LECTURE 8: RABI MODEL II

OUTLINE:

- 'Dressed states' vs. 'bare states'
- Rabi oscillations
- 'Adiatic following' and the 'sudden approximation'
- Quantum resonance: periodically driven 2-level system

Dressed States

Definition: let $H = H_0 + V$

H_0 called 'bare Hamiltonian' (unperturbed)

V called 'perturbation'

- Eigenstates of H_0 are called 'bare eigenstates'
  - often the 'physical states'
- Eigenstates of $H = H_0 + V$ are called 'dressed states'
- both sets of eigenstates can be used as a complete basis

Dressed States in Rabi Model

$$H = \frac{1}{2}\begin{pmatrix} -\Delta & \Omega^* \\ \Omega & \Delta \end{pmatrix}$$

default: $\Omega^* = \Omega$, $\hbar = 1$

- found eigenvalues: $\omega_\pm = \pm \frac{1}{2}\sqrt{\Delta^2 + \Omega^2}$
- eigenstate equation: in $\{|1\rangle, |2\rangle\}$ basis

$(H - I\omega)|\omega\rangle = 0$

$$\begin{pmatrix} -\frac{\Delta}{2} - \omega_\pm & \frac{\Omega^*}{2} \\ \frac{\Omega}{2} & \frac{\Delta}{2} - \omega_\pm \end{pmatrix}\begin{pmatrix} \langle 1|\omega_\pm\rangle \\ \langle 2|\omega_\pm\rangle \end{pmatrix} = 0$$

4 eqs. $\rightarrow$

Two equivalent choices:

A) use 1st row: $\left(-\frac{\Delta}{2}-\omega_\pm\right)\langle 1|\omega_\pm\rangle + \frac{\Omega^*}{2}\langle 2|\omega_\pm\rangle = 0$

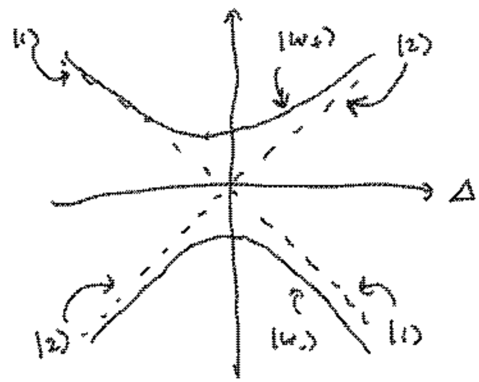
take: $\langle 1|\omega_\pm\rangle = \frac{\Omega^*}{2}$, $\langle 2|\omega_\pm\rangle = \frac{\Delta}{2}+\omega_\pm$

→ then normalize $|\omega_\pm\rangle$

B) use 2nd row: $\frac{\Omega}{2}\langle 1|\omega_\pm\rangle + \left(\frac{\Delta}{2}-\omega_\pm\right)\langle 2|\omega_\pm\rangle = 0$

take: $\langle 1|\omega_\pm\rangle = \frac{\Delta}{2}-\omega_\pm$, $\langle 2|\omega_\pm\rangle = -\frac{\Omega}{2}$

→ then normalize $|\omega_\pm\rangle$



$E_1 = -\frac{\Delta}{2}$

$E_2 = \frac{\Delta}{2}$

$\omega_\pm = \pm\frac{1}{2}|\Delta|$

-Dressed states:

$\boxed{\text{For } \Delta > 0}$ we want $|\omega_+\rangle \to |2\rangle$ as $\underbrace{\Omega \to 0 \text{ or } \Delta \to \infty}_{\Delta \gg \Omega}$

$|\omega_-\rangle \to |1\rangle$

$|\omega_+\rangle$: use A)

$$|\omega_+\rangle = \frac{\Omega^*|1\rangle + (\Delta + \sqrt{\Delta^2+|\Omega|^2})|2\rangle}{\sqrt{|\Omega|^2 + (\Delta+\sqrt{\Delta^2+|\Omega|^2})^2}}$$

$|\omega_-\rangle$: use B)

$$|\omega_-\rangle = \frac{(\Delta + \sqrt{\Delta^2+|\Omega|^2})|1\rangle - \Omega|2\rangle}{\sqrt{|\Omega|^2 + (\Delta+\sqrt{\Delta^2+|\Omega|^2})^2}}$$

$\boxed{\text{for } \Delta < 0}$ want $|\omega_+\rangle \to |1\rangle$ as $\underbrace{\Omega \to 0 \text{ or } \Delta \to -\infty}_{\Delta \ll -|\Omega|}$
$|\omega_-\rangle \to |2\rangle$

$|\omega_+\rangle$: use B)

$$|\omega_+\rangle = \frac{(\sqrt{\Delta^2 + |\Omega|^2} + |\Delta|)|1\rangle + \Omega|2\rangle}{\sqrt{\quad \sim Z_0}}$$ ← multiplied by −1

$|\omega_-\rangle$: use A)

$$|\omega_-\rangle = \frac{-\Omega^*|1\rangle + (\sqrt{\Delta^2 + |\Omega|^2} + |\Delta|)|2\rangle}{\sqrt{Z_0}}$$

Rabi Oscillations

- get dynamical solutions without finding eigenstates

• $H = -\frac{\Delta}{2}|1\rangle\langle 1| + \frac{\Delta}{2}|2\rangle\langle 2| + \frac{\Omega^*}{2}|1\rangle\langle 2| + \frac{\Omega}{2}|2\rangle\langle 1|$

• $\frac{d}{dt}|\Psi(t)\rangle = -i H |\Psi(t)\rangle$

hit with $\langle 1| \to$

$$\frac{d}{dt}\langle 1|\Psi\rangle = +i\frac{\Delta}{2}\langle 1|\Psi\rangle - i\frac{\Omega^*}{2}\langle 2|\Psi\rangle$$

hit with $\langle 2| \to$

$$\frac{d}{dt}\langle 2|\Psi\rangle = -i\frac{\Delta}{2}\langle 2|\Psi\rangle - i\frac{\Omega}{2}\langle 1|\Psi\rangle$$

let $c_1 := \langle 1|\Psi\rangle \quad c_2 := \langle 2|\Psi\rangle$

$$\boxed{\begin{aligned} \dot{c}_1 &= i\frac{\Delta}{2}c_1 - i\frac{\Omega^*}{2}c_2 \\ \dot{c}_2 &= -i\frac{\Delta}{2}c_2 - i\frac{\Omega}{2}c_1 \end{aligned}}$$

Solution:

$$\ddot{c}_1 = i\frac{\Delta}{2}\dot{c}_1 - i\frac{\Omega}{2}\dot{c}_2$$

$$= i\frac{\Delta}{2}\left(i\frac{\Delta}{2}c_1 - i\frac{\Omega}{2}c_2\right) - i\frac{\Omega}{2}\left(-i\frac{\Delta}{2}c_2 - i\frac{\Omega}{2}c_1\right)$$

$$= -\frac{\Delta^2}{4}c_1 + \cancel{\frac{\Omega\Delta}{4}c_2} - \cancel{\frac{\Omega\Delta}{4}c_2} - \frac{\Omega^2}{4}c_1$$

$$\boxed{\ddot{c}_1 = -\frac{1}{4}(\Delta^2+\Omega^2)c_1}$$ by symmetry $$\boxed{\ddot{c}_2 = -\frac{1}{4}(\Delta^2+\Omega^2)c_2}$$

let $\omega = \frac{1}{2}\sqrt{\Delta^2+\Omega^2}$

well-known solution:

$$c_1(t) = c_1(0)\cos(\omega t) + \frac{\dot{c}_1(0)}{\omega}\sin(\omega t)$$

$$c_2(t) = c_2(0)\cos(\omega t) + \frac{\dot{c}_2(0)}{\omega}\sin(\omega t)$$

$$\boxed{\begin{aligned} c_1(t) &= c_1(0)\cos(\omega t) - i\frac{(\Omega c_2(0) - \Delta c_1(0))}{\sqrt{\Delta^2+\Omega^2}}\sin(\omega t) \\ c_2(t) &= c_2(0)\cos(\omega t) - i\frac{(\Omega c_1(0) + \Delta c_2(0))}{\sqrt{\Delta^2+\Omega^2}}\sin(\omega t) \end{aligned}}$$

~ What is basic physics of 'population dynamics' in Rabi model?

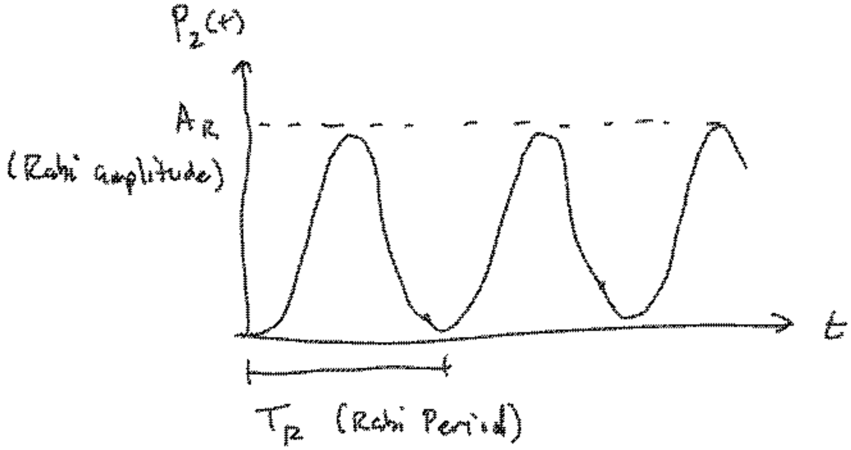
$P_n(t) = |c_n(t)|^2$

case I | let $c_1(0) = 1$, $c_2(0) = 0$ $\Big\} \to$ i.e. system in ground state ($\Delta=0$, $\Omega=0$) at $t=0$, turn on Ω

$$c_2(t) = \frac{-i\Omega}{\sqrt{\Delta^2+\Omega^2}}\sin(\omega t)$$

$$P_2(t) = |c_2(t)|^2 = \frac{\Omega^2}{\Delta^2+\Omega^2}\sin^2(\omega t)$$

$$P_1(t) = 1 - |c_2(t)|^2 = \frac{\Delta^2 + \Omega^2 \cos^2(\omega t)}{\Delta^2 + \Omega^2}$$



← Rabi Oscillations

$$T_R = \frac{\pi}{\omega} = \frac{2\pi}{\sqrt{\Delta^2 + \Omega^2}}$$

$T_R^{-1} = f_R$ frequency of Rabi Oscillations

$$= \frac{\sqrt{\Delta^2 + \Omega^2}}{2\pi}$$

$$A_R = \frac{\Omega^2}{\Delta^2 + \Omega^2}$$ ← Strength of response to V

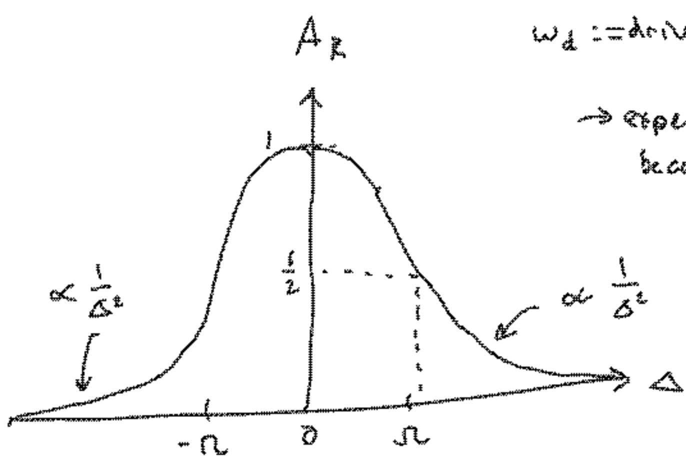
$\omega_0 :=$ Resonance freq. $= \Delta$

$\omega_d :=$ drive freq. $= 0$

let $H_0 = \begin{pmatrix} -\frac{\Delta}{2} & 0 \\ 0 & \frac{\Delta}{2} \end{pmatrix}$

$V = \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix}$

→ expect strongest response for $\Delta = 0$ because $w_d = w_0$ (on-resonance)



→ LORENTZIAN LINESHAPE

- 'standard' resonance profile
- width of the Rabi resonance is Ω (HWHM)

Δ_{HWHM} is solution to: $\frac{\Omega^2}{\Delta^2 + \Omega^2} = \frac{1}{2}$

$$\Rightarrow \Omega^2 = \frac{1}{2}\Omega^2 + \frac{1}{2}\Delta^2 \Rightarrow \Delta_{HWHM} = \Omega$$