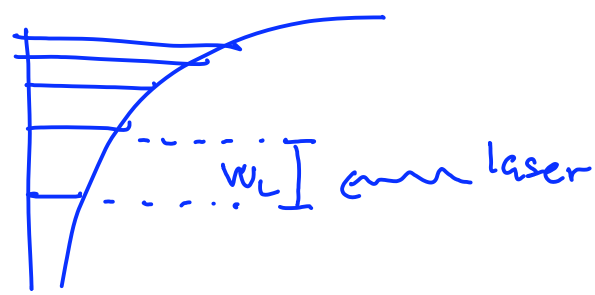
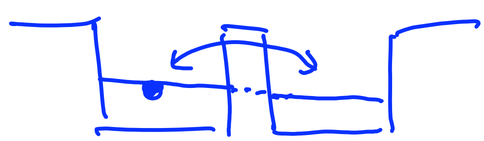
General Study of Two-Level Systems

- Goals:
  - Study ‘Quantum Resonance’ behavior
  - Discuss ‘Avoided Crossings’ and ‘Adiabatic Passage’
  - Study connection between Spin-1/2 and general two-level systems

- Examples of two level systems:
  - A Spin 1/2 particle
  - A ‘two-level atom’



    - An atom driven with an oscillating E-field whose frequency closely matches one of the atomic transition frequencies → can ignore other levels
  - Particle in a double-well potential
    - E.g. electron in a double quantum dot



    - Tunneling couples the lowest level on the left side with the lowest level on the right side

Generic Two-level Hamiltonian

- Consider a system with two quantum energy levels, and a Hamiltonian H_0
  - The eigenstates satisfy:

$$H_0|1\rangle = \hbar\omega_1|1\rangle$$
$$H_0|2\rangle = \hbar\omega_2|2\rangle$$

  - So that:

$$H_0 = \hbar\omega_1|1\rangle\langle 1| + \hbar\omega_2|2\rangle\langle 2|$$

  - In the $\{|1\rangle, |2\rangle\}$ basis, H_0 is represented by the matrix:

$$H_0 = \hbar\begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

  - The evolution of the system is then:

$$|\psi(t)\rangle = |1\rangle e^{-i\omega_1 t}\langle 1|\psi(0)\rangle + |2\rangle e^{-i\omega_2 t}\langle 2|\psi(0)\rangle$$

Adding a Perturbation

- Suppose we suddenly change the Hamiltonian, e.g. by turning on an external field
  - So that $H \rightarrow H_0 + W$, where

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{12}^* & W_{22} \end{pmatrix}$$

- Time Independant Perturbation

$$H = \begin{pmatrix} \hbar\omega_1 + W_{11} & W_{12} \\ W_{12}^* & \hbar\omega_2 + W_{22} \end{pmatrix}$$

  - Question#1: What happens to the eigenstates?
  - Question#2: Do we induce transitions from $|\omega_1\rangle$ $|1\rangle$ to $|\omega_2\rangle$?

$|2\rangle$

Choosing a Zero of Energy

- Adding a constant times the identity operator to H is equivalent to choosing a new zero of energy
  - Physical predictions unchanged
  - Lets choose $E_0 = -\tfrac{1}{2}\left(\hbar\omega_1 + W_{11} + \hbar\omega_2 + W_{22}\right)$
  - Which gives

$$H = \begin{pmatrix} \tfrac{1}{2}\left(\hbar\omega_1 + W_{11} - \hbar\omega_2 - W_{22}\right) & W_{12} \\ W_{21} & -\tfrac{1}{2}\left(\hbar\omega_1 + W_{11} - \hbar\omega_2 - W_{22}\right) \end{pmatrix}$$

  - Introduce the 'detuning' and 'Rabi frequency':

$$\Delta = \left(\omega_2 + \frac{W_{22}}{\hbar} - \omega_1 - \frac{W_{11}}{\hbar}\right)$$

$$\Omega = \frac{2W_{21}}{\hbar}$$

  - Δ is the energy spacing between the perturbed levels $|\omega_1\rangle$ and $|\omega_2\rangle$
  - Ω is the strength of the coupling between them
  - The Hamiltonian is then:

$$\boxed{H = \frac{\hbar}{2}\begin{pmatrix} -\Delta & \Omega^* \\ \Omega & \Delta \end{pmatrix}}$$

Most General Model of a Two level system (time - indep.)

  - This is the two-level **`Rabi Model'**

Physical Examples

- In a spin-1/2 system:

$$\Delta = \frac{\gamma}{\hbar} B_z \sigma_z$$ Zeeman splitting

$$\Omega = \frac{\lambda}{\hbar} B_x \sigma_x$$ Applied B-field along x-axis

- Laser-driven Two level atom:

$$\Delta = \left(E_u - E_\ell - \hbar\omega_L\right)/\hbar$$ Difference between laser and transition frequencies

$$\Omega = \frac{2dE_0}{\hbar}$$ Atomic dipole-moment times electric field amplitude

- Double-well potential

$$\Delta = \left(E_L - E_R\right)/\hbar$$ Double-well tilt

$$\Omega = \frac{\langle \varphi_L | T | \varphi_R \rangle}{\hbar}$$ Tunneling matrix element

Eigenvalues and Eigenvectors

- Lets Find the Eigenvalues and Eigenvectors:

$$\boxed{\begin{pmatrix} -\Delta/2-\omega_{\pm} & \Omega^{*}/2 \\ \Omega/2 & \Delta/2-\omega_{\pm} \end{pmatrix}\begin{pmatrix} \langle 1|\omega_{\pm}\rangle \\ \langle 2|\omega_{\pm}\rangle \end{pmatrix}=0}$$

$$H|\omega_+\rangle = \hbar\omega_+|\omega_+\rangle \rightarrow (H-\hbar\omega_+ I)|\omega_+\rangle = 0$$

$$\det\begin{vmatrix} -\Delta/2-\omega_\pm & \Omega^*/2 \\ \Omega/2 & \Delta/2-\omega_\pm \end{vmatrix} = 0$$

$$+\left(\frac{\Delta}{2}+\omega_\pm\right)\left(\omega_\pm-\frac{\Delta}{2}\right) - \frac{|\Omega|^2}{4} = 0$$

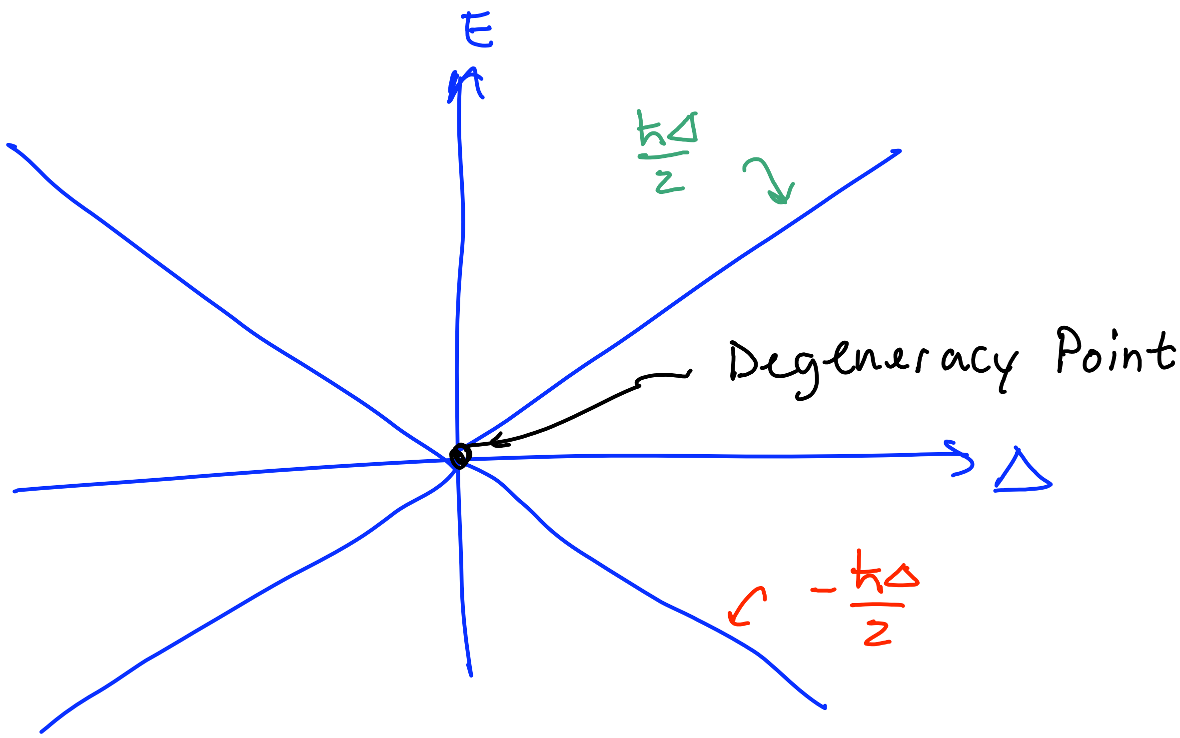
$$\omega_\pm^2 - \frac{\Delta^2}{4} - \frac{|\Omega|^2}{4} = 0$$

$$\boxed{\omega_\pm = \pm\frac{1}{2}\sqrt{\Delta^2+|\Omega|^2}}$$

Level Crossing

- Energy spectrum versus Δ for Ω=0:

  - no coupling between $|1\rangle$ and $|2\rangle$



$$H = \frac{\hbar}{2}\begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix}$$

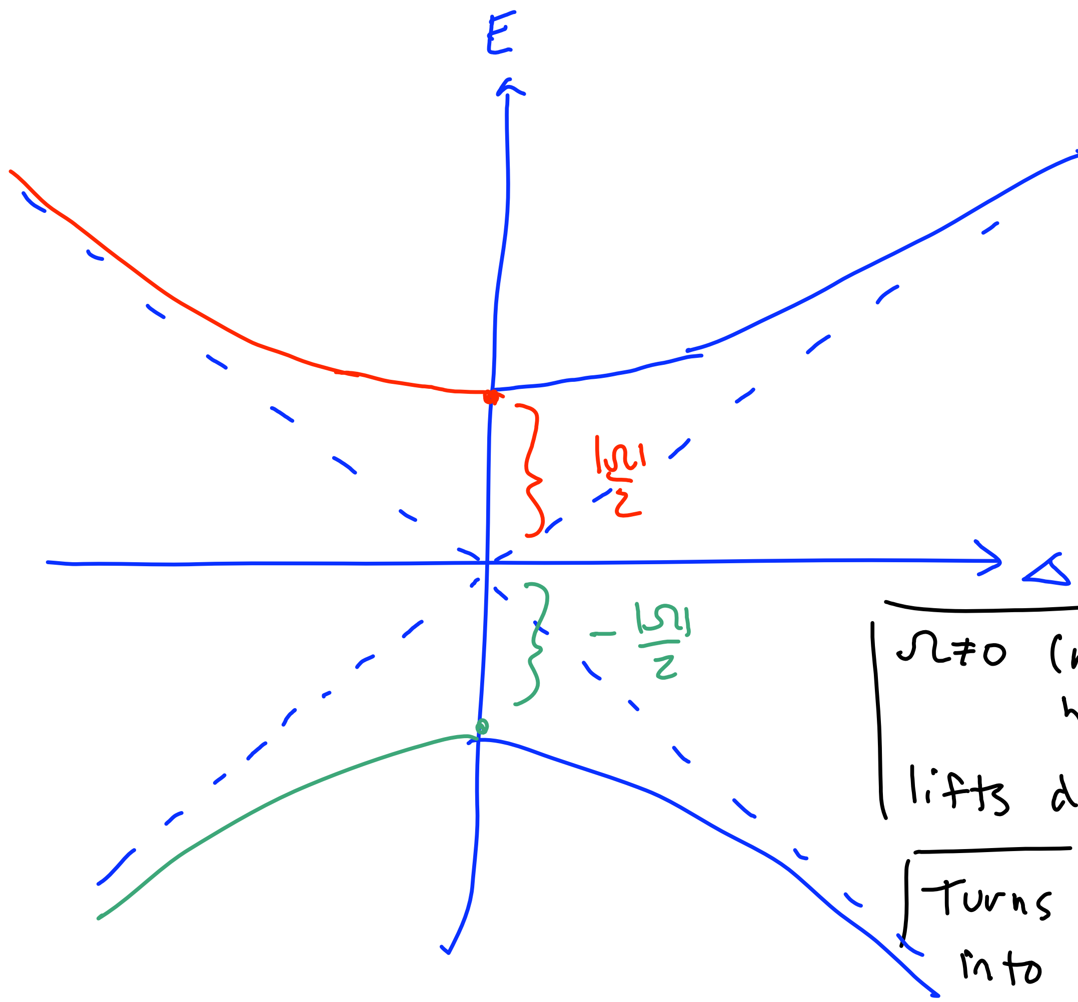
$$E_1 = -\frac{\hbar\Delta}{2} \longrightarrow |1\rangle$$

$$E_2 = \frac{\hbar\Delta}{2} \longrightarrow |2\rangle$$

- Degeneracy point → Level crossing

Avoided Level Crossing

- Energy spectrum versus Δ for $\underset{\sim}{\Omega \neq 0}$:



$\Omega \neq 0$ (no matter how small) lifts degeneracy

Turns crossing into an "avoided crossing"

$$\omega_\pm = \pm\frac{1}{2}\sqrt{\Delta^2 + |\Omega|^2}$$

for $\Delta = 0$

$\omega_+ = \frac{|\Omega|}{2}$ ———— $|\omega_+\rangle$ $E_+ = \hbar\omega_+$

$\omega_- = -\frac{|\Omega|}{2}$ ———— $|\omega_-\rangle$ $E_- = \hbar\omega_-$

for $\Delta \to \pm\infty$

$\omega_+ \to \frac{|\Delta|}{2}$

$\omega_- \to -\frac{|\Delta|}{2}$

$E_{gap} = \hbar|\Omega|$