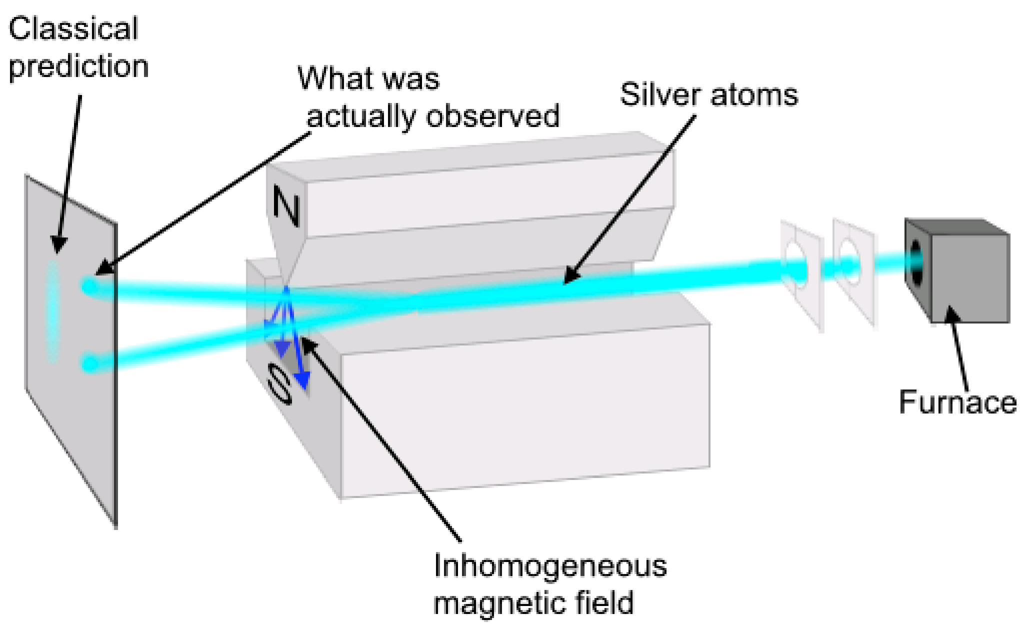
Lecture 33: Quantum Mechanical Spin

Phy851 Fall 2009

Intrinsic Spin

- Empirically, we have found that most particles have an additional *internal* degree of freedom, called ‘spin’

- The Stern-Gerlach experiment (1922):



- Each type of particle has a discrete number of internal states:
  - 2 states --> spin _
  - 3 states --> spin 1
  - Etc….

Interpretation

- It is best to think of spin as just an additional quantum number needed to specify the state of a particle.
  - Within the Dirac formalism, this is relatively simple and requires no new physical concepts

- The physical meaning of spin is not well-understood

- Fro Dirac eq. we find that for QM to be Lorentz invariant requires particles to have both anti-particles and spin.

- The ‘spin’ of a particle is a form of angular momentum

Spin Operators

- Spin is described by a vector operator:

$$\vec{S} = S_x \vec{e}_x + S_y \vec{e}_y + S_z \vec{e}_z$$

- The components satisfy angular momentum commutation relations:

$$[S_x, S_y] = i\hbar S_z$$

$$[S_y, S_z] = i\hbar S_x$$

$$[S_z, S_x] = i\hbar S_y$$

- This means simultaneous eigenstates of S^2 and S_z exist:

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

$$S^2 \left| s, m_s \right\rangle = \hbar^2 s(s+1) \left| s, m_s \right\rangle$$

$$S_z \left| s, m_s \right\rangle = \hbar m \left| s, m_s \right\rangle$$

Allowed quantum numbers

- For any set of 3 operators satisfying the angular momentum algebra, the allowed values of the quantum numbers are:

$$j \in \left\{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots\right\}$$

$$m_j \in \left\{-j, -j+1, \ldots, j\right\}$$

- For orbital angular momentum, the allowed values were further restricted to only integer values by the requirement that the wavefunction be single-valued

- For spin, the quantum number, *s*, can only take on one value
  - The value depends on the type of particle
  - ***S*=0:** Higgs
  - ***s*=1/2:** Electrons, positrons, protons, neutrons, muons,neutrinos, quarks,…
  - ***s*=1:** Photons, W, Z, Gluon
  - ***s*=2:** graviton

$$m_s \in \left\{-s, -s+1, \ldots, s\right\}$$

Complete single particle basis

- A set of 5 commuting operators which describe the independent observables of a single particle are:

$$\vec{R}, S^2, S_z$$

  - Or equivalently:

$$R, L^2, L_z, S^2, S_z$$

- Some possible basis choices:

$$\left\{\left|\vec{r}, s, m_s\right\rangle\right\}$$

$$\left\{\left|\vec{p}, s, m_s\right\rangle\right\}$$

$$\left\{\left|r, \ell, m_\ell, s, m_s\right\rangle\right\}$$

$$\left\{\left|n, \ell, m_\ell, s, m_s\right\rangle\right\}$$

- When dealing with a single-particle, it is permissible to drop the *s* quantum number

Intrinsic Magnetic Dipole Moment

- Due to spin, an electron has an *intrinsic magnetic dipole moment*:

$$\vec{\mu}_e = -\frac{g_e |e|}{2m_e}\vec{S}$$

  - g_e is the electron g-factor
  - For an electron, we have:

$$g_e = 2.0023193043622 \pm 0.0000000000015$$

  - The is the most precisely measured physical quantity
- For most purposes, we can take $g_e \approx 2$, so that

$$\vec{\mu}_e = -\frac{|e|}{m_e}\vec{S}$$

- For any charged particle we have:

$$\vec{\mu} = \frac{g\, q}{2M}\vec{S}$$

Each particle has a different g-factor

Hamiltonian for an electron in a magnetic field

- Because the electron is a point-particle, the dipole-approximation is always valid for the spin degree of freedom

- Any `kinetic' energy associated with S^2 is absorbed into the rest mass

- To obtain the full Hamiltonian of an electron, we must add a single term:

$V = -\vec{\mu}\cdot\vec{B}$

$$H \rightarrow H + \frac{|e|}{m_e}\vec{S}\cdot\vec{B}(\vec{R})$$

$$H = \frac{1}{2m_e}\left[\vec{P} + |e|\vec{A}(\vec{R})\right]^2 - |e|\ddot{O}(\vec{R}) + \frac{|e|}{m_e}\vec{S}\cdot\vec{B}(\vec{R})$$

Uniform Weak Magnetic Field as a perturbation

- For a weak uniform field, we find

$$H = \frac{P^2}{2m_e} + \frac{|e|B_0}{2m_e}\left(L_z + 2S_z\right)$$

- With the addition of a spherically symmetric potential, this gives:

$$H = \frac{P^2}{2m_e} + V(R) + \frac{|e|B_0}{2m_e}\left(L_z + 2S_z\right)$$

  - If the zero-field eigenstates are known

$$H_0\left|n,\ell,m_\ell\right\rangle = E_{0,n}\left|n,\ell,m_\ell\right\rangle$$

  - The weak-uniform-field eigenstates are:

$$\left\{\left|n,\ell,m_\ell,m_s\right\rangle\right\}$$

$$H\left|n,\ell,m_\ell,m_s\right\rangle = E_{n,m_\ell,m_s}\left|n,\ell,m_\ell,m_s\right\rangle$$

$$E_{n,m_\ell,m_s} = E_{0,n} + \mu_B B_0\left(m_\ell + 2m_s\right)$$

$$\mu_B = \frac{|e|\hbar}{2m_e}$$ 'Bohr Magneton'

Wavefunctions

- In Dirac notation, all spin does is add two extra quantum numbers
- The separate concept of a 'spinor' is unnecessary

- Coordinate basis: $\left\{ \left| \vec{r}, s, m_s \right\rangle \right\}$
  - Eigenstate of $\vec{R}, S^2, S_z$

- Projector:

$$I = \sum_{m_s=-s}^{s} \int_V d^3r \left| \vec{r}, s, m_s \right\rangle \left\langle \vec{r}, s, m_s \right|$$

- Wavefunction:

$$\psi_{m_s}(\vec{r}) := \left\langle \vec{r}, s, m_s \middle| \psi \right\rangle$$

Spinor Notation:

$$\psi_{m_s}(\vec{r}) := \langle \vec{r}, s, m_s | \psi \rangle$$

- We think of them as components of a length 2s+1 vector, where each component is a wavefunction
- Example: s=1/2

$$\psi_{\uparrow}(\vec{r}) := \langle \vec{r}, s, \tfrac{1}{2} | \psi \rangle$$

$$\psi_{\downarrow}(\vec{r}) := \langle \vec{r}, s, -\tfrac{1}{2} | \psi \rangle$$

- Spinor wavefunction definition:

$$[\psi](\vec{r}) := \begin{pmatrix} \psi_{\uparrow}(\vec{r}) \\ \psi_{\downarrow}(\vec{r}) \end{pmatrix} = \begin{pmatrix} \psi_{\frac{1}{2}}(\vec{r}) \\ \psi_{-\frac{1}{2}}(\vec{r}) \end{pmatrix}$$

- If external and internal motions are not entangled, we can factorize the spinor wavefunction:

$$[\psi](\vec{r}) := \begin{pmatrix} c_{\uparrow} \\ c_{\downarrow} \end{pmatrix} \psi(\vec{r})$$

$$\begin{pmatrix} c_{\uparrow} \\ c_{\downarrow} \end{pmatrix}$$ Is then a *pure spinor*

Schrödinger's Equation

- We start from:

$$i\hbar \frac{d}{dt}\left|\psi\right\rangle = H\left|\psi\right\rangle$$

- Hit from left with with $\left\langle \vec{r}, m_s \right|$

$$i\hbar \frac{d}{dt}\left\langle \vec{r}, m_s \middle| \psi \right\rangle = \left\langle \vec{r}, m_s \middle| H \middle| \psi \right\rangle$$

- Insert the projector

$$i\hbar \frac{d}{dt}\left\langle \vec{r}, m_s \middle| \psi \right\rangle = \sum_{m_s'=-s}^{s} \int_V d^3 r' \left\langle \vec{r}, m_s \middle| H \middle| \vec{r}\,', m_s' \right\rangle \left\langle \vec{r}\,', m' \middle| \psi \right\rangle$$

- Let: $H = \frac{P^2}{2M}[[I]] + [[V]](\vec{R})$

  - For s=1/2:

$$[[I]] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad [[V]](\vec{r}) = \begin{pmatrix} V_{\uparrow\uparrow}(\vec{r}) & V_{\uparrow\downarrow}(\vec{r}) \\ V_{\downarrow\uparrow}(\vec{r}) & V_{\downarrow\downarrow}(\vec{r}) \end{pmatrix}$$

$$i\hbar \frac{d}{dt}[\psi](\vec{r}) = -\frac{\hbar^2}{2M}\nabla^2[\psi](\vec{r}) + [[V]](\vec{r})[\psi](\vec{r})$$

Example: Electron in a Uniform Field

$$i\hbar \frac{d}{dt}[\psi](\vec{r}) = -\frac{\hbar^2}{2M}\nabla^2[\psi](\vec{r}) + [[V]](\vec{r})[\psi](\vec{r})$$

$$i\hbar \frac{d}{dt}\begin{pmatrix}\psi_\uparrow(\vec{r})\\ \psi_\downarrow(\vec{r})\end{pmatrix} = \left[-\frac{\hbar^2}{2m_e}\nabla^2 - i\mu_B B_0 \frac{\partial}{\partial\phi}\right]\begin{pmatrix}\psi_\uparrow(\vec{r})\\ \psi_\downarrow(\vec{r})\end{pmatrix} + \mu_B B_0 \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_\uparrow(\vec{r})\\ \psi_\downarrow(\vec{r})\end{pmatrix}$$

- This is just a representation of two separate equations:

$$i\hbar \frac{d}{dt}\psi_\uparrow(\vec{r}) = \left[-\frac{\hbar^2}{2m_e}\nabla^2 - i\mu_B B_0 \frac{\partial}{\partial\phi}\right]\psi_\uparrow(\vec{r}) + \mu_B B_0 \psi_\uparrow(\vec{r})$$

$$i\hbar \frac{d}{dt}\psi_\downarrow(\vec{r}) = \left[-\frac{\hbar^2}{2m_e}\nabla^2 - i\mu_B B_0 \frac{\partial}{\partial\phi}\right]\psi_\downarrow(\vec{r}) - \mu_B B_0 \psi_\downarrow(\vec{r})$$

- We would have arrived at these same equations using Dirac notation, without ever mentioning ‘Spinors’

Pauli Matrices

$$i\hbar \frac{d}{dt}\begin{pmatrix}\psi_\uparrow(\vec r)\\ \psi_\downarrow(\vec r)\end{pmatrix} = \left[-\frac{\hbar^2}{2m_e}\nabla^2 - i\mu_B B_0 \frac{\partial}{\partial\phi}\right]\begin{pmatrix}\psi_\uparrow(\vec r)\\ \psi_\downarrow(\vec r)\end{pmatrix} + \mu_B B_0 \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_\uparrow(\vec r)\\ \psi_\downarrow(\vec r)\end{pmatrix}$$

- Here we see that we have recovered one of the Pauli Matrices:

$$\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

- The other Pauli matrices are:

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

- Then in the basis of eigenstates of S_z we have:

$$\vec S = \frac{\hbar}{2}\vec\sigma$$

$$i\hbar \frac{d}{dt}[\psi](\vec r) = \left(-\frac{\hbar^2}{2M}\nabla^2 + \mu_B B_0\left(-i\frac{\partial}{\partial\phi} + \sigma_z\right)\right)[\psi](\vec r)$$

- If we only care about spin dynamics:

$$i\frac{d}{dt}[c] = \frac{|e|}{2m_e}\vec\sigma\cdot\vec B[c]$$

$\sigma \cdot B\; c$

$B_x\sigma_x + B_y\sigma_y + B_z$

Two particles with spin

- How do we treat a system of two particles with masses M_1 and M_2, charges q_1 and q_2, and spins s_1 and s_2?
  - Basis:

$$\left|\vec{r}_1, s_1, m_{s1}; \vec{r}_2, s_2, m_{s2}\right\rangle$$

  - Wavefunction:

$$\psi_{s_1, m_{s1}, s_2, m_{s2}}(\vec{r}_1, \vec{r}_2) := \left\langle \vec{r}_1, s_1, m_{s1}; \vec{r}_2, s_2, m_{s2} \middle| \psi \right\rangle$$

  - Hamiltonian w/out motional degrees of freedom:

$$H = -\frac{q_1}{M_1}\vec{S}_1 \cdot \vec{B}(\vec{R}_1) - \frac{q_2}{M_2}\vec{S}_2 \cdot \vec{B}(\vec{R}_2)$$

  - Hamiltonian w/ motional degrees of freedom:

$$H = \frac{1}{2M_1}\left(\vec{P}_1 - q_1\vec{A}(\vec{R}_1)\right) + q_1\Phi(\vec{R}_1) - \frac{q_1}{M_1}\vec{S}_1 \cdot \vec{B}(\vec{R}_1)$$

$$\frac{1}{2M_2}\left(\vec{P}_2 - q_2\vec{A}(\vec{R}_2)\right) + q_2\Phi(\vec{R}_2) - \frac{q_2}{M_2}\vec{S}_2 \cdot \vec{B}(\vec{R}_2)$$

Example #1

- A spin _ particle is in the ↑ state with respect to the z-axis. What is the probability of finding it in the ↓-state with respect to the x-axis?

- Let: $\left|\psi\right\rangle = \left|\uparrow_z\right\rangle$

- In the basis, $\left\{\left|\uparrow_z\right\rangle, \left|\downarrow_z\right\rangle\right\}$ the operator for the x-component of spin is:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- By symmetry, σ_x must have eigenvalues +1 and -1
- The eigenvector corresponding to -1 is defined by:

$$\sigma_x\left|\downarrow_x\right\rangle = -\left|\downarrow_x\right\rangle$$

Example #1 continued:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_x \left| \downarrow_x \right\rangle = -\left| \downarrow_x \right\rangle$$

$$\left| \downarrow_x \right\rangle = -\sigma_x \left| \downarrow_x \right\rangle$$

$$\left\langle \uparrow_z \middle| \downarrow_x \right\rangle = -\left\langle \uparrow_z \middle| \sigma_x \middle| \downarrow_x \right\rangle = -1$$

$$= -\left\langle \downarrow_z \middle| \downarrow_x \right\rangle$$

- This implies that:

$$\left| \downarrow_x \right\rangle = \frac{1}{\sqrt{2}} \left(\left| \uparrow_z \right\rangle - \left| \downarrow_z \right\rangle \right)$$

$$P = \left| \left\langle \downarrow_x \middle| \uparrow_z \right\rangle \right|^2 = \frac{1}{2}$$

Example #2

- Two identical spin-1/2 particles are placed in a uniform magnetic field. Ignoring motional degrees of freedom, what are the energy-levels and degeneracies of the system?

- States: $\left\{ \left|\uparrow\uparrow\right\rangle, \left|\uparrow\downarrow\right\rangle, \left|\downarrow\uparrow\right\rangle, \left|\downarrow\downarrow\right\rangle \right\}$

  – Z-axis chosen along B-field

- Hamiltonian: $H = -\frac{gqB_0}{2M}\left(S_{1z} + S_{2z}\right)$

- Basis states are already eigenstates:

$$H\left|\uparrow\uparrow\right\rangle = -\frac{\hbar gqB_0}{2M}\left|\uparrow\uparrow\right\rangle \qquad E_1 = -\frac{\hbar gqB_0}{2M}; \quad d_1 = 1$$

$$H\left|\uparrow\downarrow\right\rangle = H\left|\downarrow\uparrow\right\rangle = 0 \qquad E_2 = 0; \quad d_2 = 2$$

$$H\left|\downarrow\downarrow\right\rangle = \frac{\hbar gqB_0}{2M}\left|\downarrow\downarrow\right\rangle \qquad E_3 = \frac{\hbar gqB_0}{2M}; \quad d_3 = 1$$