# **Remark: Commutation of Operators**

• Since operators are matrices, they do not necessarily commute

$$AB \neq BA$$

• We define the commutator of the operators A and B as

$$\begin{bmatrix} A, B \end{bmatrix} = AB - BA$$

• The properties of most physically important operators (e.g. *X*, *P*, *L*, *S*,...) can generally be deduced solely from their commutation relations

Definition: 
$$\Psi(x) := \langle x | \Psi \rangle$$
  
T x can be any  
wavefinishion observable  
in x-representation  $\psi(x) = \langle x | \Psi \rangle$   
 $\therefore \Psi(x) = \langle x | \Psi \rangle$   $f(x) = \langle x | \Psi \rangle$   
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 $\therefore \Psi(x) = \langle x | \Psi \rangle$   $f(x) = \langle x | \Psi \rangle$   
 $\therefore \Psi(x) = \langle x | \Psi \rangle$  are two representations  
of the same state  $\langle x | \Psi \rangle$   
 $\Psi(x) = \langle x | \Psi \rangle = \int dx \langle x | \Psi \rangle \langle x | \Psi \rangle$   
 $\Psi(x) = \langle x | \Psi \rangle \langle x | \Psi \rangle = \int dx \langle x | \Psi \rangle \langle x | \Psi \rangle$ 

Lecture 5: Postulates of QM

#### **MOTIVATION TO STUDY QM:**

- Quantum mechanics underlies Nuclear, Particle, Condensed Matter, and Atomic physics (and is thus very important for Astronomy and Astrophysics)
- Quantum mechanics explains the periodic table and Chemistry
- When tested, Quantum Mechanics has always been found to be correct
  - Some predictions of QM tested to ten decimal places of precision
- Quantum Mechanics is self-consistent, there are no 'paradoxes'
  - So called 'paradoxes of QM' are merely points where its predictions conflict with 'classical' intuition about the nature of reality
- The 'meaning' of Quantum Mechanics is not 'understood'
  - The key difficulty is the origin of the randomness inherent in QM
  - Is true randomness logically tenable?
  - Very different interpretations are equally valid
    - Many-worlds, Bohmian mechanics,...

#### Question #1

• Suppose a particle has the wavefunction:

$$\psi(x) = \frac{1}{\sqrt[4]{\pi\sigma^2}} e^{-(x-x_0)^2/2\sigma^2}$$

• If the position of the particle is measured, what will the result of the measurement be?

is it 
$$\langle X \rangle := \int dx \, V(x) \, x \, V(x) = x_0$$
?  

$$\frac{WRONG}{5} , b.t meaningfull with respect}{5} a broad class of actual experiments}$$

$$CX is average result over many measurements$$

$$\frac{Correct:}{5} a \frac{BANDOM}{5} \frac{NUMDER}{5}$$

$$\frac{dP(x) = |V(x)|^2}{5}$$
with  $P = 687$ ,  $x_0 - c < x_{result} < x_0 + c$ 

$$P : 95%, x_0 - 26 < x_{result} < x_1 + 2c$$

$$P = \int dx |V(u)|^2 \quad x_1 < x_{result} < x_2$$

#### Statement of the Postulates

- 1. At a fixed time  $t_q$ , the state of a physical system is defined by specifying a ket  $|\psi(t_q)\rangle$  belonging to the state space of the system
- 2. Every measureable physical quantity is described by a Hermitian operator *A* acting in the state space of the system. This operator is called an 'observable'
- 3. The only possible result of the measurement of the physical property associated with the observable *A* is one of the eigenvalues of *A*.
- 4. When the observable *A* is measured on a system in the normalized state  $|\psi\rangle$ , the probability of obtaining eigenvalue *a* is:\*  $P(a) = \langle I_a \rangle$
- 5. If the measurement of the observable *A* on the system in stqte  $|\psi\rangle$  gives the result *a* the state of the system immediately after the measurement is

$$\left|\psi'\right\rangle = \frac{I_{a}\left|\psi\right\rangle}{\sqrt{\left\langle\psi\left|I_{a}\right|\psi\right\rangle}}$$

6. The time evolution of the state vector is governed by Schrödinger's equation:  $i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle$ 

where H is the operator associated with the total energy of the system

#### Postulate #1

At a fixed time  $t_0$ , the state of an isolated physical system is defined by specifying a ket  $|\psi(t_0)\rangle$  belonging to the state space of the system

L+(1) =1 - what is state space - what is an isolated system - no evergy flow in or out and no interaction at all between system and nest of the universe - no such thing as an isolated system · QM can be applied to approximately 'isolated' systems - what about QM of open systems? or T=0 -> LATER - Postulate #1 is for T=0 ASSUME (414)=

#### Postulate #2

Every measureable physical quantity is described by a Hermitian operator A acting in the state space of the system. This operator is called an 'observable'

- to describe system need: - state of system (YCH)> - set of observables

## Postulate #3

The only possible result of the measurement of the physical property associated with the observable A is one of the eigenvalues of A.

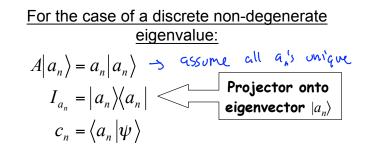
discrete non-desenvente $\rightarrow A   a_n$ , degenerate	$ \begin{array}{l} a_n \rangle = a_n   a_n \rangle \\ m \rangle = a_n   a_n, m \rangle \end{array} $	some different forms of
continuous -> A	$ \begin{array}{c} \alpha \rangle = \alpha  \alpha\rangle \\ m \rangle = \alpha  \alpha, m\rangle \\ \end{array} $	eigenvalue equation
discrete desenaracy.	$A \alpha,\beta\rangle = \alpha \alpha,$ $A \alpha_n,\alpha_n\rangle = \alpha_n $	

#### Postulate #4

• When an observable is measured, the probability of obtaining a particular eigenvalue is given by the expectation value of the projector onto the subspace associated with that eigenvalue.

$$P(a) = \langle I_a \rangle$$
  
 $I_a$  is projector onto  
subspace associated  
with a

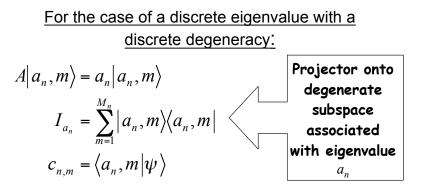
#### Postulate #4a



When the observable *A* is measured on a system in the normalized state  $|\psi\rangle$ , the probability of obtaining eigenvalue  $a_n$  is

$$P(a_n) = \langle I_{a_n} \rangle$$
$$= \langle \psi | a_n \rangle \langle a_n | \psi \rangle$$
$$= |\langle a_n | \psi \rangle|^2$$
$$= |c_n|^2$$

#### Postulate #4b



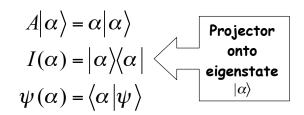
When the observable *A* is measured on a system in the normalized state  $|\psi\rangle$ , the probability of obtaining eigenvalue  $a_n$  is

$$P(a_n) = \langle I_{a_n} \rangle$$
  
=  $\sum_{m=1}^{d_n} \langle \psi | a_n, m \rangle \langle a_n, m | \psi \rangle$   
=  $\sum_{m=1}^{d_n} |\langle a_n, m | \psi \rangle|^2$   
=  $\sum_{m=1}^{d_n} |c_{n,m}|^2$ 

Here  $d_n$  is the degree of degeneracy of the eigenvalue  $a_n$ 

#### Postulate #4c

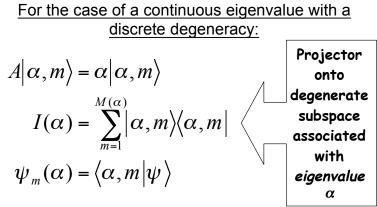
For the case of a continuous non-degenerate eigenvalue:



When the observable *A* is measured on a system in the normalized state  $|\psi\rangle$ , the differential probability  $dP(\alpha)$  of obtaining a result between  $\alpha$ and  $\alpha + d\alpha$  is:

$$dP(\alpha) = \langle I(\alpha) \rangle d\alpha$$
$$= \langle \psi | \alpha \rangle \langle \alpha | \psi \rangle d\alpha$$
$$= |\langle \alpha | \psi \rangle|^2 d\alpha$$
$$= |\psi(\alpha)|^2 d\alpha$$

#### Postulate #4d



When the observable *A* is measured on a system in the normalized state  $|\psi\rangle$ , the probability  $dP(\alpha)$ of obtaining a result between  $\alpha$  and  $\alpha + d\alpha$  is:

$$dP(\alpha) = \langle I(\alpha) \rangle \, d\alpha$$
  
=  $\sum_{m=1}^{M(\alpha)} \langle \psi | \alpha, m \rangle \langle \alpha, m | \psi \rangle d\alpha$   
=  $\sum_{m=1}^{M(\alpha)} |\langle \alpha, m | \psi \rangle|^2 d\alpha$   
=  $\sum_{m=1}^{M(\alpha)} |\psi_m(\alpha)|^2 d\alpha$ 

## Postulate #4e

- Additional cases:]
  - A discrete eigenvalue with a continuous degeneracy
  - A continuous eigenvalue with a continuous degeneracy
- Additional topics
  - Imperfect measurements -> "Low" resolution
  - Strong versus weak measurements
  - Single measurements of ensembles of identical systems
- These topics will be covered in the homework

# Postulate #5

- Once the measurement is complete and the result is obtained, the state of the system after the measurements is given by the projection of the state before the measurement onto the subspace associated with the resulting eigenvalue.
  - The new state should be re-normalized to unity.

## Postulate #5a

#### The case of a discrete non-degenerate spectrum

•The spectrum of an operator is a list of its eigenvalues and associated degeneracies

If the measurement of the observable *A* on the system in state  $|\psi\rangle$  gives the result  $a_n$ , the state of the system immediately after the measurement is the normalized projection onto the subspace associated with  $a_n$ :

#### Postulate #5b

# The case of a discrete spectrum with a discrete degeneracy

If the measurement of the observable *A* on the system in state  $|\psi\rangle$  gives a result between  $\alpha_1$  and  $\alpha_2$ , the state of the system immediately after he measurement is:

$$\begin{split} |\psi\rangle &\to |\psi'\rangle \\ |\psi'\rangle &= \frac{I_{a_n}|\psi\rangle}{\sqrt{\langle\psi|I_{a_h}^{(0)}|\psi\rangle}} \\ I_{a_n} &= \sum_{m=1}^{d_n} |a_n,m\rangle \langle a_n,m| \\ \psi'\rangle &= \underbrace{\underset{m=1}{\overset{d_n}{\underset{m=1}{\overset{(q_{n_1}m)}{\underset{m=1}{\overset{(before)}{\underset{m=1}{\overset{(c_{n_1}m)}{\underset{m=1}{\underset{m=1}{\overset{(c_{n_1}m)}{\underset{m=1}{\underset{m=1}{\overset{(c_{n_1}m)}{\underset{m=1}{\underset{m}}{\underset{m}}{\underset{m}}{\underset{m}}{\underset{m}}}}}$$

#### Postulate #5c

The case of a continuous non-degenerate spectrum

If the measurement of the observable *A* on the system in state  $|\psi\rangle$  gives a result between  $\alpha_1$  and  $\alpha_2$ , the state of the system immediately after he measurement is:

– The interval ( $\alpha_{l_1} \alpha_2$ ) would correspond to the resolution of the detector

$$\begin{split} |\psi\rangle &\to |\psi'\rangle \\ |\psi'\rangle &= \frac{I_{\alpha_1}^{\alpha_2} |\psi\rangle}{\sqrt{\langle\psi |I_{\alpha_1}^{\alpha_2}|\psi\rangle}} \\ I_{\alpha_1}^{\alpha_2} &= \int_{\alpha_1}^{\alpha_2} d\alpha |\alpha\rangle \langle\alpha| \end{split}$$

## Postulate #5d

- Additional cases:
  - Discrete spectrum with continuous degeneracy
  - Continuous spectrum with discrete degeneracy
  - Continuous spectrum with continuous degeneracy

#### Postulate #6

The time evolution of the state vector  $|\psi(t)\rangle$  is governed by the Schrödinger equation:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

Where H is the observable associated with the total energy of the system

- H is called the 'Hamiltonian'

How to evolve a state in time: it d (4(+)) = H(4(+)) given (4(+)) what is (4(+)) - Method I: project onto eigenstates {wn} of H : H(wn) = twn (wn) - assume you know the [wn3, E(wn)] - H = H<sup>+</sup>  $\rightarrow w_n^{+} = w_n$ H(wn) = twn (wn)  $\rightarrow (w_n) H = twn (w_n)$ 

$$i\frac{d}{dt}(i\frac{d}{dt}) = H(i\frac{d}{dt})$$

$$-hit \quad from \quad left \quad with \quad Cw_n(i):$$

$$i\frac{d}{dt}(w_n(i\frac{d}{dt}) = Cw_n(i\frac{d}{dt}))$$

$$i\frac{d}{dt}(w_n(i\frac{d}{dt}) = Cw_n(i\frac{d}{dt}))$$

$$i\frac{d}{dt}(w_n(i): = Cw_n(i\frac{d}{dt}))$$

$$i\frac{d}{dt}(w_n(i) = hw_n(w_n(i\frac{d}{dt})))$$

$$\frac{d}{dt}(w_n(i) = hw_n(w_n(i\frac{d}{dt})))$$

$$\frac{d}{dt}(w_n(i) = -iw_n(w_n(i\frac{d}{dt})))$$

$$-ist \quad order \quad diff. \quad eq.$$

$$-sigle \quad variable$$

$$\frac{d}{dt}C_{n} = -iv_{n}C_{n}$$

$$\frac{d}{dt}C_{n} = -iw_{n}dt$$

$$\int_{c_{n}}^{t} \frac{1}{C_{n}}dc_{n} = -i\int_{w_{n}}^{t}w_{n}dt$$

$$\int_{t_{0}}^{t} \frac{1}{C_{n}}dc_{n} = -i\int_{t_{0}}^{t}w_{n}dt$$

$$\int_{t_{0}}^{t} \frac{1}{C_{n}}dc_{n} = -iw(t-t_{0})$$

$$-iw_{n}(t-t_{0})$$

$$\int_{c_{n}}^{c}(t_{0}) = -iw(t-t_{0})$$

$$\int_{c_{n}}^{c}(t_{0}) = -iw(t-t_{0})$$

