Remark: Commutation of Operators

- Since operators are matrices, they do not necessarily commute

$$AB \neq BA$$

- We define the commutator of the operators A and B as

$$[A, B] = AB - BA$$

- The properties of most physically important operators (e.g. *X, P, L, S,…*) can generally be deduced solely from their commutation relations

The 'wavefunction'
- as opposed to a 'state vector' $:= |\psi\rangle$

- not necessary, but we can define a 'wavefunction' for systems with continuous degrees of freedom

Definition: $\psi(x) := \langle x | \psi \rangle$

wavefunction in x-representation

x can be any observable

$\phi(x) = \langle x | \phi \rangle$

$\therefore \quad \psi(p) = \langle p | \psi \rangle$

$f(x) = \langle x | f \rangle$

if x is position
" p " momentum

$\psi(x)$ and $\psi(p)$ are two 'representations' of the same state $|\psi\rangle$

$$\psi(p) = \langle p | \psi \rangle = \int dx \langle p | x \rangle \langle x | \psi \rangle = \int dx \langle p | x \rangle \psi(x)$$

need $\langle p | x \rangle$

Lecture 5: Postulates of QM

MOTIVATION TO STUDY QM:

- Quantum mechanics underlies Nuclear, Particle, Condensed Matter, and Atomic physics (and is thus very important for Astronomy and Astrophysics)
- Quantum mechanics explains the periodic table and Chemistry
- When tested, Quantum Mechanics has always been found to be correct
  - Some predictions of QM tested to ten decimal places of precision
- Quantum Mechanics is self-consistent, there are no 'paradoxes'
  - So called 'paradoxes of QM' are merely points where its predictions conflict with 'classical' intuition about the nature of reality
- The 'meaning' of Quantum Mechanics is not 'understood'
  - The key difficulty is the origin of the randomness inherent in QM
  - Is true randomness logically tenable?
  - Very different interpretations are equally valid
    - Many-worlds, Bohmian mechanics,…

Global Phase Invariance

$|\psi\rangle$ (physical state) is only defined up to a 'global phase'

$|\psi\rangle \rightarrow e^{i\phi}|\psi\rangle$ for any ϕ

is a gauge transformation

→ all observable properties are unchanged

Proof: all observable/measureable is described in QM by some

"expectation value" $\rightarrow \langle \hat{O} \rangle := \langle \psi | O | \psi \rangle$ — O is some operator

":=" defined as

$\rightarrow \langle \psi | e^{-i\phi} O e^{i\phi} | \psi \rangle$

$= \langle \psi | \hat{O} | \psi \rangle$

→ choose ϕ for convenience

Question #1

- Suppose a particle has the wavefunction:

$$\psi(x) = \frac{1}{\sqrt[4]{\pi\sigma^2}} e^{-(x-x_0)^2/2\sigma^2}$$

- If the position of the particle is measured, what will the result of the measurement be?

is it $\langle x \rangle := \int dx\, \psi^*(x)\, x\, \psi(x) = x_0$?

WRONG, but meaningful with respect to a broad class of actual experiments

$\langle x \rangle$ is average result over many measurements

Correct: a RANDOM NUMBER

$dP(x) = |\psi(x)|^2$

with $P = 68\%$ $\quad x_0 - \sigma < x_{result} < x_0 + \sigma$

$P = 95\%$ $\quad x_0 - 2\sigma < x_{result} < x_0 + 2\sigma$

$P = \int_{x_1}^{x_2} dx\, |\psi(x)|^2$ $\quad x_1 < x_{result} < x_2$

Statement of the Postulates

1. At a fixed time t_0, the state of a physical system is defined by specifying a ket $|\psi(t_0)\rangle$ belonging to the state space of the system
2. Every measureable physical quantity is described by a Hermitian operator A acting in the state space of the system. This operator is called an 'observable'
3. The only possible result of the measurement of the physical property associated with the observable A is one of the eigenvalues of A.
4. When the observable A is measured on a system in the normalized state $|\psi\rangle$, the probability of obtaining eigenvalue a is:* $\quad P(a) = \langle I_a \rangle$
5. If the measurement of the observable A on the system in stqte $|\psi\rangle$ gives the result a the state of the system immediately after the measurement is
$$|\psi'\rangle = \frac{I_a|\psi\rangle}{\sqrt{\langle\psi|I_a|\psi\rangle}}$$
6. The time evolution of the state vector is governed by Schrödinger's equation: $\quad i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$
where H is the operator associated with the total energy of the system

Postulate #1

At a fixed time t_0, the state of an $\boxed{\text{isolated}}$ physical system is defined by specifying a ket $|\psi(t_0)\rangle$ belonging to the state space of the system

$\langle\psi|\psi\rangle = 1$

- what is state space
- what is an isolated system
  - no energy flow in or out and no interaction at all between system and rest of the universe
- no such thing as an isolated system
  - QM can be applied to approximately 'isolated' systems.
- what about QM of open systems? or $T \neq 0$
  - → LATER
- Postulate #1 is for $T = 0$

ASSUME $\langle\psi|\psi\rangle = 1$

Postulate #2

Every measureable physical quantity is described by a Hermitian operator A acting in the state space of the system. This operator is called an 'observable'

- to describe system need:
  - state of system $|\psi(t)\rangle$
  - set of observables

Postulate #3

The only possible result of the measurement of the physical property associated with the observable A is one of the eigenvalues of A.

discrete non-degenerate → $A|a_n\rangle = a_n|a_n\rangle$

discrete degenerate → $A|a_n, m\rangle = a_n|a_n, m\rangle$

continuous non-degenerate → $A|\alpha\rangle = \alpha|\alpha\rangle$

continuous discrete degeneracy → $A|\alpha, m\rangle = \alpha|\alpha, m\rangle$

some different forms of eigenvalue equation

or

$A|\alpha, \beta\rangle = \alpha|\alpha, \beta\rangle$

$A|a_n, \alpha\rangle = a_n|a_n, \alpha\rangle$

Postulate #4

- When an observable is measured, the probability of obtaining a particular eigenvalue is given by the expectation value of the projector onto the subspace associated with that eigenvalue.

$P(a) = \langle I_a \rangle$

I_a is projector onto **subspace** associated with a

Postulate #4a

For the case of a discrete non-degenerate eigenvalue:

$$A|a_n\rangle = a_n|a_n\rangle$$ → assume all a_n's unique

$$I_{a_n} = |a_n\rangle\langle a_n|$$

Projector onto eigenvector $|a_n\rangle$

$$c_n = \langle a_n|\psi\rangle$$

When the observable A is measured on a system in the normalized state $|\psi\rangle$, the probability of obtaining eigenvalue a_n is

$$\boxed{\begin{aligned} P(a_n) &= \langle I_{a_n}\rangle \\ &= \langle\psi|a_n\rangle\langle a_n|\psi\rangle \\ &= |\langle a_n|\psi\rangle|^2 \\ &= |c_n|^2 \end{aligned}}$$

Postulate #4b

For the case of a discrete eigenvalue with a discrete degeneracy:

$$A|a_n,m\rangle = a_n|a_n,m\rangle$$

$$I_{a_n} = \sum_{m=1}^{M_n}|a_n,m\rangle\langle a_n,m|$$

Projector onto degenerate subspace associated with eigenvalue a_n

$$c_{n,m} = \langle a_n,m|\psi\rangle$$

When the observable A is measured on a system in the normalized state $|\psi\rangle$, the probability of obtaining eigenvalue a_n is

$$\boxed{\begin{aligned} P(a_n) &= \langle I_{a_n}\rangle \\ &= \sum_{m=1}^{d_n}\langle\psi|a_n,m\rangle\langle a_n,m|\psi\rangle \\ &= \sum_{m=1}^{d_n}|\langle a_n,m|\psi\rangle|^2 \\ &= \sum_{m=1}^{d_n}|c_{n,m}|^2 \end{aligned}}$$

Here d_n is the degree of degeneracy of the eigenvalue a_n

Postulate #4c

For the case of a continuous non-degenerate eigenvalue:

$$A|\alpha\rangle = \alpha|\alpha\rangle$$
$$I(\alpha) = |\alpha\rangle\langle\alpha|$$
$$\psi(\alpha) = \langle\alpha|\psi\rangle$$

Projector onto eigenstate $|\alpha\rangle$

When the observable A is measured on a system in the normalized state $|\psi\rangle$, the differential probability $dP(\alpha)$ of obtaining a result between α and $\alpha + d\alpha$ is:

$$\boxed{\begin{aligned} dP(\alpha) &= \langle I(\alpha)\rangle d\alpha \\ &= \langle\psi|\alpha\rangle\langle\alpha|\psi\rangle d\alpha \\ &= |\langle\alpha|\psi\rangle|^2 d\alpha \\ &= |\psi(\alpha)|^2 d\alpha \end{aligned}}$$

Postulate #4d

For the case of a continuous eigenvalue with a discrete degeneracy:

$$A|\alpha,m\rangle = \alpha|\alpha,m\rangle$$
$$I(\alpha) = \sum_{m=1}^{M(\alpha)} |\alpha,m\rangle\langle\alpha,m|$$
$$\psi_m(\alpha) = \langle\alpha,m|\psi\rangle$$

Projector onto degenerate subspace associated with *eigenvalue* α

When the observable A is measured on a system in the normalized state $|\psi\rangle$, the probability $dP(\alpha)$ of obtaining a result between α and $\alpha + d\alpha$ is:

$$\boxed{\begin{aligned} dP(\alpha) &= \langle I(\alpha)\rangle\, d\alpha \\ &= \sum_{m=1}^{M(\alpha)} \langle\psi|\alpha,m\rangle\langle\alpha,m|\psi\rangle d\alpha \\ &= \sum_{m=1}^{M(\alpha)} |\langle\alpha,m|\psi\rangle|^2 d\alpha \\ &= \sum_{m=1}^{M(\alpha)} |\psi_m(\alpha)|^2 d\alpha \end{aligned}}$$

Postulate #4e

- Additional cases:]
  - A discrete eigenvalue with a continuous degeneracy
  - A continuous eigenvalue with a continuous degeneracy
- Additional topics
  - Imperfect measurements → "Low" resolution
  - Strong versus weak measurements
  - Single measurements of ensembles of identical systems

- These topics will be covered in the homework

Postulate #5

- Once the measurement is complete and the result is obtained, the state of the system after the measurements is given by the projection of the state before the measurement onto the subspace associated with the resulting eigenvalue.
  - The new state should be re-normalized to unity.

Postulate #5a

The case of a discrete non-degenerate spectrum

•The spectrum of an operator is a list of its eigenvalues and associated degeneracies

If the measurement of the observable A on the system in state $|\psi\rangle$ gives the result a_n, the state of the system immediately after the measurement is the normalized projection onto the subspace associated with a_n :

$$|\psi\rangle \to |\psi'\rangle$$

$$|\psi'\rangle = \frac{I_{a_n}|\psi\rangle}{\sqrt{\langle\psi|I_{a_n}|\psi\rangle}}$$

$$|\psi'\rangle = \frac{|a_n\rangle\langle a_n|\psi\rangle}{\sqrt{\langle\psi|a_n\rangle\langle a_n|\psi\rangle}}$$

$$|\psi'\rangle = |a_n\rangle\frac{\langle a_n|\psi\rangle}{|\langle a_n|\psi\rangle|}$$

Just a global phase factor

$$\boxed{|\psi'\rangle = |a_n\rangle}$$

Postulate #5b

The case of a discrete spectrum with a discrete degeneracy

If the measurement of the observable A on the system in state $|\psi\rangle$ gives a result between α_1 and α_2, the state of the system immediately after he measurement is:

$$|\psi\rangle \to |\psi'\rangle$$

$$|\psi'\rangle = \frac{I_{a_n}|\psi\rangle}{\sqrt{\langle\psi|I_{a_n}|\psi\rangle}}$$

$$I_{a_n} = \sum_{m=1}^{d_n}|a_n,m\rangle\langle a_n,m|$$

$$|\psi'\rangle = \frac{\sum_{m=1}^{d_n}|a_n,m\rangle\, c_{n,m}^{(\text{before})}}{\sqrt{\sum_{m=1}^{d_n}|c_{n,m}|^2}}$$

Postulate #5c

<u>The case of a continuous non-degenerate spectrum</u>

If the measurement of the observable A on the system in state $|\psi\rangle$ gives a result between α_1 and α_2, the state of the system immediately after he measurement is:

- The interval (α_1, α_2) would correspond to the resolution of the detector

$$|\psi\rangle \rightarrow |\psi'\rangle$$

$$|\psi'\rangle = \frac{I_{\alpha_1}^{\alpha_2}|\psi\rangle}{\sqrt{\langle\psi|I_{\alpha_1}^{\alpha_2}|\psi\rangle}}$$

$$I_{\alpha_1}^{\alpha_2} = \int_{\alpha_1}^{\alpha_2} d\alpha |\alpha\rangle\langle\alpha|$$

Postulate #5d

- Additional cases:
  - Discrete spectrum with continuous degeneracy
  - Continuous spectrum with discrete degeneracy
  - Continuous spectrum with continuous degeneracy

Postulate #6

The time evolution of the state vector $|\psi(t)\rangle$ is governed by the Schrödinger equation:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

Where H is the observable associated with the total energy of the system

- H is called the 'Hamiltonian'

How to evolve a state in time:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

given $|\psi(t_0)\rangle$

what is $|\psi(t)\rangle$

- Method I: project onto eigenstates $\{\omega_n\}$ of H: $H|\omega_n\rangle = \hbar\omega_n|\omega_n\rangle$

  - assumes you know the $\{\omega_n\}, \{|\omega_n\rangle\}$

- $H = H^\dagger \rightarrow \omega_n^* = \omega_n$

  $H|\omega_n\rangle = \hbar\omega_n|\omega_n\rangle \rightarrow \langle\omega_n|H = \hbar\omega_n\langle\omega_n|$

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

- hit from left with $\langle \omega_n|$:

$$i\hbar \frac{d}{dt}\langle\omega_n|\psi(t)\rangle = \langle\omega_n|H|\psi(t)\rangle$$

let $c_n(t) := \langle \omega_n|\psi(t)\rangle$

$$i\hbar \frac{d}{dt} c_n(t) = \hbar\omega_n \langle\omega_n|\psi(t)\rangle$$

$$\frac{d}{dt}c_n(t) \overset{*}{=} -i\omega_n c_n(t)$$

- 1st order diff. eq.
- single variable

$$\frac{d}{dt}c_n = -i\omega_n c_n$$

$$\frac{1}{c_n}\, dc_n = -i\omega_n\, dt$$

$$\int_{t_0}^{t} \frac{1}{c_n}\, dc_n = -i\int_{t_0}^{t} \omega_n\, dt$$

$$\ln(c_n(t)) - \ln(c_n(t_0)) = -i\omega(t-t_0)$$

$$\frac{c_n(t)}{c_n(t_0)} = e^{-i\omega_n(t-t_0)}$$

$$\boxed{c_n(t) = c_n(t_0)\, e^{-i\omega_n(t-t_0)}}$$

$$c_n(t_0) = \langle\omega_n|\psi(t_0)\rangle$$

$$
\begin{aligned}
|\psi(t)\rangle &= |\psi(t)\rangle \\
&= I\,|\psi(t)\rangle \\
&= \sum_n |\omega_n\rangle\langle\omega_n|\psi(t)\rangle \\
&= \sum_n |\omega_n\rangle\, c_n(t)
\end{aligned}
$$

$$
\boxed{|\psi(t)\rangle = \sum_n |\omega_n\rangle\, c_n(t_0)\, e^{-i\omega_n(t-t_0)}}
$$