Operators

- In QM, an operator is an object that acts on a ket, transforming it into another ket
  - Let A represent a generic operator
  - An operator is a linear map

$$A : \mathcal{H} \rightarrow \mathcal{H}$$

$$A|\psi\rangle = |\psi'\rangle$$

  - Operators are linear:

$$A\left(a\,|\psi_1\rangle + b\,|\psi_2\rangle\right) = aA\,|\psi_1\rangle + bA|\psi_2\rangle$$

    - a and b are arbitrary c-numbers

Notation:

- Generally, we will follow Cohen-Tannoudji, and use capital letters for operators and lower-case letters for c-numbers.
- Another common convention is to distinguish operators by giving them a '^'

$$\hat{a} \qquad \hat{A} \qquad \hat{\psi}$$

  - I may use this occasionally

Matrix representations

- Just as kets are vectors, operators are matrices
- Let the set $\{|1\rangle, |2\rangle, |3\rangle, \ldots, |M\rangle\}$ be a set of orthogonal unit vectors which spans an entire *M*-dimensional Hilbert space
- The c-number $\langle j|\psi\rangle$ is thus the j^{th} component of the vector $|\psi\rangle$
- Matrix Representation of an operator:
  - Start from the equation: $|\psi'\rangle = A|\psi\rangle$
  - Hit it from the left with the bra $\langle j|$: $\langle j|\psi'\rangle = \langle j|A|\psi\rangle$
  - Insert `the identity' between A and $|\psi\rangle$:
    $$\langle j|\psi'\rangle = \sum_{k=1}^{M} \langle j|A|k\rangle\langle k|\psi\rangle$$
  - Use the replacements: $c_j \equiv \langle j|\psi\rangle$, $d_j \equiv \langle j|\psi'\rangle$ and $A_{jk} \equiv \langle j|A|k\rangle$ to get:
    $$d_j = \sum_{k=1}^{M} A_{jk} c_k$$
    - This is just the standard formula for matrix multiplication:
    $$\vec{d} = A\vec{c}$$

Defining states and operators

- A state (vector) is specified by giving its components in some physically meaningful basis
- An operator is defined by giving its matrix elements in some physically meaningful basis
- Operators and/or states can alternatively be defined as the solution to a particular equation
  - Gives components implicitly instead of explicitly

Projectors

- Note that $|j\rangle\langle j|\psi\rangle = c_j|j\rangle$
- Thus $|j\rangle\langle j|$ is an operator
  - We call it the 'projector' onto the state $|j\rangle$

$$I_j = |j\rangle\langle j|$$

- The sum of projectors onto a set of *M* orthonormal states in an *M*-dimensional Hilbert space is called the 'identity operator'

$$I = \sum_{j=1}^{M} |j\rangle\langle j|$$

  - If the sum is incomplete, the resulting operator is the projector onto the subspace spanned by the included unit vectors.
    - It is the identity operator inside that subspace

$$I_s = \sum_{j=3}^{5} |j\rangle\langle j|$$

- The projector onto state $|\psi\rangle$ is:

$$I_\psi = |\psi\rangle\langle\psi|$$

- All projectors satisfy:

$$I_s^{\,2} = I_s$$

Outer product

- Clearly the `outer product' of any two state vectors is an operator:

$$|\psi\rangle\langle\phi|$$

- An operator can be 'expanded' in a given basis, and expressed in terms of its matrix elements:

Eigenvalues and Eigenvectors

- Since operators are matrices, they have eigenvalues and eigenvectors.
- All operators in an M-dimensional Hilbert space have M eigenvalues, but they may not all be distinct
- Definition:
  - Let a_n be the nth eigenvalue of the operator A
  - Let $|a_n\rangle$ be the corresponding eigenvector
  - They are related via the eigenvalue equation:

$$A|a_n\rangle = a_n|a_n\rangle$$

    - Much of the course will be spent solving various versions of this equation via a variety of methods

Determining the eigenvalues and eigenvectors of an operator

- Method 1: express the operator in matrix form, then use standard matrix methods:

$$\text{Det}\,|\,A - aI\,| = 0$$

  - Can use Mathematica or other numerical software for large matrices
- There are also analytic methods that work in some cases
  - called ‘algebraic solutions’

Hermitian Conjugation of Operators

- Recall that '†' symbolizes 'Hermitian conjugation'
  - Note: The H.c. is sometimes called the 'adjoint'
  - † = T and * (transpose plus complex conjugation)
  - The bra $\langle\psi|$ is the H.c. of the ket $|\psi\rangle$
  - The operator $A^\dagger$ is the Hermitian conjugate of A.
    - This means that $\left(A^\dagger\right)_{jk} = (A_{kj})^*$
    - Or equivalently $\langle j|A^\dagger|k\rangle = \langle k|A|j\rangle^*$
  - The operator $B^\dagger A^\dagger$ is the Hermitian conjugate of the operator product AB: $(AB)^\dagger = B^\dagger A^\dagger$
    - This reverse ordering is the same as for the ordinary Transpose:
  - What is the conjugate of $A|\psi\rangle$?

$$(A|\psi\rangle)^\dagger = \langle\psi|A^\dagger$$

Rule of thumb for H.c.

1. Reverse order of all terms
2. Turn bras into kets and vice versa
3. Replace all operators with their Hermitian conjugates

Hermitian Operators

- Definition: an operator is said to be Hermitian if it satisfies: $A^{\dagger}=A$
  - Alternatively called 'self adjoint'
  - In QM we will see that all observable properties must be represented by Hermitian operators

- Theorem: all eigenvalues of a Hermitian operator are real
  - Proof:

Eigenvectors of a Hermitian operator

- Note: all eigenvectors are defined only up to a multiplicative c-number constant

$$A|a_m\rangle = a_m|a_m\rangle \quad \rightarrow A\left(c|a_m\rangle\right) = a_m\left(c|a_m\rangle\right)$$

  - Thus we can choose the normalization $\langle a_m|a_m\rangle=1$

- Theorem: all eigenvectors corresponding to distinct eigenvalues are orthogonal
  - Proof:

Completeness of Eigenvectors of a Hermitian operator

- Theorem: If an operator in an M-dimensional Hilbert space has M distinct eigenvalues (i.e. no degeneracy), then its eigenvectors form a `complete set' of unit vectors (i.e a complete 'basis')
  - Proof:

    M orthonormal vectors must span an M-dimensional space.

- Thus we can use them to form a representation of the identity operator:

Degeneracy

- Definition: If there are at least two linearly independent eigenvectors associated with the same eigenvalue, then the eigenvalue is degenerate.
  - The `degree of degeneracy' of an eigenvalue is the number of linearly independent eigenvectors that are associated with it

- Example: d=2
  - Let's refer to the two linearly independent eigenvectors $|\omega_n\rangle$ and $|\Omega_n\rangle$
  - Linear independence means $\langle\omega_n|\Omega_n\rangle \neq 1$.
  - If they are not orthogonal ($\langle\omega_n|\Omega_n\rangle \neq 0$), we can always use Gram-Schmidt Orthogonalization to get an orthonormal set

Gram-Schmidt Orthogonalization

- Procedure:
  - Let $$|\omega_n,1\rangle \equiv |\omega_n\rangle$$
  - A second orthogonal vector is then

$$|\omega_n,2\rangle \equiv \frac{|\Omega_n\rangle - |\omega_n\rangle\langle\omega_n|\Omega_n\rangle}{\left\| |\Omega_n\rangle - |\omega_n\rangle\langle\omega_n|\Omega_n\rangle \right\|}$$

    - Proof:

$$\langle\omega_n,1|\omega_n,2\rangle \equiv \frac{\langle\omega_n|\Omega_n\rangle - \langle\omega_n|\omega_n\rangle\langle\omega_n|\Omega_n\rangle}{\left\| |\Omega_n\rangle - |\omega_n\rangle\langle\omega_n|\Omega_n\rangle \right\|}$$

      - but $\langle\omega_n|\omega_n\rangle = 1$
      - Therefore $\langle\omega_n,1|\omega_n,2\rangle = 0$
  - Can be continued for higher degree of degeneracy

- Result: From M linearly independent degenerate eigenvectors we can always form M orthonormal unit vectors which span the M-dimensional degenerate subspace.
  - If this is done, then the eigenvectors of a Hermitian operator form a complete basis even with degeneracy present