Lecture I: Dirac Notation

- To describe a physical system, QM assigns a complex number (`amplitude') to each distinct available physical state.
  - (Or alternately: two real numbers)
  - What is a `distinct physical state'?

- Consider a system with M distinct available states
  - The 2M real numbers can be viewed as a vector in an 2M-dimensional real-valued vector space
  - Or alternatively as a vector in an M-dimensional complex-valued vector space
  - We will refer to this abstract vector space as `Hilbert Space' or `state space'
  - Any vector in this space corresponds to a possible quantum-mechanical state. The number of such quantum states is uncountable infinity

- Just as *calculus* provides the mathematical basis for Classical Mechanics, the mathematical basis for QM is *linear algebra*
  - Vectors, matrices, eigenvalues, rotations, etc… are key concepts

Various common vector notations:

1. Vector notation: $\vec{r}(t)$
   - Just a name, an abstraction that refers to something physical
2. Unit vectors: $\vec{r}(t) = r_1(t)\vec{e}_1 + r_2(t)\vec{e}_2 + r_3(t)\vec{e}_3$
   - Unit vectors are predefined in physical terms $r_1(t) = \vec{r}\cdot\vec{e}_1$
   - Components are projections onto unit vectors $\vec{e}_1\cdot\vec{e}_2 = 0 \quad \vec{e}_1\cdot\vec{e}_1 = 1$
   - Unit vectors are orthonormal
3. Column vector:
   - Unit vectors are *implied* $\vec{r}(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{pmatrix}$

`Dirac notation':

- Just new symbols for same concepts

`ket'

$$\vec{r}(t) \to |\psi(t)\rangle, |\varphi(t)\rangle, |\Psi(t)\rangle, \ldots$$

`bra'

$$\vec{r}^{\,T}(t) \to \langle\psi(t)|, \langle\varphi(t)|, \langle\Psi(t)|, \ldots$$

$$\vec{e}_j \to |j\rangle, |n\rangle, |a_n\rangle, |r\rangle, |p\rangle, |n,m\rangle, |E_n\rangle, |E_n, m\rangle, \ldots$$

$$\vec{r}t) = r_1(t)\vec{e}_1 + r_2(t)\vec{e}_2 + r_3(t)\vec{e}_3 \to |\psi(t)\rangle = c_1(t)|1\rangle + c_2(t)|2\rangle + c_3(t)|3\rangle$$

$$\vec{a}\cdot\vec{b} = \vec{a}^{\,T}\vec{b} = \langle a|b\rangle$$ `inner product'

$$r_j(t) = \vec{e}_j\cdot\vec{r}(t) \to c_j(t) = \langle j|\psi(t)\rangle$$

Added catch since QM vectors are complex

- Transpose operation replaced by 'Hermitian conjugation' or 'dagger' operation

$$\langle b|a\rangle = \langle a|b\rangle^*$$

- '†' is transpose plus complex conjugation

$$\vec{r}^T = (\vec{r})^T \rightarrow \langle\psi| = (|\psi\rangle)^\dagger$$

- Projectors and Closure relations:

$$\vec{r} = r_1\vec{e}_1 + r_2\vec{e}_2 + \ldots + \vec{r}_M\vec{e}_M$$
$$= \vec{e}_1(\vec{e}_1 \cdot \vec{r}) + \vec{e}_2(\vec{e}_2 \cdot \vec{r}) + \ldots + \vec{e}_M(\vec{e}_M \cdot \vec{r})$$

$$|\psi\rangle = c_1|1\rangle + c_1|1\rangle + \ldots + c_M|M\rangle$$
$$= |1\rangle\langle 1|\psi\rangle + |2\rangle\langle 2|\psi\rangle + \ldots + |M\rangle\langle M|\psi\rangle$$
$$= \left(|1\rangle\langle 1| + |2\rangle\langle 2| + \ldots + |M\rangle\langle M|\right)|\psi\rangle$$

- This proves the `closure relation': $\sum_{j=1}^{M}|j\rangle\langle j| = 1$

The summation is over a complete set of unit vectors that spans any Hilbert sub-space is equal to the identity operator in that sub-space

- The entire Hilbert space is a trivial sub-space

- Norm of a vector:
  - a.k.a. magnitude, length

$$\|r\| = \sqrt{\vec{r}\cdot\vec{r}} = \sqrt{\vec{r}^T\vec{r}} \longrightarrow \|\psi\| = \sqrt{\langle\psi|\psi\rangle}$$

- To compute the norm in terms of the components along a set of orthogonal unit vectors:
  - Insert the identity

$$\langle\psi|\psi\rangle = \langle\psi|1|\psi\rangle$$
$$= \langle\psi|\left(\sum_{j=1}^{M}|j\rangle\langle j|\right)|\psi\rangle$$
$$= \sum_{j=1}^{M}\langle\psi|j\rangle\langle j|\psi\rangle$$
$$= \sum_{j=1}^{M}c_j^*c_j = \sum_{j=1}^{M}|c_j|^2$$

Old notation: $\vec{r}\cdot\vec{r} = r_1^2 + r_2^2 + \ldots = \sum_j r_j^2$

Avoid being confused by implied meanings of various symbols

- To avoid confusion, keep in mind that '$|\ \rangle$' indicates a Hilbert-space vector, the 'ψ' in $|\psi\rangle$' is just a label
  - We could call it anything
    - $|\psi\rangle$, $|\varphi\rangle$, $|\phi\rangle$, $|3\rangle$, $|Alice\rangle$
  - We just need to clearly define our labels
    - "let $|\psi(t)\rangle$ be the state of our system at time *t*."
    - "let $|x\rangle$ be the state in which the particle lies at position *x*."
      - Here *x* is a placeholder which could take on any numerical value. I.e. defining the state $|x\rangle$ as above actually defines an infinite set of vectors, one for each point on the real axis.
      - This is exactly how the symbol '*x*' is used when you say ` *f(x)* = cos(*x*)'
    - 'let $|j\rangle$ be the state in which our system is in the j^{th} quantized energy level.
      - Here *j* is a placeholder for an arbitrary integer

Summary

- There are 'ket's and 'bra's:
  - ket: $|\psi\rangle$
    - A ket is a vector in an M dimensional Hilbert space, where M is the number of distinct physical states of a system
  - bra: $\langle\psi|$
    - A bra is a transposed, conjugated ket
- Put a bra and a ket together to get a c-number
  - $\langle\psi|\varphi\rangle$:= a c-number
    - c-number := complex number
- Unit vectors:
  - An M dimensional Hilbert space is spanned by M orthonormal unit vectors
    - $\{|j\rangle\}=\{|1\rangle, |2\rangle, |3\rangle, \ldots, |M\rangle\}$ ({ } = 'the set of')
    - $\langle j|k\rangle = \delta_{jk}$ (δ_{jk} is 'Kronecker delta function')
      - » 1 if j=k
      - » 0 else
  - Closure relation:

$$\sum_{j=1}^{M} |j\rangle\langle j| = 1$$