rightarrow \leftarrow |\psi_2\rangle^{(1)} \Rightarrow \langle\psi_1|\psi_2\rangle$$

  - BUT, *Bras* and *kets* in different Hilbert spaces ***do not*** **'attach'**

$$\langle\psi_1|^{(1)} \rightarrow \leftarrow |\varphi_1\rangle^{(2)} \Rightarrow |\varphi_1\rangle^{(2)}\langle\psi_1|^{(1)}$$

Schmidt Basis

- The easiest way to find a good basis for a tensor product space is to use tensor products of basis states from each sub-space

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If: $\{|n_1\rangle^{(1)}\}$; n_1=1,2,…,N_1 is a basis in $\mathcal{H}_1$

$\{|n_2\rangle^{(2)}\}$; n_2=1,2,…,N_2 is a basis in $\mathcal{H}_2$

It follows that:

$\{|n_1, n_2\rangle^{(12)}\}$; $|n_1, n_2\rangle^{(12)} = |n_1\rangle^{(1)} \otimes |n_2\rangle^{(2)}$ is a basis in $\mathcal{H}_{12}$.

If System 1 is in state: $\left|\psi_1\right\rangle^{(1)} = \sum_{n_1=1}^{N_1} a_{n_1} \left|n_1\right\rangle^{(1)}$

and System 2 is in state: $\left|\psi_2\right\rangle^{(2)} = \sum_{n_2=1}^{N_2} b_{n_2} \left|n_2\right\rangle^{(2)}$

Then the combined system is in state:

$$\left|\psi_1,\psi_2\right\rangle^{(12)} = \left|\psi_1\right\rangle^{(1)} \otimes \left|\psi_2\right\rangle^{(2)} = \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} a_{n_1} b_{n_2} \left|n_1, n_2\right\rangle^{(12)}$$

- **Schmidt Decomposition Theorem:**

All states in a tensor-product space can be expressed as a linear combination of tensor product states

$$\left|\psi_{12}\right\rangle^{(12)} = \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} c_{n_1 n_2} \left|n_1, n_2\right\rangle^{(12)}$$

Entangled States

- The essence of quantum `weirdness' lies in the fact that *there exist states* in the tensor-product space of *physically distinct systems* that are *not tensor product states*

- A tensor-product state is of the form

$$|\psi\rangle^{(12)} = |\psi_1\rangle^{(1)} \otimes |\psi_2\rangle^{(2)}$$

  - Tensor-product states are called 'factorizable'

- The most general state is

$$|\psi_{12}\rangle^{(12)} = \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} c_{n_1 n_2} |n_{1,} n_2\rangle^{(12)}$$

  - This may or may-not be 'factorizable'

- Non-factorizable states are called 'entangled'

  - For an `entangled state', each sub-system has *no independent objective reality*

Configuration Space

- The state of a quantum system of N particles in 3 dimensions lives in 'configuration space'
  - There are three quantum numbers associated with each particle
    - It takes 3N quantum numbers to specify a state of the full system
  - Coordinate Basis:

$$\left\{\left|x_1,y_1,z_1,x_2,y_2,z_2,\ldots,x_N,y_N,z_N\right\rangle\right\}$$

or we could just write

$$\left\{\left|\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N\right\rangle\right\}$$

This form is the coordinate system independent representation

  - Wavefunction:

$$\psi\left(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N\right)=\left\langle\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N\middle|\psi\right\rangle$$

  - To specify a state of *N* particles in *d* dimensions requires *d·N* quantum numbers

Tensor Products of Operators

- THEOREM:

  Let $A^{(1)}$ act in $\mathcal{H}_1$, and $B^{(2)}$ act in $\mathcal{H}_2$,

  Then the tensor product operator $C^{(12)}=A^{(1)}\otimes B^{(2)}$ acts in $\mathcal{H}_{12}$.

- PROOF:

$$A^{(1)} = \sum_m a_m \left|a_m\right\rangle^{(1)} \left\langle a_m\right|^{(1)}$$

$$B^{(2)} = \sum_m b_m \left|b_m\right\rangle^{(2)} \left\langle b_m\right|^{(2)}$$

$$A^{(1)} \otimes B^{(2)} = \sum_{m,n} a_m b_n \left|a_m\right\rangle^{(1)} \left\langle a_m\right|^{(1)} \otimes \left|b_n\right\rangle^{(2)} \left\langle b_n\right|^{(2)}$$

$$= \sum_{m,n} a_m b_n \left(\left|a_m\right\rangle^{(1)} \otimes \left|b_n\right\rangle^{(2)}\right)\left(\left\langle a_m\right|^{(1)} \otimes \left\langle b_n\right|^{(2)}\right)$$

$$C^{(12)} = \sum_{m,n} a_m b_n \left|a_m, b_n\right\rangle^{(12)} \left\langle a_m, b_n\right|^{(12)}$$

- The action of $C^{(12)}$ on a tensor-product state:

$$\left|\psi_1,\psi_2\right\rangle^{(12)} := \left|\psi_1\right\rangle^{(1)} \otimes \left|\psi_2\right\rangle^{(2)}$$

$$C^{(12)}\left|\psi_1,\psi_2\right\rangle^{(12)} = \sum_{m,n} a_m b_n \left|a_m, b_n\right\rangle^{(12)} \left\langle a_m \middle| \psi_1\right\rangle^{(1)} \left\langle b_n \middle| \psi_2\right\rangle^{(2)}$$

General form of Operators in Tensor-product spaces

- The most general form of an operator in $\mathcal{H}_{12}$ is:

$$C^{(12)} = \sum_{m,n} c_{m,n;m',n'} \left| m,n \right\rangle^{(12)} \left\langle m',n' \right|^{(12)}$$

$$c_{m,n;m',n'} := \left\langle m,n \right|^{(12)} C^{(12)} \left| m',n' \right\rangle^{(12)}$$

  - Here $|m,n\rangle$ may or may not be a tensor product state. The important thing is that it takes two quantum numbers to specify a basis state in $\mathcal{H}_{12}$
    - A basis that is not formed from tensor-product states is an 'entangled-state' basis

- In the beginning, you should always start with a tensor-product basis as your 'physical basis'
  - Then all operators are well-defined
  - Just expand states and operators onto tensor-product states
  - Then match up the bras and kets with their proper partners when taking inner products

Upgrading Subspace Operators

- Any operator in H_1 can be upgraded to an operator in H_{12} by taking the tensor product with the identity operator in H_2 :

$$A^{(12)} = A^{(1)} \otimes I^{(2)}$$

  - If A_1 is an observable in H_1, then it is also an observable in H_{12} (since it remains Hermitian when upgraded).
  - The spectrum of A_1 remains the same after upgrading

Proof:

Let
$$A^{(1)}\left|a_m\right\rangle^{(1)} = a_m\left|a_m\right\rangle^{(1)}$$

Then:

$$A^{(1)} \otimes I^{(2)}\left(\left|a_m\right\rangle^{(1)} \otimes \left|\psi\right\rangle^{(2)}\right) = \left(A^{(1)}\left|a_m\right\rangle^{(1)}\right) \otimes \left(I^{(2)}\left|\psi\right\rangle^{(2)}\right)$$
$$= a_m\left|a_m\right\rangle^{(1)} \otimes \left|\psi\right\rangle^{(2)}$$
$$= a_m\left(\left|a_m\right\rangle^{(1)} \otimes \left|\psi\right\rangle^{(2)}\right)$$

Note that $|\psi_2\rangle$ is completely arbitrary, but $|a_1\rangle \otimes |\psi_2\rangle$ is an eigenstate of $A^{(12)}$

Product of two Upgraded Operators

- Let $A^{(1)}$ and $B^{(2)}$ be observables in their respective Hilbert spaces
- Let $A^{(12)} = A^{(1)} \otimes I^{(2)}$ and $B^{(12)} = I^{(1)} \otimes B^{(2)}$.

- The product $A^{(12)}B^{(12)}$ is given by $A^{(1)} \otimes B^{(2)}$

  – Proof:

$$\begin{aligned}\left(A_1 \otimes I_2\right)\left(I_1 \otimes B_2\right) &= A_1 I_1 \otimes I_2 B_2 \\ &= A_1 \otimes B_2\end{aligned}$$

Compatible observables

- Let $A^{(1)}$ and $B^{(2)}$ be observables in their respective Hilbert spaces
- Let $A = A^{(1)} \otimes I^{(2)}$ and $B = I^{(1)} \otimes B^{(2)}$.
- **Theorem:** $[A,B]=0$

  Proof:
$$\begin{aligned}[A,B] &= AB - BA \\ &= (A_1 \otimes I_2)(I_1 \otimes B_2) - (I_1 \otimes B_2)(A_1 \otimes I_2) \\ &= (A_1 I_1) \otimes (I_2 B_2) - (I_1 A_1) \otimes (B_2 I_2) \\ &= A_1 \otimes B_2 - A_1 \otimes B_2 = 0\end{aligned}$$

- Conclusion: any operator in $\mathcal{H}_1$, is 'compatible' with any operator in $\mathcal{H}_2$,.
- I.e. simultaneous eigenstates exist.
  - Let $A_1 |a_1\rangle = a_1 |a_1\rangle$ and $B_2 |b_2\rangle = b_2 |b_2\rangle$.
  - Let $a = a_1$ and $b = b_2$
  - Let $|ab\rangle = |a_1\rangle \otimes |b_2\rangle$.
  - Then $AB|ab\rangle = ab|ab\rangle$.

‘And’ versus ‘Or’

- The tensor product correlates with a system having property A ***and*** property B
  - Dimension of combined Hilbert space is product of dimensions of subspaces associated with A and B
  - **Example:** start with a system having 4 energy levels. Let it interact with a 2 level system. The Hilbert space of the combined system has 8 possible states.

- Hilbert spaces are added when a system can have either property A ***or*** property B
  - Dimension of combined Hilbert space is sum of dimensions of subspaces associated with A and B
  - **Example:** start with a system having 4 energy levels. Add 2 more energy levels to your model, and the dimension goes from 4 to 6

Example #1 : Particle in Three Dimensions

- Let $\mathcal{H}_1$ be the Hilbert space of functions in one dimension
  - The projector is: $I_1 = \int dx |x\rangle\langle x|$
  - So a basis is: $\{|x\rangle\}$
- Then $\mathcal{H}_3 = (\mathcal{H}_1)^3$ would then be the Hilbert space of square integrable functions in three dimensions.
  - Proof: $I_3 = I_1 \otimes I_1 \otimes I_1$

$$= \int dx |x\rangle\langle x| \otimes \int dy |y\rangle\langle y| \otimes \int dz |z\rangle\langle z|$$

$$= \int dxdydz |x\rangle\langle x| \otimes |y\rangle\langle y| \otimes |z\rangle\langle z|$$

$$= \int dxdydz \big(|x\rangle \otimes |y\rangle \otimes |z\rangle\big)\big(\langle x| \otimes \langle y| \otimes \langle z|\big)$$

$$= \int dxdydz |x,y,z\rangle\langle x,y,z|$$

$$|x,y,z\rangle \equiv |x\rangle \otimes |y\rangle \otimes |z\rangle \qquad \psi(x,y,z) = \langle x,y,z|\psi\rangle$$

$$|\psi\rangle = \int dxdydz |x,y,z\rangle \psi(x,y,z)$$

- **Note:** $\mathcal{H}_3$ is also the Hilbert space of three particles in one-dimensional space

Three-dimensional Operators

- We can define the vector operators:

$$\vec{R} = X\,\hat{x} + Y\,\hat{y} + Z\,\hat{z}$$

$$\vec{P} = P_x\,\hat{x} + P_y\,\hat{y} + P_z\,\hat{z}$$

$$\vec{R}\left|x,y,z\right\rangle = \left(x\,\hat{x} + y\,\hat{y} + z\,\hat{z}\right)\left|x,y,z\right\rangle$$

or $\vec{R}\left|\vec{r}\right\rangle = \vec{r}\left|\vec{r}\right\rangle$

- Note that: $X = X^{(1)} \otimes I^{(2)} \otimes I^{(3)}$ and $P_y = I^{(1)} \otimes P^{(2)} \otimes I^{(3)}$ so that $[X, P_y]=0$.

- With

$$R_1 = X \qquad P_1 = P_x$$
$$R_2 = Y \qquad P_2 = P_y$$
$$R_3 = Z \qquad P_3 = P_z$$

- We can use:

$$\left[R_j, R_k\right] = \left[P_j, P_k\right] = 0$$

$$\left[R_j, P_k\right] = i\hbar\delta_{jk}$$

Example #2: Two particles in One Dimension

- For two particles in one-dimensional space, the Hilbert space is $(\mathcal{H}_1)^2$.

$$|x_1, x_2\rangle = |x_1\rangle^{(1)} \otimes |x_2\rangle^{(2)}$$

$$\psi(x_1, x_2) = \langle x_1, x_2 | \psi \rangle$$

$$I = \int dx_1 dx_2 |x_1, x_2\rangle\langle x_1, x_2| = 1$$

$$X_1 = X_1 \otimes I_2$$
$$P_2 = I_1 \otimes P_2$$
$$etc...$$

$$[X_j, X_k] = [P_j, P_k] = 0$$
$$[X_j, P_k] = i\hbar\delta_{jk}$$

Hamiltonians

- One particle in three dimensions:
  - Each component of momentum contributes additively to the Kinetic Energy

$$H = \frac{1}{2m}\left(P_x^2 + P_y^2 + P_z^2\right) + V(X, Y, Z)$$

$$= \frac{1}{2m}\vec{P}\cdot\vec{P} + V(\vec{R})$$

- Two particles in one dimension:

$$H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + V(X_1, X_2)$$

Conclusions

- The 'take home' messages are:
  - The combined Hilbert space of two systems, dimensions d_1 and d_2, has dimension $d_{1\otimes 2}=d_1\cdot d_2$
  - A physical basis set for the combined Hilbert space, $\mathcal{H}_{1\otimes 2}$ can be formed by taking all possible products of one basis state from space $\mathcal{H}_1$ with one basis state from $\mathcal{H}_2$.

$$\left\{\left|n_1\right\rangle^{(1)}\right\}\supseteq\mathcal{H}_1,\quad\left\{\left|n_2\right\rangle^{(2)}\right\}\supseteq\mathcal{H}_2$$

$$\left|n_1,n_2\right\rangle:=\left|n_1\right\rangle^{(1)}\otimes\left|n_2\right\rangle^{(2)}$$

$$\therefore\left\{\left|n_1,n_2\right\rangle\right\}\supseteq\mathcal{H}_{12}:=\mathcal{H}_1\otimes\mathcal{H}_2$$

  - In a tensor product space, a bra from one subspace can only attach to a ket from the same subspace:

$$\left\langle\psi_1\right|^{(1)}\rightarrow\leftarrow\left|\psi_2\right\rangle^{(1)}\Rightarrow\quad\left\langle\psi_1\middle|\psi_2\right\rangle$$

$$\left\langle\psi_1\right|^{(1)}\rightarrow\leftarrow\left|\varphi_1\right\rangle^{(2)}\Rightarrow\quad\left|\varphi_1\right\rangle^{(2)}\left\langle\psi_1\right|^{(1)}$$

  - For N particles (spin 0) in d dimensions, $d\cdot N$ quantum numbers are required to specify a unit-vector in any basis

$$\left|\vec{r}_1,\vec{r}_2,\ldots\vec{r}_N\right\rangle$$

$$\left|n_{1x},n_{1y},n_{1z},n_{2x},n_{2y},n_{2z},\ldots,n_{Nx},n_{Ny},n_{Nz}\right\rangle$$