# LECTURE 9: RABI MODEL III

- 'Dressed States' vs. 'bare states' ✓
- Rabi oscillations ✓
    - Pulse Area theorem
- 'Adiabatic Following' and 'Sudden Approximation'
- Periodically driven 2×2 system (Quantum Resonance)

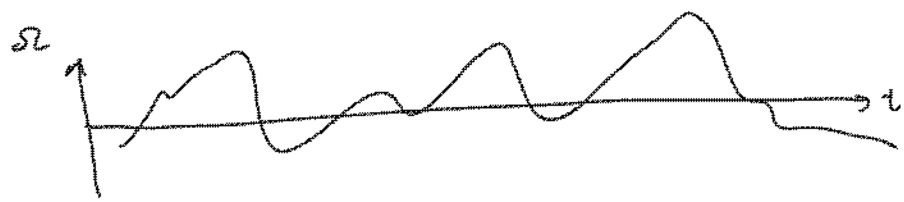
---

**Case II** let $\Delta = 0$ (on-resonance)

$c_2(0) = 0$

$c_1(0) = 1$

let $\Omega \to \Omega(t)$ (real)




- now derive 'pulse area theorem'
    - claim: final state depends only on pulse area $= \int_{t_0}^{t} dt' \Omega(t')$

$t_0$ = start time

$$\frac{d}{dt} c_1 = i\frac{\Delta}{2} c_1 - i\frac{\Omega}{2} c_2$$

$$\frac{d}{dt} c_2 = -i\frac{\Delta}{2} c_2 - i\frac{\Omega}{2} c_1$$

$\Delta \to 0 \quad \Omega \to \Omega(t)$

$$\frac{d}{dt}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\frac{i\Omega(t)}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\frac{d}{dt}\vec{y} = -i\frac{\Omega(t)}{2}M\vec{y}$$

note that : $[H(t), H(t')] = \frac{\Omega^2(t)}{2}[M, M] = 0$

therefore we have: $\vec{y}(t) = e^{\overbrace{-\frac{i}{2}\int_{t_0}^{t} dt'\,\Omega(t')\,M}} \vec{y}(0)$

2×2 matrix
- the time propagator

let $\theta = \int_{t_0}^{t} dt'\,\Omega(t')$ ← 'pulse area'

$\vec{y}(t) = e^{-i\frac{\theta}{2}M}\,\vec{y}(0)$ → theorem is now proven.

$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , $M^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\vec{y}(t) = \left[\cos\left(\frac{\theta}{2}M\right) - i\sin\left(\frac{\theta}{2}M\right)\right]\vec{y}(0)$

Aside: for numbers $(a+b)^2 = a^2 + 2ab + b^2$

but watch out $(A+B)^2 = A^2 + AB + BA + B^2$

- so don't trust familiar relations between functions.

$\cos\left(\frac{\theta}{2}M\right) = \left(\frac{\theta}{2}M\right)^0 - \frac{1}{2}\left(\frac{\theta}{2}M\right)^2 + \ldots$ (even powers only)

$= I - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 M^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 M^4 + \ldots$

$$= I - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 I + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 I + \ldots$$

$$= I \cos\left(\frac{\theta}{2}\right)$$

$$\sin\left(\frac{\theta}{2} M\right) = \left(\frac{\theta}{2} M\right) - \frac{1}{3!}\left(\frac{\theta}{2} M\right)^3 + \ldots \quad \text{odd powers}$$

$$= M\left[\frac{\theta}{2} I - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 M^2 + \ldots\right]$$

$$= M\left[\frac{\theta}{2} I - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 I + \ldots\right]$$

$$= M \cdot I \cdot \sin\left(\frac{\theta}{2}\right)$$

$$= M \sin\left(\frac{\theta}{2}\right)$$

$$\vec{y}(t) = \left[I \cos\left(\frac{\theta}{2}\right) - i M \sin\frac{\theta}{2}\right] \vec{y}(0)$$

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -i \sin\frac{\theta}{2} \\ -i \sin\frac{\theta}{2} & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

just a choice
- not part of PA theorem

$$\boxed{\begin{aligned} c_1(t) &= \cos\left(\frac{\theta}{2}\right) \\ c_2(t) &= -i \sin\left(\frac{\theta}{2}\right) \end{aligned}}$$

---

- common pulse areas:

let $\theta = \pi$ "$\pi$-pulse"

$c_1(t) = 0 \rightarrow |c_1|^2 = 0$

$c_2(t) = -i \quad |c_2|^2 = 1$

$\rightarrow$ Flipped Probabilities

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -i \end{pmatrix}$$

let $\underline{\theta = 2\pi}$ "$2\pi$ - pulse"

$c_1(t) = -1$
$c_2(t) = 0$
→ Probabilities are same but added global phase shift.

let $\underline{\theta = 3\pi}$

$c_1(t) = 0$

$c_2(t) = i$

let $\underline{\theta = 4\pi}$

$c_1(t) = 1$
$c_2(t) = 0$
→ back to where we started

let $\underline{\theta = \frac{\pi}{2}}$

$c_1 = \frac{1}{\sqrt{2}}$
$c_2 = -\frac{i}{\sqrt{2}}$
→ even split of probability

---

FOR $\Delta = 0$, FINAL STATE DEPENDS ONLY ON PULSE AREA

- makes state preparation much easier experimentally

---

NOTE: for $\Delta \neq 0$ $[H(t), H(t')] \neq 0$

→ no theorem

→ need numerical methods

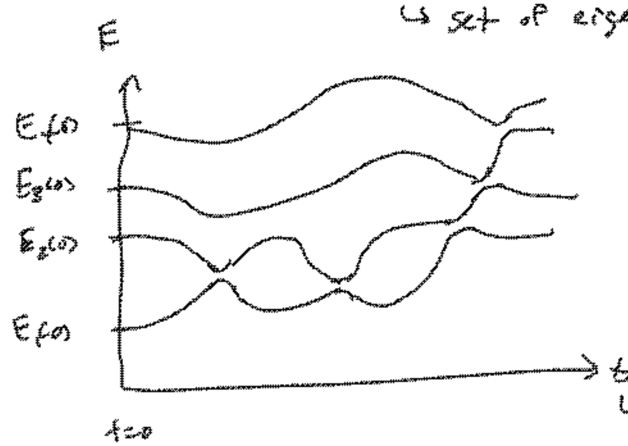
# ADIABATIC FOLLOWING

- let $H = H(t)$

- at any instant, there is a set of eigenstates $\{|E_n(t)\rangle\}$ and eigenvalues $\{E_n(t)\}$

they satisfy: $H(t)|E_n(t)\rangle = E_n(t)|E_n(t)\rangle$

→ If a parameter in $H$ is changed with time we can plot the spectrum vs time
↳ set of eigenvalues (and degeneracies)




- All crossings are really avoided crossings "no crossing theorem"

↳ or any parameter in H

- let $T$ be the interval over which a parameter is varied

→ Two limiting cases:

$T \to \infty$ : Adiabatic theorem

$T \to 0$ : Sudden Approximation

## Adiabatic Theorem: $T \to \infty$

- A system initially in state $|E_n(0)\rangle$ at $t=0$ it will remain in state $|E_n(t)\rangle$ (n is conserved)

- Energy not conserved, but $H = H(t)$ so not surprising

- $|\psi(0)\rangle = |E_n(0)\rangle \to |\psi(t)\rangle = |E_n(t)\rangle$

- how slow is slow enough?

  - let $\Delta E_{gap}$ = minimum energy difference between $E_n(t)$ and $E_{n+1}(t)$ and $E_n(t)$ and $E_{n-1}(t)$

    - typically this is $E_{gap}$ for the 'tightest' avoided crossing

if $\boxed{T \gg \frac{\hbar}{\Delta E_{gap}}}$ → theorem is valid

"rule of thumb"

## SUDDEN APPROXIMATION

- in case $T \ll \frac{\hbar}{\Delta E_{gap}}$ $T \approx 0$

- physical state is unchanged

$$\boxed{|\psi(T)\rangle = |\psi(0)\rangle}$$

let $|\psi(0)\rangle = |E_n(0)\rangle$

$$|\psi(T)\rangle = |\psi(0)\rangle = \sum_n |E_n(T)\rangle \langle E_n(T)|E_n(0)\rangle$$

induced transitions to other levels

- lots of oscillations in observables for $t > T$

- kicked the system $\rightarrow$ it jiggles

- Energy also not conserved