Lecture 26:
Angular Momentum II

Phy851 Fall 2009

The Angular Momentum Operator

- The angular momentum operator is defined as:

$$\vec{L} = \vec{R} \times \vec{P}$$

- It is a vector operator:

$$\vec{L} = L_x \vec{e}_x + L_y \vec{e}_y + L_z \vec{e}_z$$

- According to the definition of the cross-product, the components are given by:

$$L_x = YP_z - ZP_y$$

$$L_y = ZP_x - XP_z$$

$$L_z = XP_y - YP_x$$

Commutation Relations

- The commutation relations are given by:

$$[L_x, L_y] = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y$$

- These are not definitions, they are just a consequence of $[X,P]=i\hbar$

- Any three operators which obey these relations are considered as 'generalized angular momentum operators'

- Compact notation:

$$[L_j, L_k] = i\hbar \varepsilon_{jk\ell} L_\ell$$

Summation over ℓ is implied

'Levi Cevita tensor'

$$\varepsilon_{jk\ell} \begin{cases} 0 & \text{if any two indices are the same} \\ 1 & \text{cyclic permutations of } x,y,z \text{ (or 1,2,3)} \\ -1 & \text{cyclic permutations of } z,y,x \text{ (or 3,2,1)} \end{cases}$$

Simultaneous Eigenstates

- In the HW will see that:

$$[L^2, L_x] = 0$$

$$[L^2, L_y] = 0$$

$$[L^2, L_z] = 0$$

  - Where:

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

- This means that simultaneous eigenstates of L^2 and L_z exist
  - Let:

$$L^2 \left| a, b \right\rangle = a \left| a, b \right\rangle$$

$$L_z \left| a, b \right\rangle = b \left| a, b \right\rangle$$

  - We want to find the allowed values of a and b.

Algebraic solution to angular momentum eigenvalue problem

- In a analogy to what we did for the Harmonic Oscillator, we now define raising and lowering operators:

$$L_+ = L_x + iL_y$$

$$L_- = L_x - iL_y$$

- Lets consider the action of L_+ first:

recall our approach for SHO: $$A^\dagger A\left(A^\dagger|\varepsilon\rangle\right) = (\varepsilon+1)\left(A^\dagger|\varepsilon\rangle\right)$$

$$L_z\left(L_+|a,b\rangle\right) = L_zL_x|a,b\rangle + iL_zL_y|a,b\rangle$$

$$[L_z, L_x] = i\hbar L_y \quad \rightarrow \quad L_zL_x = i\hbar L_y + L_xL_z$$

$$[L_z, L_y] = -i\hbar L_x \quad \rightarrow \quad L_zL_y = -i\hbar L_x + L_yL_z$$

$$L_z\left(L_+|a,b\rangle\right) = i\hbar L_y|a,b\rangle + L_xL_z|a,b\rangle + \hbar L_x|a,b\rangle + iL_yL_z|a,b\rangle$$

$$= (L_x + iL_y)L_z|a,b\rangle + \hbar(L_x + iL_y)|a,b\rangle$$

$$L_z\left(L_+|a,b\rangle\right) = (b+\hbar)L_+|a,b\rangle$$

Ladder Argument

$$L_z\left(L_+|a,b\rangle\right) = (b+\hbar)L_+|a,b\rangle$$

- Conclusion: if $|a,b\rangle$ is an eigenstate of L_z with eigenvalue b, then there is also an eigenstate $|a,b+\hbar\rangle$ with eigenvalue $b+\hbar$

- Thus we can say:

$$L_+|a,b\rangle = C_{+ab}|a,b+\hbar\rangle$$

- Similarly, we can readily show that:

$$L_z\left(L_-|a,b\rangle\right) = (b-\hbar)L_-|a,b\rangle$$

  – So that we must have also:

$$L_-|a,b\rangle = C_{-ab}|a,b-\hbar\rangle$$

- This allows us to say:

  1. If $|a,b\rangle$ exists, then $|a,b+\hbar\rangle$ exists or $C_{+ab}=0$
  2. If $|a,b\rangle$ exists, then $|a,b-\hbar\rangle$ exists or $C_{-ab}=0$

Establishing Some Lower Bounds

- Starting from:

$$L_x = YP_z - ZP_y$$

- It follows that

$$\begin{aligned} L_x^\dagger &= P_z^\dagger Y^\dagger - Z^\dagger P_y^\dagger \\ &= P_z Y - Z P_y \\ &= Y P_z - P_y Z \\ &= L_x \end{aligned}$$

  - Which makes sense because L_x is an observable

- Therefore, we have:

$$\begin{aligned} \langle a,b | L_x^2 | a,b \rangle &= \langle a,b | L_x^\dagger L_x | a,b \rangle \\ &= \left| L_x | a,b \rangle \right|^2 \end{aligned}$$

- Result:

$$\langle a,b | L_x^2 | a,b \rangle \geq 0$$

$$\langle a,b | L_y^2 | a,b \rangle \geq 0$$

The Trick

- We note that:

$$L^2 - L_z^2 = L_x^2 + L_y^2$$

$$\langle a,b|\left(L^2 - L_z^2\right)|a,b\rangle = \langle a,b|L_x^2|a,b\rangle + \langle a,b|L_y^2|a,b\rangle$$

- This means that:

$$\langle a,b|\left(L^2 - L_z^2\right)|a,b\rangle \ge 0$$

- By definition, we have:

$$\left(L^2 - L_z^2\right)|a,b\rangle = \left(a - b^2\right)|a,b\rangle$$

- Combining the two gives us:

$$a - b^2 \ge 0$$

$$a \ge b^2$$

$$a \ge 0$$

$$-\sqrt{a} \le b \le \sqrt{a}$$

The conclusion

$$a \geq 0 \qquad -\sqrt{a} \leq b \leq \sqrt{a}$$

- Conclusion:
  1. There is a $b_{\min}$ for every allowed a
  2. There is a $b_{\max}$ for every allowed a

$$-\sqrt{a} \leq b_{\min}(a) \qquad b_{\max}(a) \leq \sqrt{a}$$

- Thus we must have:

$$L_+ \left| a, b_{\max}(a) \right\rangle = 0$$

  - So that no states with higher b can exist
  - Which requires:

$$C_{+ab_{\max}(a)} = 0$$

  - Similarly,

$$C_{-ab_{\min}(a)} = 0$$

Just a Little Further to Go…

$$C_{+ab_{\max}(a)} = 0 \qquad C_{-ab_{\min}(a)} = 0$$

- If we have:

$$L_+ \left| a, b_{\max}(a) \right\rangle = 0$$

- Then we must have:

$$L_- L_+ \left| a, b_{\max}(a) \right\rangle = 0$$

- But:

$$L_- L_+ = \left(L_x - iL_y\right)\left(L_x + iL_y\right)$$

$$= L_x^2 + iL_x L_y - iL_y L_x + L_y^2$$

$$= L^2 - L_z^2 - \hbar L_z$$

$$= L_x^2 + L_y^2 + i[L_x, L_y]$$

Starting to see the light at the end of the tunnel…

$$L_-L_+ = L^2 - L_z^2 - \hbar L_z$$

$$L_-L_+ \left| a, b_{\max}(a) \right\rangle = 0$$

- This means:

$$\left(L^2 - L_z^2 - \hbar L_z\right)\left| a, b_{\max} \right\rangle = 0$$

$$\left(a - b_{\max}^2 - \hbar b_{\max}\right)\left| a, b_{\max} \right\rangle = 0$$

$$a - b_{\max}^2 - \hbar b_{\max} = 0$$

$$a = b_{\max}^2(a) - \hbar b_{\max}(a)$$

Not quite done yet…

$$L_- L_+ \left| a, b_{\max}(a) \right\rangle = 0 \quad \Rightarrow \quad a = b_{\max}^2(a) - \hbar b_{\max}(a)$$

- Similarly, we can show that:

$$L_+ L_- \left| a, b_{\min}(a) \right\rangle = 0$$

- Which together with:

$$\begin{aligned} L_+ L_- &= \left(L_x + iL_y\right)\left(L_x - iL_y\right) \\ &= L_x^2 - iL_x L_y + iL_y L_x + L_y^2 \\ &= L_x^2 + L_y^2 - i[L_x, L_y] \\ &= L^2 - L_z^2 + \hbar L_z \end{aligned}$$

- Gives us:

$$a = b_{\min}^2(a) + \hbar b_{\min}(a)$$

Keep moving forward…

$$a = b_{\max}(a)\left(b_{\max}(a) + \hbar\right)$$

$$a = b_{\min}(a)\left(b_{\min}(a) - \hbar\right)$$

- From a=a we get:

$$b_{\max}^2(a) + \hbar b_{\max}(a) = b_{\min}^2(a) - \hbar b_{\min}(a)$$

$$b_{\min}^2(a) - \hbar b_{\min}(a) - b_{\max}^2(a) - \hbar b_{\max}(a) = 0$$

- The solution is:

$$b_{\min} = \frac{1}{2}\left(\hbar \pm \sqrt{\hbar^2 + 4b_{\max}^2 + 4\hbar b_{\max}}\right)$$

$$b_{\min} = \frac{1}{2}\left(\hbar \pm \left(\hbar + 2b_{\max}\right)\right)$$

+

$$b_{\min} = \hbar + b_{\max}$$

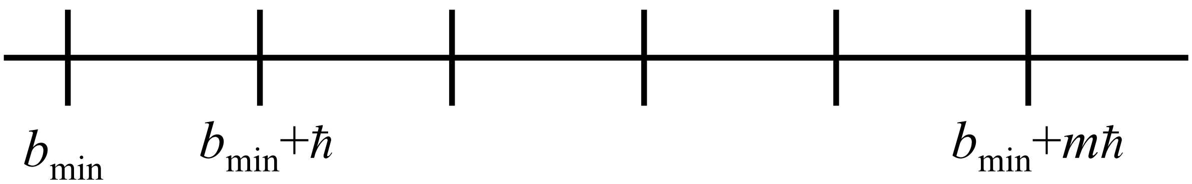
$$b_{\min} < b_{\max}$$

-

$$b_{\min} = -b_{\max}$$

Ladder Termination Requirement

- MAIN POINT SO FAR:
  - For a given a, $b_{\min}(a)$ and $b_{\max}(a)$ are the only b values from which the lowering/raising operators do not create new states.
- So we can build a ladder by starting at $b_{\min}$ and acting with L_+ to create a new state at $b_{\min}+\hbar$, etc…



- The only way the ladder will terminate is if we have

$$b_{\max} = b_{\min} + N\hbar$$

  - For some integer N
- With $b_{\min}=-b_{\max}$, this leads to: $$2b_{\max} = N\hbar$$

$$b_{\max} = \frac{N}{2}\hbar$$

- This means that the maximum z-component of angular momentum must always be a half-integer ***or*** whole integer times $\hbar$

Quantization of angular momentum

- Lets replace b_{max} with the symbol ℓ → $\ell = N/2$.
- The allowed eigenvalues of L_z are then:

$$b = m\hbar; \quad m = -\ell, -\ell+1, -\ell+2, \ldots, +\ell$$

- From the condition:

$$a = b_{\max}(a)\left(b_{\max}(a) + \hbar\right)$$

- We see that the corresponding eigenvalue of L^2 is:

$$a = \hbar\ell\left(\hbar\ell + \hbar\right)$$

$$a = \hbar^2 \ell\left(\ell + 1\right)$$

- Conclusion:
  - We can relabel the states as: $|a, b\rangle \rightarrow |\ell, m\rangle$
  - Then we have:

$$L^2 |\ell, m\rangle = \hbar^2 \ell(\ell+1) |\ell, m\rangle$$

$$L_z |\ell, m\rangle = \hbar m |\ell, m\rangle$$

The FINAL SLIDE

- Now we have:

$$(L_+)^\dagger = L_-$$

$$\left| L_+ | a,b \rangle \right|^2 = \left| C_{+ab} \right|^2$$

$$\left| L_+ | a,b \rangle \right|^2 = \langle a,b | L_- L_+ | a,b \rangle$$

$$= a - b^2 - \hbar b$$

$$C_{+ab} = \sqrt{a - b^2 - \hbar b}$$

  - Likewise:

$$C_{-ab} = \sqrt{a - b^2 + \hbar b}$$

$$a = \hbar^2 \ell(\ell + 1)$$

$$b = \hbar m$$

$$L_\pm | \ell, m \rangle = \hbar \sqrt{\ell(\ell+1) - m(m \pm 1)} \, | \ell, m \pm 1 \rangle$$