PHY852 – Chapter 4: Harmonic Oscillator

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Consider a particle of mass \boldsymbol{m} in a one-dimensional harmonic oscillator potential with fundamental frequency $\boldsymbol{\omega}$,

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

(a) Show that the Hamiltonian can be written as:

$$H = \hbar \omega \left(a^{\dagger} a + \frac{1}{2} \right)$$

- (b) Using your answer from part (a), find the ground state energy of the harmonic oscillator.
- (c) To second order in perturbation theory, find the correction to the ground state energy when the perturbation

$$V = \alpha X^2$$

is added to the system.

(a) Show: $H = \hbar w (a^+ a + 1/2)$		
Recalling:	$a = \int \frac{mw}{2\hbar} \left(\hat{X} + \frac{L}{mw} \hat{P} \right) \qquad a^{+} = \int \frac{mw}{2\hbar} \left(\hat{X} - \frac{L}{mw} \hat{P} \right)$	
Con	idev ata:	
• • • • •	$\Lambda^+ \alpha = \frac{m\omega}{(\hat{\mathbf{x}} - \hat{\mathbf{L}} \hat{\mathbf{p}})} (\hat{\mathbf{x}} + \hat{\mathbf{L}} \hat{\mathbf{p}})$	
· · · · · ·	2ħ (^ mw')(^ mw')	
• • • • •	$= \frac{mW}{2\hbar} \left[\chi^2 + \frac{i}{mW} \hat{\chi} \hat{P} - \frac{i}{mW} \hat{P} \hat{\chi} + \frac{1}{m^2 W^2} P^2 \right]$	
	$[\hat{\mathbf{X}},\hat{\mathbf{p}}] = i\hbar$	
• • • • •	$= \frac{m\omega}{2\hbar} \left[\chi^2 + \frac{\omega}{m\omega} \left[\chi_j^2 \hat{\rho} \right] + \frac{p^2}{m^2 \omega^2} \right]$	
• • • • •	$\int M \hat{\nu} \left[2 + \frac{1}{2} \right]^{2} = 0$	
• • • • •	$= \frac{1}{2\hbar} \left[\chi^2 - \frac{1}{mw} + \frac{1}{m^2 W^2} \right]$	
• • • • •	$ \xrightarrow{\longrightarrow} a^{\dagger}a = \frac{m\omega}{2\hbar}\chi^{2} + \frac{p^{2}}{2\hbar m\omega} - \frac{1}{2} \longrightarrow \left(a^{\dagger}a + \frac{1}{2}\right) = \frac{1}{\hbar\omega}\left(\frac{1}{2}m\omega^{2}\chi^{2} + \frac{p^{2}}{2m}\right) $	
• • • • •		
· · · · · ·	$\Rightarrow \hat{H} = \hbar w (a^{\dagger}a + \frac{1}{2})$	
(b) To fin	$\Rightarrow \hat{H} = \hbar w (a^{\dagger}a + \frac{1}{2})$ $A + he G.S. energy,$	
(b) To fin i	$\Rightarrow \hat{H} = \pi w (a^{\dagger}a + \frac{1}{2})$ $A + he G.S. energy,$ $\exists n\rangle = E n\rangle$ $\begin{bmatrix} a^{\dagger} = \sqrt{n+1} n+1\rangle \\ a = \sqrt{n} \ln -1 \rangle \end{bmatrix}$	
(b) To fin f	$\Rightarrow \hat{H} = \pi w (a^{\dagger}a + \frac{1}{2})$ $A + He G.S. energy,$ $\exists n\rangle = E n\rangle$ $\begin{cases} a^{\dagger} = \sqrt{n+1} n+1\rangle \\ a = \sqrt{n} n-1\rangle \end{cases}$ $\begin{cases} a^{\dagger} = \sqrt{n} n-1\rangle \end{cases}$	
(b) To fin i	$\Rightarrow H = \pi w (a^{\dagger}a + \frac{1}{2})$ $A + he G.S. energy,$ $\exists h = \pi w (a^{\dagger}a + \frac{1}{2})$ $\exists h = \pi w (a^{\dagger}a + \frac{1}{2}) n >$ $\exists (\pi w a^{\dagger}a) n > + (\pi w/2) n >$	
(b) To fin	$\Rightarrow \hat{H} = \pi w (a^{\dagger}a + \frac{1}{2})$ $A + he G.S. energy,$ $\exists n\rangle = E n\rangle \qquad \qquad$	
(b) To fin	$ = H $ $ \Rightarrow H = \pi w (a^{\dagger}a + \frac{1}{2}) $ $ A + He G.S. energy, $ $ = \ln 2 \qquad \left[a^{\dagger} = \sqrt{n+1} n+1\rangle \right] $ $ = \sqrt{n} n-1\rangle = \pi w (a^{\dagger}a + \frac{1}{2}) n\rangle $ $ = (\pi w a^{\dagger}a) n\rangle + (\pi w/2) n\rangle $ $ = (\pi w) \sqrt{n} \sqrt{n} n\rangle + (\pi w/2) n\rangle $ $ = \pi w (n+\frac{1}{2}) $	
(b) To fin i i i i i i i i i i i i i i i i i i	$\Rightarrow \hat{H} = \pi w (a^{\dagger}a + \frac{y_{2}}{2})$ $a + he (g. S. energy),$ $f(n) = E(n),$ $a^{\dagger} = \sqrt{n+1} n+1\rangle,$ $a = \sqrt{n-1} n-1\rangle,$ $f(n) = \pi w (a^{\dagger}a + \frac{y_{2}}{2}) n\rangle,$ $= (\pi w) \sqrt{n} (n-1) + (\pi w/2) n\rangle,$ $= (\pi w) \sqrt{n} (n-1) + (\pi w/2) n\rangle = \pi w (n+\frac{y_{2}}{2}),$ $\Rightarrow E_{n} = \pi w (n+\frac{y_{2}}{2}),$ $and for the around state = n = 0, F = \pi w/2$	
	$\Rightarrow \hat{H} = \hbar w (a^{\dagger}a + \frac{y_{2}}{2})$ $a + he G.S. energy,$ $A^{\dagger} = \sqrt{n+1} n+1\rangle$ $a^{\dagger} = \sqrt{n+1} n+1\rangle$ $a^{\dagger} = \sqrt{n-1} n-1\rangle$ $a^{\dagger} = \sqrt{n-1} n-1\rangle$ $= (\hbar w) \sqrt{n} a^{\dagger} n-1\rangle + (\hbar w/2) n\rangle$ $= (\hbar w) \sqrt{n} \sqrt{n} (n+1) + (\hbar w/2) n\rangle = \hbar w (n+\frac{y_{2}}{2})$ $\Rightarrow E_{n} = \hbar w (n+\frac{y_{2}}{2})$ and for the ground state, $n=0$, $E_{0} = \frac{\hbar w}{2}$	
	$\Rightarrow \hat{H} = \pi w (a^{\dagger}a + \frac{1}{2})$ A the G.S. energy, $\exists n\rangle = E n\rangle \qquad [a^{\dagger} = \sqrt{n+1} n+1\rangle]$ $\exists n\rangle = \pi w (a^{\dagger}a + \frac{1}{2}) n\rangle = (\pi w a^{\dagger}a) n\rangle + (\pi w/2) n\rangle = \sqrt{n} n-1\rangle = (\pi w) \sqrt{n} \sqrt{n} n\rangle + (\pi w/2) n\rangle = \pi w (n+\frac{1}{2})$ $= (\pi w) \sqrt{n} \sqrt{n} n\rangle + (\pi w/2) n\rangle = \pi w (n+\frac{1}{2})$ $= (\pi w) \sqrt{n} \sqrt{n} n\rangle + (\pi w/2) n\rangle = \pi w (n+\frac{1}{2})$ and for the ground state, $n=0$, $E_0 = \frac{\pi w}{2}$	

<u>(C)</u>	V=qX ²
• • •	(i) get X in terms of $a \cdot a^{t}$ and plug in for $V = \alpha X^{2}$
• • •	$a = \sqrt{\frac{n\omega}{2n}} \hat{X} + i \sqrt{\frac{1}{2\pi m\omega}} \hat{P}$
• • • • • •	$+ \frac{\alpha^{+}}{2\hbar} = \sqrt{\frac{m\omega}{2\hbar}} \hat{\chi} - i \frac{1}{\sqrt{2\hbar m\omega}} \hat{\rho}$
• • •	$\left(A + A^{+}\right) = \int \frac{2m\omega}{\hbar} \hat{X} \longrightarrow \hat{X} = \int \frac{\hbar}{2m\omega} \left(A + A^{+}\right)$
• • •	$\Longrightarrow \chi^{2} = \frac{\pi}{2m\omega} \left(\lambda^{2} + \lambda \lambda^{2} + \lambda^{2} \right)$
· · ·	$\implies \forall = \alpha' \chi^2 = \frac{\alpha \pi}{2 m w} \left(\alpha^2 + \alpha \alpha^{\dagger} + \alpha^{\dagger} \alpha + \alpha^{\dagger^2} \right)$
• • •	(ii) Find first order energy correction to G.S.
• • •	$E_{n}^{(1)} = \langle N^{(0)} V N^{(0)} \rangle$
• • • •	$E_{0}^{(1)} = \langle 0 dX^{2} 0 \rangle = \langle 0 \frac{d\pi}{2mw} (a^{2} + aa^{\dagger} + a^{\dagger}a + a^{\dagger}a + a^{\dagger}a^{2}) 0 \rangle = \frac{d\pi}{2mw} \langle 0 aa^{\dagger} 0 \rangle$ only surviving term
• • •	$\Rightarrow \begin{bmatrix} E_{0}^{(1)} = \frac{a_{10}}{2mW} \end{bmatrix}$
• • •	(iii) Find second order energy correction:
• • • • • •	$E_{n}^{(2)} = \sum_{M(\neq n)}^{\gamma} \frac{ \langle M^{(\theta)} V N^{(\theta)} \rangle ^{2}}{E_{n} - E_{M}}$
• • • • • • • • •	For some $\langle m V 0\rangle$ $/ m\neq 0$, so a^{+2} only surviving term and $\underline{m=2}$. $L_{\Rightarrow} = \frac{\alpha t_{1}}{2m\omega} \langle m (\rho^{2} + a\rho t^{4} + a^{+2}) 0\rangle \rightarrow \frac{\alpha t_{2}}{2m\omega} \langle 2 a^{+2} 0\rangle$
· · · ·	$ \frac{d\pi}{2mw} \langle 2 a^{\dagger}a^{\dagger} 0\rangle = \frac{d\pi}{2mw} \langle 2 (\sqrt{1})a^{\dagger} 1\rangle = \frac{d\pi}{2mw} \langle 2 \sqrt{2} 2\rangle $
· · · ·	$= \frac{d t_{v}}{\sqrt{2} mW}$

 $E_{0}^{(2)} = \frac{|\langle 2| V|0\rangle|^{2}}{E_{0} - E_{2}} = \left(\frac{d^{2}\hbar^{2}}{2m^{2}W^{2}}\right) \left(\frac{1}{\pi w/2} - 5\pi w/2}\right) = \left(\frac{d^{2}\hbar^{2}}{2m^{2}W^{2}}\right) \left(\frac{1}{-2\pi W}\right)$ $\frac{\pi w/2}{E_{0}} + \frac{\pi w(n+1/2)}{E_{2} = 5\pi w/2}$ $E_{1}^{(2)} = \frac{-d^{2}\pi}{4m^{2}W^{3}}$ $\implies E_0 = \frac{mw}{2} + \frac{at}{2mw} - \frac{a^2t}{4m^2w^3}$