Lecture 25:
Introduction to the Quantum Theory of Angular Momentum

Phy851 Fall 2009

Goals

1. Understand how to use different coordinate systems in QM (cartesian, spherical,…)

2. Derive the quantum mechanical properties of Angular Momentum
   - Use an algebraic approach similar to what we did for the Harmonic Oscillator

3. Use the resulting theory to treat spherically symmetric problems in three dimensions
   - Calculating the Hydrogen atom energy levels will be our target goal

Motion In 3Dimensions

- For a particle moving in three dimensions, there is a *distinct quantum state for every point* in space.
  - Thus each position state is now labeled by a vector

$$X \rightarrow \vec{R}$$

$$|x\rangle \rightarrow |\vec{r}\rangle$$

  - Vector operators are really three operators

$$\vec{R} = X\,\vec{e}_x + Y\,\vec{e}_y + Z\,\vec{e}_z$$

Scalar Operators	Ordinary Vectors
X, Y, Z	$\vec{e}_x, \vec{e}_y, \vec{e}_z$

- Coordinate system not unique:
  - For example, we could use spherical coordinates

$$\vec{R} = R\,\vec{e}_r(\Theta, \Phi)$$

Scalar Operators	Vector Operator
R, Θ, Φ	$\vec{e}_r(\Theta, \Phi)$

$$\vec{e}_r(\Theta, \Phi) = \sin\Theta\cos\Phi\,\vec{e}_x + \sin\Theta\sin\Phi\,\vec{e}_y + \cos\Theta\,\vec{e}_z$$

You can never go wrong with Cartesian Coordinates

- In all other coordinate systems, the unit vectors are also operators
  - so must be treated carefully

- Eigenstates: $\vec{R}|\vec{r}\rangle = \vec{r}|\vec{r}\rangle$

- For each point in space there is a position eigenstate
- How we want to label these points is up to us:

$$X|\vec{r}\rangle = \vec{e}_x \cdot \vec{R}|\vec{r}\rangle = \vec{e}_x \cdot \vec{r}|\vec{r}\rangle = x|\vec{r}\rangle$$

$$Y|\vec{r}\rangle = y|\vec{r}\rangle \qquad Z|\vec{r}\rangle = z|\vec{r}\rangle$$

$$|\vec{r}\rangle = |x, y, z\rangle$$

$$|\vec{r}\rangle = |r, \theta, \phi\rangle$$

$$|\vec{r}\rangle = |\rho, z, \phi\rangle$$

Different ways to refer to the same state

Orthogonality

- In three dimensions, the orthogonality condition becomes:

$$\langle \vec{r} | \vec{r}\,' \rangle = \delta^3(\vec{r} - \vec{r}\,')$$

  - Cartesian coordinates:

$$\langle xyz | x'y'z' \rangle = \delta(x - x')\delta(y - y')\delta(z - z')$$

  - Spherical coordinates:

$$\langle r\theta\phi | r'\theta'\phi' \rangle = \delta(r - r')\frac{\delta(\theta - \theta')}{r\sin\theta}\frac{\delta(\phi - \phi')}{r}$$

  - Mixing coordinates:

$$\langle xyz | r\theta\phi \rangle = \delta(x - r\sin\theta\cos\phi)\delta(y - r\sin\theta\sin\phi)\delta(z - r\cos\theta)$$

Wavefunctions

- Wavefunctions are defined in the usual way as:

$$\psi(\vec{r}) = \langle \vec{r} | \psi \rangle$$

$$\int d^3 r \, |\psi(\vec{r})|^2 = 1$$

$$\psi(x, y, z) = \langle xyz | \psi \rangle$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, |\psi(x,y,z)|^2 = 1$$

$$\psi(r, \theta, \phi) = \langle r\theta\phi | \psi \rangle$$

$$\int_0^{\infty} dr \int_0^{\pi} r \sin\theta d\theta \int_0^{2\pi} r d\phi \, |\psi(r,\theta,\phi)|^2 = 1$$

Momentum

- The three-dimensional momentum vector operator is:

$$\vec{P} = \vec{e}_x P_x + \vec{e}_y P_y + \vec{e}_z P_z$$

- The three-dimensional Hamiltonian is

$$H = \frac{\vec{P} \cdot \vec{P}}{2m} + V(\vec{R})$$

- In Cartesian components, this becomes:

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m} + V(\vec{R})$$

$$\langle xyz | H | \psi \rangle = \left[-\frac{\hbar^2}{2m} \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \right) + V(x, y, z) \right] \langle xyz | \psi \rangle$$

or

$$\langle \vec{r} | H | \psi \rangle = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) \right] \langle \vec{r} | \psi \rangle$$

Angular Momentum

- We can decompose the momentum operator onto spherical components as:

$$\vec{P} = \vec{e}_r(\Theta,\Phi)P_r + \vec{e}_\theta(\Theta,\Phi)P_\theta + \vec{e}_\phi(\Theta,\Phi)P_\phi$$

$$P_r = \vec{e}_r(\Theta,\Phi)\cdot\vec{P}$$

$$P_\theta = \vec{e}_\theta(\Theta,\Phi)\cdot\vec{P}$$

$$P_\phi = \vec{e}_\phi(\Theta,\Phi)\cdot\vec{P}$$

- The unit-vector operators are:

$$\vec{e}_r(\Theta,\Phi) = \sin\Theta\cos\Phi\vec{e}_x + \sin\Theta\sin\Phi\vec{e}_y + \cos\Theta\vec{e}_z$$

$$\vec{e}_\theta(\Theta,\Phi) = -\cos\Theta\cos\Phi\vec{e}_x - \cos\Theta\sin\Phi\vec{e}_y + \sin\Theta\vec{e}_z$$

$$\vec{e}_\phi(\Theta,\Phi) = -\sin\Phi\vec{e}_x + \cos\Phi\vec{e}_y$$

- From the above decomposition, we are led to define the Angular momentum as:

$$\vec{L} = \vec{e}_\theta(\Theta,\Phi)P_\theta + \vec{e}_\phi(\Theta,\Phi)P_\phi$$

- This is not the usual definition (L=RxP) but is equivalent.