GENERATION OF ACOUSTIC AND GRAVITY WAVES
BY TURBULENCE IN AN ISOTHERMAL
STRATIFIED ATMOSPHERE

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(Received 8 June, 1967)

Abstract. Lighthill's method of calculating the aerodynamic emission of sound waves in a homogeneous atmosphere is extended to calculate the acoustic and gravity-wave emission by turbulent motions in a stratified atmosphere. The acoustic power output is

\[ P_{ac} \approx 10^3 \frac{\rho_0 u_0^5}{l_0} M^5 \text{ ergs/cm}^3 \text{ sec,} \]

and the upward gravity wave flux is

\[ F_{zgr} \approx 10^2 \frac{\rho_0 u_0^3}{l_0} \left( \frac{l_0}{H} \right)^5 \text{ ergs/cm}^3 \text{ sec.} \]

Here \( u_0 \) is the turbulence velocity scale, \( l_0 \) is its length scale, and \( H \) the scale height at the atmosphere. \( M = u_0 / c_0 \) is the Mach number of the turbulence. The acoustic power output is proportional to the maximum value of the turbulence spectrum, and inversely to its rate of falloff at high frequencies. The stratification cuts off the acoustic emission at low Mach numbers. The gravity emission occurs near the critical angle to the vertical \( \theta = \cos^{-1} \omega / \omega_a \), where \( \omega_a^2 = (\gamma - 1) / \gamma^2 (c_0 / H) \), and at very short wavelengths. It is proportional to the large wave number tail of the turbulence spectrum. On the sun, gravity-wave emission is much more efficient than acoustic, but can occur only from turbulent motions in stable regions, whereas acoustic waves are produced by turbulence in the convection zone.

A theory of sound-wave emission by fluid motions in a homogeneous medium has been developed by Lighthill (1952, 1960) and others (Proudman, 1952; Parker, 1953; Meecham and Ford, 1955; Muller and Matschat, 1958; Ffowcs Williams, 1963). In the atmosphere of the sun, however, the characteristic size of turbulent elements is considered to be comparable to the scale height of the stratification produced by gravity. Thus it is necessary to extend Lighthill's theory to include the effects of stratification.

Lighthill's approach is to combine the equations of motion, with the linear and non-linear terms separated, into an inhomogeneous wave equation for one of the variables. The linear terms are the wave propagation operator, and the non-linear terms are a quadrupole type source function. If the wave motions are so weak compared to the turbulent motions that their contribution to the source function can be neglected and, in addition, they produce no back reaction on the turbulent flow, then the inhomogeneous wave equation with the source function determined by the given flow can be solved, and the resulting oscillations in the far field are the emitted

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acoustic radiation. The acoustic energy flux is proportional to the turbulence-energy density \(\rho u^2\), where \(u\) is the turbulence-velocity scale) divided by the characteristic period of a turbulent eddy \(l/u\) where \(l\) is the turbulence length scale) times the Mach number \((u/c)\) to the fifth power.

In a gravitationally stratified atmosphere, gravity as well as pressure acts as a restoring force for fluid oscillations, and two types of waves – acoustic and gravity – occur, depending on the dominant restoring force. The gravitational restoring force in the wave equation produces a critical frequency \(\omega_1=c_0/2H\) (Lamb, 1945) below which acoustic waves cannot propagate, and another \(\omega_2=(\sqrt{\gamma-1/\gamma})c_0/H\) (Väisälä, 1925; Brunt, 1927) above which gravity waves cannot propagate. (Here \(H\) is the scale height, \(c_0\) the sound velocity and \(\gamma\) the ratio of specific heats \(c_p/c_v\)). In addition, stratification acts as a boundary surface in the homogeneous case (Curle, 1955) giving rise to dipole and monopole terms in the source function. These effects of stratification have been considered in a preliminary way by Unno and Kato (1964) and more rigorously by Unno (1964) for the special case of isothermal propagation, which excludes gravity waves. Moore and Spiegel (1964) and Kato (1966) have considered emission from a harmonic point source.

In the present work we extend Lighthill’s approach to stratified atmospheres. We first derive and solve the inhomogeneous wave equation for the pressure fluctuations in an isothermal stratified atmosphere; then we derive the expression for the emitted acoustic and gravity wave energy flux to lowest order in the Mach number; we conclude with a discussion of the emission and its dependence on the turbulence spectrum. Our calculation show that stratification cuts off the acoustic emission at low Mach numbers (because the characteristic turbulent frequency \((u/l)\) becomes much less than \(\omega_1\)), and that turbulence-spectrum variations alter the emission by two orders of magnitude. Our lowest-order expression for the gravity-wave emission has, unfortunately, a logarithmic low-frequency divergence, so a higher-order calculation, including the interaction of gravity waves and turbulence, is necessary to evaluate the gravity-wave emission. However, our values for the vertical component of the flux should be meaningful since the very low frequencies, emitted in the horizontal direction, do not contribute. The vertical component of the gravity-wave flux is proportional to the turbulent energy density divided by the characteristic period of the turbulence, as for the acoustic emission, but is multiplied by the ratio of the turbulent length scale to the scale height \((l/H)\), rather than by the Mach number, to the fifth power. Thus, gravity-wave emission in laboratory situations, where the turbulence length scale is much less than the scale height, will be negligible. However, in the sun gravity-wave emission is much more efficient than acoustic, because the characteristic size of turbulent eddies is the scale height, while the turbulence Mach number is about 0.2.

1. The Inhomogeneous Wave Equations

A. Derivation of the Inhomogeneous Wave Equation

We start the derivation of the wave equation from the equations of momentum and
mass conservation, with the linear terms separated on the left-hand side and the non-linear terms on the right-hand side:

$$\rho_0 \frac{\partial u}{\partial t} + \nabla P_1 - \rho_1 g = f,$$

(1)

where

$$f = -\frac{\partial (\rho u u_j)}{\partial x_j} - \frac{\partial (\rho_1 u)}{\partial t},$$

(2)

and

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 u) = q,$$

(3)

where

$$q = -\nabla \cdot (\rho_1 u),$$

(4)

and the equation of adiabatic motion,

$$\frac{\partial P_1}{\partial t} + u \cdot \nabla (P_0 + P_1) = c_0^2 \left[ \frac{\partial \rho_1}{\partial t} + u \cdot \nabla (\rho_0 + \rho_1) \right],$$

which we rewrite, using the mass conservation equation and again separating linear and non-linear terms, as

$$\frac{\partial P_1}{\partial t} + \rho_0 u \cdot g + \rho_0 c_0^2 \nabla \cdot u = h,$$

(5)

where

$$h = -u \cdot \nabla P_1 - \gamma P_1 \nabla \cdot u.$$

(6)

Here, $P_1$ and $\rho_1$ represent perturbations of pressure and density about the undisturbed pressure $P_0$ and density $\rho_0$ of an unbounded isothermal atmosphere in hydrostatic equilibrium. Hence, the undisturbed pressure and density are stratified with height as exp($-z/H$), with scale height $H = P_0/\rho_0 g$. The gravitational acceleration is $g = (0, 0, -g)$, and the other notations are standard.

Eliminating $\rho_1$ and $u$ from the linear terms in Equations (1), (3) and (5) (see Appendix A for the details), we obtain the following inhomogeneous wave equation for $P_1$:

$$\frac{\partial^4 P_1}{\partial t^4} - c_0^2 \nabla^2 \frac{\partial^2 P_1}{\partial t^2} + \gamma g \cdot \nabla \frac{\partial^2 P_1}{\partial t^2} - c_0^2 \omega_2^2 \nabla^2 P_1 = 0,$$

(7)

where

$$\omega_2 = (\sqrt{\gamma - 1/\gamma}) c_0 / H.$$

(8)
is the Väisälä (1925) – Brunt (1927) critical frequency, and

\[ \nabla_i^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \]

The stratification produces a height variation of the perturbation amplitude, in order to assure constancy of the wave energy flux

\[ F = P_1 u \approx \rho_0 u^2 c_0, \]

such that, \( u \propto P_0^{-\frac{1}{2}} \), \( P_1 \propto P_0^{\frac{1}{2}} \), and \( \rho_1 \propto P_0^{\frac{1}{2}} \). Therefore, to remove this height dependence, and at the same time eliminate the first-order space derivative term in the wave equation, we introduce, and from here on work in terms of, the variable

\[ P' = \frac{P_1}{P_0^{\frac{1}{2}}}. \] (9)

The inhomogeneous wave equation (7) becomes, in terms of \( P' \) (Kato, 1966)

\[ \left[ \frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2 + \omega_1^2 - \left( \frac{\partial}{\partial t} \right)^{-2} c_0^2 \omega_1^2 \nabla_1^2 \right] P' = \gamma^4 c_0 S, \] (10)

where the source function is

\[ \gamma^4 c_0 S = P_0^{-\frac{1}{2}} \left[ c_0^2 \left( 1 + \left( \frac{\partial}{\partial t} \right)^{-2} \omega_1^2 \right) \left( -\nabla \cdot \mathbf{f} + \frac{\partial q}{\partial t} \right) \right. \]

\[ + \left. \left( 1 + \left( \frac{\partial}{\partial t} \right)^{-2} \mathbf{g} \cdot \nabla \right) \left\{ (\gamma - 1) \mathbf{g} \cdot \mathbf{f} - \frac{\partial}{\partial t} (c_0^2 q - h) \right\} \right], \] (11)

and

\[ \omega_1 = c_0 / 2H \] (12)

is the other critical frequency (Lamb, 1945). Equation (10) is the basic equation for calculating the generation of acoustic and gravity waves by given fluid motions. It has the form of the usual wave equation but with two extra terms. The term \( \omega_1^2 \) arises from the effect of the stratification on the compressional restoring force, and the term \( (\partial/\partial t)^{-2} c_0^2 \omega_1^2 \nabla_1^2 \) arises from the gravitational restoring force.

B. SOLUTION OF THE INHOMOGENEOUS WAVE EQUATION

The fluid fluctuations are composed of wave motions and a turbulent flow field, which we assume is confined to a finite region of space. The radiated waves are the solution of the wave equation far from the source region. If we write the inhomogeneous wave equation as

\[ LP' = S, \]

where \( L \) is the linear wave operator and \( S \) the source function, then the solution for \( P' \) is

\[ P' = L^{-1} S + C\delta (L), \]

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where $C$ is a constant determined by the boundary condition that the waves be outgoing. This expression is an integral equation because, in general, the source function $S$ depends on the wave motions as well as on the given flow field. However, if the amplitude of the wave motions is much smaller than that of the turbulent motions, their contribution to the source function is negligible, and we have an expression for the wave motions in the far field in terms of the given turbulent flow field. Our method is thus restricted to weak wave emission.

To achieve the reduction of the problem from an intractable integral equation to a quadrature in a consistent fashion, we make a Mach-number expansion of the source function and retain only the lowest-order terms. This excludes terms dependent on the wave emission (which make the problem non-linear), and also excludes terms trilinear in the turbulent velocity (which lead to the appearance of fifth-order turbulent velocity correlations, that we are unable to evaluate, in the energy-flux expression).

For wave motions small compared to the turbulent motions

$$\overline{u^2_{\text{wave}}} \ll \overline{u^2_{\text{turb}}} \sim M^2,$$

and

$$\sqrt{\overline{\rho^2_{\text{wave}}}} \ll \rho_0,$$

while the relative turbulent density fluctuations are

$$\sqrt{\overline{\rho^2_{\text{turb}}}/\rho_0} \sim M^2.$$

Thus the source function terms arising from the Reynolds stress become

$$- \nabla \cdot f + \frac{\partial q}{\partial t} = \frac{\partial^2 (\rho u u_j)}{\partial x_i \partial x_j} \rightarrow \frac{\partial^2 (\rho_0 u_{\text{turb}} u_{\text{turb}})}{\partial x_i \partial x_j}$$

and

$$g \cdot f = g \frac{\partial}{\partial x_j} (\rho w u_j) + g \frac{\partial}{\partial t} (\rho_1 w) \rightarrow g \frac{\partial}{\partial x_j} (\rho_0 w_{\text{turb}} u_{\text{turb}}).$$

The last source-function term $(\partial/\partial t) (c^2_0 q - h)$ arises from the non-adiabatic nature of the turbulence and the stratification, and is generally smaller. From Equations (4) and (6)

$$c^2_0 q - h = u \cdot \nabla (P_1 - c^2_0 \rho_1) + (\gamma P_1 - c^2_0 \rho_1) \nabla \cdot u
= u \cdot \nabla (P_1 - c^2_0 \rho_1) - (\gamma P_1 - c^2_0 \rho_1) u \cdot \nabla \ln \rho_0,$$

since to lowest order in the Mach number $\nabla \cdot u_{\text{turb}} = -u_{\text{turb}} \cdot \nabla \ln \rho_0$. Therefore the ratio of the last source-function term to that arising from the Reynolds stress is

$$\frac{\partial}{\partial t} (c^2_0 q - h) \sim \left( \frac{\omega P_1 u}{l} \right) \left( \frac{g \rho_0 u^2}{l} \right) \sim \frac{ku_k^2}{g},$$
since for an eddy of the size $2\pi/k$, $\omega \sim ku_k$ and $P_1 \sim \rho_0 u_k^2$. For the dominant, large-size eddies the ratio is of order $M^2$, since $k_0 \sim 1/H \sim g/c_0^2$. If $u_k$ decreases slower than $k^{-\frac{1}{2}}$ this ratio increases for small eddies. Of various turbulence spectra which we consider, the Kolmogorov spectrum has the slowest decrease in velocity,

$$u_k = u_0 \left( \frac{k_0}{k} \right)^{\frac{1}{2}}$$

out to a maximum wave number of $(k/k_0) \sim Re^\frac{1}{2}$, where dissipation by viscosity produces a rapid decrease in the velocity amplitude. Here $Re$ is the Reynolds number. Thus the ratio of the non-adiabatic to the Reynolds stress terms in the source function goes as

$$M^2 \left( \frac{k}{k_0} \right)^{\frac{1}{2}}$$

for the Kolmogorov turbulence spectrum, with a maximum value of

$$M^2 Re^\frac{1}{2}.$$

In the solar atmosphere $Re$ is of order $10^{11}$, so at worst, for the Kolmogorov spectrum, the non-adiabatic term is negligible for Mach numbers less than 0.04. For other turbulence spectra the Reynolds stresses already dominate at larger Mach numbers. Thus the source function to lowest order in the Mach number is

$$\gamma^\frac{1}{4}c_0 S = P_0^{-\frac{1}{2}} \left[ c_0^2 \left( 1 + \left( \frac{\partial}{\partial t} \right)^{-2} \omega^2 \right) \nabla \cdot \nabla \cdot \left( \rho_0 uu \right) + (\gamma - 1)g \left( 1 + \left( \frac{\partial}{\partial t} \right)^{-2} g \cdot \nabla \right) \nabla \cdot \left( \rho_0 wu \right) \right],$$

where $u$ is the turbulent velocity and $w$ its vertical component.

The inhomogeneous wave equation (10) can now be solved for the pressure fluctuations (since the source function to lowest order in the Mach number depends only on the turbulent motions) by making a Fourier transform to wave-number frequency space, which yields

$$P'(k, \omega) = \frac{\gamma^\frac{1}{4}}{c_0 n^2 + \left( 1 - \frac{\omega^2}{\omega_2^2} \right) \left( l^2 + m^2 \right) - \frac{\omega^2 - \omega_1^2}{c_0^2} + i \varepsilon},$$

where $S(k, \omega)$ is the Fourier transform of the source function and $k = (l, m, n)$. Contributions to the pressure fluctuation come from those wave numbers and frequencies where the denominator vanishes, i.e. where the dispersion relation is satisfied.
To obtain the pressure fluctuation, we invert the Fourier transform (14), which gives (KATO, 1966)

\[ P'(x, t) = \frac{\gamma^2}{8\pi^2 c_0 \left( \omega^2 - \omega_2^2 \right)^\frac{3}{2}} \int \frac{S(x', t)}{r \left( \omega^2 - \omega_2^2 \cos^2 \theta \right)^\frac{3}{2}} \times \exp \left\{ i\omega(t - t') - i \frac{(\omega - \omega_1) \left( \omega^2 - \omega_2^2 \cos^2 \theta \right)^\frac{3}{2}}{c_0} r \right\} d^3x' dt', \]

where \( r = |x - x'| \) and \( \theta \) is the angle between \( r \) and the vertical. From this expression for the pressure fluctuation, without considering any specific source function, we see that waves propagate for \( \omega > \omega_1 \) (acoustic waves), and \( \omega_2 > \omega > |\omega_2 \cos \theta| \) (gravity waves), while waves are attenuated for \( \omega_1 > \omega > \omega_2 \) and \( \omega < |\omega_2 \cos \theta| \) (MOORE and SPIEGEL, 1964). Moreover, the amplitude diverges for \( \omega = \omega_2 \cos \theta \), so gravity wave emission is peaked near the critical angle and acoustic-wave emission is predominantly in the vertical direction at low frequencies (KATO, 1966).

2. The Emitted Energy Flux

The mechanical energy flux in a stratified atmosphere is, in lowest order, the same as in a homogeneous medium (ECKART, 1960), namely

\[ F = P_1 u. \]

We have obtained a solution for the pressure fluctuation and can express the fluid velocity in terms of it by means of Equations (1), (3) and (5) (see Equations 54 and 55), with the result, retaining only lowest-order terms, that the energy flux is

\[ F = -\frac{c_0^2}{\gamma} P' \left( \frac{\partial}{\partial t} \right)^{-1} \left\{ \nabla_1 P' + \left( \frac{V_z - \frac{2 - \gamma}{2} \frac{g}{c_0^2} \omega_2^2}{1 + \left( \frac{\partial}{\partial t} \right)^2 \frac{\omega_2^2}{\omega_2^2}} \right) P' \right\}, \]

where \( V_1 \) is the gradient in the horizontal and \( V_z \) in the vertical direction. The quantity that we are interested in is the mean or time average energy flux.

The mean mechanical energy flux is evaluated by expressing each factor \( P'(x, t) \) in Equation (17) in terms of its Fourier transform \( P'(k, \omega) \), which is given by Equation (14), and performing a time average. This requires the frequency of both factors to be the same. The remaining spatial Fourier transform can be evaluated in the far field by means of LIGHTHILL'S (1960) formula for the asymptotic value of Fourier transforms:

\[ \int \frac{S(k)}{G(k)} e^{ik \cdot x} d^3k = \frac{4\pi^2}{|x|} \sum_k \frac{S(k) e^{ik \cdot x}}{|V_k G| \sqrt{|k|}}, \]
where the sum is over the set of wave numbers on the slowness surface $G = 0$ where the normal $\nabla G$ is in the direction of $x$. $K$ is the Gaussian curvature of $G$. The slowness surface,

$$
G(k, \omega) = n^2 + \left(1 - \frac{\omega^2}{\omega_2} \right) \left( l^2 + m^2 \right) - \frac{\omega^2 - \omega_1^2}{c_0^2} = 0,
$$

(19)

is the surface in wave number space where the dispersion relation is satisfied (Figure 1). The group velocity is

$$
u_g = \nabla_k \omega = - \nabla_k G \left( \frac{\partial G}{\partial \omega} \right)^{-1},
$$

so the normal to the slowness surface is the direction of propagation. The shape of the slowness surface or dispersion relation is essential in determining the effect of the (stratified) medium on the emission. Evaluating $\nabla_k G$ and $K$ (Equations 66 and 67) and expressing the results in terms of the frequency and angle $\theta$ between the direction of propagation and the vertical, we get (Equation 75)

$$
F(x, \omega) = \lim_{r \to \infty} \frac{1}{T|x|^3 c_0} \left\{ \frac{\partial}{\partial \omega} \left[ \frac{\omega^4}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_1^2 \cos^2 \theta)} \right] \right\}^2

- 2 \frac{\partial^2}{\partial \gamma} \left[ \frac{\omega \omega_1}{\omega^2 - \omega_1^2} \right] \left[ \frac{\omega^4}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_1^2 \cos^2 \theta)} \right] |S(k, \omega)|^2,
$$

(20)

where $\partial^2 = \omega^2 - \omega_1^2$.

The first term is real for acoustic waves ($\omega > \omega_1$) and for gravity waves ($\omega_2 > \omega > |\omega_2 \cos \theta|$), and imaginary for non-propagating waves ($\omega_1 > \omega > \omega_2$ and $\omega < |\omega_2 \cos \theta|$). The second term is always imaginary and so can be neglected. The flux vanishes as $\omega \to \omega_1$ and has peaks at $\omega \approx \omega_2$, and for gravity waves, as anticipated, at the critical angle $\theta_c = \cos^{-1} \omega/\omega_2$. For $\omega$ slightly greater than $\omega_1$, the flux is peaked in the vertical angle direction. These properties depend only on the form of the slowness surface or dispersion relation (19) and are due to the stratification. We have not yet considered the specific nature of the source function. Mathematically, the large flux near the critical angle is due to the asymptotic approach of the slowness (hyperboloid) surface to a cone (Figure 1), so that an infinite arc of the surface emits waves in the same direction.

By the same reasoning, we may expect a large flux at very low frequencies, where the slowness surface becomes a plane. We may also anticipate that the gravity-wave emission will be greater than the acoustic emission, since the slowness surface shows no such degeneracies for the acoustic mode. At the critical angle, the above expression (18) for the asymptotic value of the Fourier transform is invalid, for the curvature $K$ (Equation 67) vanishes. The appropriate expression (LIGHTHILL, 1960) has no singularity, but instead of decreasing as $r^{-1}$ with distance, as above, it only falls off as $r^{-4}$. However, since, as we shall see later, gravity waves are emitted close to, but not at, the critical angle, our expression (20) for the emitted flux is valid.
We must now finally consider the explicit form of the source function (13), which may be transformed into a multipole expansion (Equation 80):

\[
S = \left[ \left( 1 - \frac{\omega^2}{\omega^2_2} + \frac{\omega^2}{\omega^2_2} \frac{\delta_{j3}}{\delta x_j} \frac{\partial^2 (\rho_0^\frac{\gamma}{c_0} u_j w)}{\partial x_j} \right) - 2 \frac{\omega^2}{c_0} \left( \frac{1}{\gamma} - \frac{1}{2 \omega^2} + \frac{1}{2 \omega^2} \frac{\delta_{j3}}{\delta x_j} \right) \frac{\partial (\rho_0^\frac{\gamma}{c_0} w)}{\partial x_j} + \frac{\omega^2}{c_0^2} \left( \frac{2 - \gamma}{\gamma} \right) (\rho_0^\frac{\gamma}{c_0} w w) \right].
\]

(UNNO (1964) derived the multipole expansion for the case of isothermal propagation, \(\gamma = 1, \omega_2 = 0\).) The source-function term arising from internal fluid (Reynolds) stresses is quadrupole (the second space derivative of a tensor), because these stresses produce equal and opposite forces on opposite sides of a given element of fluid; that is, the volume of the fluid element remains constant, its center of mass moves uniformly, but its surface distorts. The external gravitational force produces a stratification and introduces additional dipole and monopole source terms.

The absolute value squared of the Fourier transform of the source function, \(|S(k, \omega)|^2\), which appears in the energy-flux expression (20), is evaluated as a product of two transforms. Since the source function (21) is expressed as a multipole expansion, we can integrate by parts to replace the derivatives by powers of the wave vector \(k\).
Then we transform to the average and relative coordinates of the two source elements and integrate over the mean time, which reintroduces a time average (Equation \ref{eq:86}). We obtain (\ref{eq:88})

\[
\mathbf{F}(\mathbf{x}, \omega) = \frac{\hat{x}_{\pi}}{2|\mathbf{x}|^2 c_0 \omega} \omega^4 \left( \frac{1}{\omega^2 - \omega_1^2} \left( \frac{1}{\omega^2 - \omega_2^4} \right) \right) \nabla \int \frac{1}{2\pi^4} \rho_0(\mathbf{x}_0) \left( \mathbf{x}_0 - \mathbf{r}, \tau \right) e^{i \omega \tau - ik \cdot \mathbf{r}} \, d^3 \mathbf{x}_0 \, d^3 \mathbf{r} \, d\tau ,
\]

(22)

where \( \mathbf{x}_0 \) is the mean position, \( \mathbf{r} \) and \( \tau \) the space and time interval between the two source points, and \( \zeta(\mathbf{x}_0, \mathbf{r}, \tau) \) is the correlation (\ref{eq:87}) and (\ref{eq:89})

\[
\zeta(\mathbf{x}_0, \mathbf{r}, \tau) = \frac{1}{\rho_0(\mathbf{x}_0)} \left\langle S \left( \mathbf{x}_0 - \frac{\mathbf{r}}{2}, \frac{t - \tau}{2} \right) S \left( \mathbf{x}_0 + \frac{\mathbf{r}}{2}, \frac{t + \tau}{2} \right) \right\rangle
\]

\[
= \left( 1 - \frac{\omega_1^2}{\omega^2} + \frac{1}{2} \frac{\omega_2^2}{\omega^2} (\delta_{13} + \delta_{13}) \right) \left( 1 - \frac{\omega_1^2}{\omega^2} + \frac{1}{2} \frac{\omega_2^2}{\omega^2} (\delta_{13} + \delta_{33}) \right)
\]

\[
\times k_i k_j k_i k_m \langle u_i u_j u_i u_m \rangle \\
+ 4 \left( \frac{1}{\gamma} - \frac{\omega_1^2}{2 \omega^2} + \frac{1}{2} \frac{\omega_2^2}{\omega^2} \delta_{13} \right) \left( \frac{1}{\gamma} - \frac{\omega_1^2}{2 \omega^2} + \frac{1}{2} \frac{\omega_2^2}{\omega^2} \delta_{33} \right)
\]

\[
\times \frac{\omega_1^2}{c^2} k_i k_j k_m \langle u_i w^j u_m w^m \rangle
\]

\[
- \left( \frac{2 - \gamma}{\gamma} \right) \frac{\omega_1^2}{c^2} \left( \left( 1 - \frac{\omega_1^2}{\omega^2} + \frac{1}{2} \frac{\omega_2^2}{\omega^2} (\delta_{13} + \delta_{13}) \right) \right) k_i k_j \langle u_i u_j w^i w^j \rangle
\]

\[
+ \left( 1 - \frac{\omega_1^2}{\omega^2} + \frac{1}{2} \frac{\omega_2^2}{\omega^2} (\delta_{13} + \delta_{33}) \right) k_i k_m \langle w^j w^j u_i u_m \rangle
\]

\[
+ \left( \frac{2 - \gamma}{\gamma} \right)^2 \frac{\omega_1^4}{c^4} \langle w^j w^j w^j w^j \rangle.
\]

(23)

The correlation \( \zeta \) is an expression in terms of the fourth-order velocity correlations, because each source-function term (\ref{eq:21}) is a product of two turbulent velocity components at a point.

To proceed, we must now consider the properties of the turbulent velocity correlations and their Fourier transforms. We have, as yet, no theory of turbulence that determines the fourth-order velocity correlations, but we can, on general grounds, write down some plausible expressions for the second-order velocity correlation or turbulence energy spectrum. We therefore reduce the fourth-order correlations to second-order correlations, by assuming that

\[
\langle u_1 u_2 u_3 u_4 \rangle = \langle u_1 u_2 \rangle \langle u_3 u_4 \rangle + \langle u_1 u_3 \rangle \langle u_2 u_4 \rangle + \langle u_1 u_4 \rangle \langle u_2 u_3 \rangle.
\]

(24)
Kraichnan (1957) has pointed out that this is a bad approximation, because the neglected cumulants are not small. On the other hand, it is the only means of analytically evaluating the fourth-order correlations, and hopefully will give a reasonable estimate for the purpose of calculating the wave emission (Batchelor, 1953). The first term in the reduction (24) is independent of \( r \) and \( \tau \), so its Fourier transform is \( \delta^3(\mathbf{k}) \delta(\omega) \) and it makes no contribution to the emission.

When we have reduced the correlation (23) to the product of second-order velocity correlations by (24), then the expression (22) for the energy-flux emission involves the Fourier transform of a product, which is a convolution:

\[
\frac{1}{2\pi^2} \int d^3 \mathbf{r} \int d\tau e^{-i\omega t + i\mathbf{k} \cdot \mathbf{r}} \langle \mathbf{v}_1 \cdot \mathbf{v}_2 \rangle \langle \mathbf{v}_3 \cdot \mathbf{v}_4 \rangle = \int \Phi_{12}(\mathbf{k} - \mathbf{p}, \omega - \sigma) \Phi_{34}(\mathbf{p}, \sigma) \, d^3 \mathbf{p} \, d\sigma, \tag{25}
\]

where \( \Phi \) is the Fourier transform of the velocity correlation.

In suggesting forms for the velocity correlation we guide ourselves by the results for isotropic, homogeneous, incompressible turbulence. These conditions certainly do not apply to the largest eddies in a stratified atmosphere, of sizes comparable to the scale height. However, it is only for this simple case that we have any guide as to the form of the correlation. For incompressible, homogeneous, isotropic turbulence, the Fourier transform of the velocity correlation has the form (Batchelor, 1953)

\[
\Phi_{ij}(\mathbf{k}, \omega) = \frac{E(\mathbf{k}, \omega)}{4\pi k^2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right). \tag{26}
\]

A turbulent eddy of size \( \lambda_k = 2\pi/k \) will contain frequencies of the order of \( u_k/\lambda_k \), where \( u_k \) is the velocity perturbation of the eddy \( k \). The turbulence energy spectrum \( E(\mathbf{k}, \omega) \) may therefore, plausibly, be factored into a spectrum \( E(k) \) and a frequency-dependent factor \( \Delta(\omega/k u_k) \), which will have the form of a smoothed delta function centered about \( \omega = k u_k \) (Kraichnan, 1965). We therefore write

\[
E(\mathbf{k}, \omega) = E(k) \Delta(\omega/k u_k) \tag{27}
\]

and consider three forms of the frequency factor \( \Delta(\omega/k u_k) \):

(i)

\[
\Delta(\omega/k u_k) = \frac{1}{k u_k} e^{-\omega/k u_k}, \tag{28}
\]

which is large for all frequencies less than \( k u_k \) and decreases rapidly for higher frequencies,

(ii)

\[
\Delta(\omega/k u_k) = \frac{2}{\pi k^2 u_k} e^{-(\omega/k u_k)^2}, \tag{29}
\]

which falls off faster at high frequencies, and

(iii)

\[
\Delta(\omega/k u_k) = \frac{4}{\pi^{3/2} (k u_k)^3} e^{-(\omega/k u_k)^2}, \tag{30}
\]
which has a maximum at \( \omega = k u_k \) and falls off for both lower and higher frequencies. We hope that this range of functional forms will adequately explore the effect on the wave emission of the frequency dependence of the turbulence spectrum.

The velocity fluctuation \( u_k \) of the \( k \)th eddy is the velocity fluctuation in a system moving with the mean velocity of the larger eddies (since uniform motion of a fluid generates no waves, only velocity fluctuations are significant), and (KRAICHTAN, 1965) is given by

\[
u_k = \left[ \int \frac{E(k) \, dk}{k} \right]^{\frac{1}{2}}. \tag{31}\]

Our velocity correlations are, therefore, Lagrangian, not Eulerian, correlations. (An Eulerian correlation, for which \( u_k = u_0 \) for all \( k \), implies that there are no convective effects, i.e. that there is only one size eddy.)

To finally obtain an explicit expression for the emitted flux, we substitute the expression (72) and (71) for \( k^2 \) and \( k_z^2 \) in terms of the frequency and direction of propagation into Equation (23) for the correlation \( \zeta(x_0, r, \tau) \), reduce the fourth-order velocity correlations to second-order correlations by (24), and perform the Fourier transform in the energy-flux expression (22), which yields the convolutions (25). We use the expressions (26) and (27) for the turbulence spectrum with one of the forms (28)–(30) for the frequency factor \( \Delta(\omega|ku_k) \) and perform the integration over the frequency \( \sigma \) (94)–(97), and the azimuthal angle \( \varphi \) of the wave vector (101)–(106). The final result is (Equation 113)

\[
F(x, \omega) = \frac{\hat{x}}{4\pi |x|^2 c_0^3 \omega} \iint_{-1}^{+1} \frac{\omega^4}{(\omega^2 - \omega_z^2)(\omega^2 - \omega_z^2 \cos^2 \theta)} \, d\theta \, dp \int_0^\infty \frac{E(p) E(q)}{q^2} g(p, q, \omega) f(\omega, \theta, k, p, \mu), \tag{32}\]

where \( g(p, q, \omega) \) arises from the convolution (25) of the frequency factors in the turbulence spectra over \( \sigma \), and \( f(\omega, \theta, k, p, \mu) \) is the angular part of the correlations \( \zeta \). The frequency part is

(i) for

\[
\Delta(\omega|ku_k) = \frac{1}{k u_k} e^{-\omega|ku_k}: \\

g(p, q, \omega) = \frac{2}{(pu_p + qu_q)} \frac{pu_p e^{-\omega|pu_p} - qu_q e^{-\omega|qu_q}}{(pu_p + qu_q)(pu_p - qu_q)},
\]

(ii) for

\[
\Delta(\omega|ku_k) = \frac{2}{\pi^4 k u_k} e^{-\omega^2|ku_k|^2}:
\]

\[
g(p, q, \omega) = \frac{4}{\pi(p^2 u_p^2 + q^2 u_q^2)} e^{-\omega^2((p^2 u_p^2 + q^2 u_q^2))} \]
and

\[
\Delta (\omega/k u_k) = \frac{4 \omega^2}{\pi^\frac{3}{2} (k u_k)^3} e^{-\omega^2/(p^2 k^2 + q^2 k^2)}
\]

\[
g(p, q, \omega) = \frac{4}{\pi (p^2 u_p^2 + q^2 u_q^2)^\frac{3}{2}} e^{-\omega^2/(p^2 k^2 + q^2 k^2)}
\times \left[ 3 \left( \frac{p u_p q u_q}{p^2 u_p^2 + q^2 u_q^2} \right)^2 + 2 \frac{\omega^2}{(p^2 u_p^2 + q^2 u_q^2)^3} \right]
\times \left( p^4 u_p^4 - 4 p^2 u_p^2 q^2 u_q^2 + q^4 u_q^4 + 4 \frac{\omega^4}{(p^2 u_p^2 + q^2 u_q^2)^4} \right).
\]  

(33)

The angular factor is:

\[
f(\omega, \theta, k, p, \mu) = k^4 \left( \frac{\omega^2 - \omega_2^2}{\omega^2} \right)^2 \frac{p^2}{q^2} (1 - \mu^2)^2
\]

\[
- 2 k^4 \frac{\omega_2^2}{\omega^2} \left( \frac{\omega^2 - \omega_2^2}{\omega^2} \right) \cos^2 \theta_k \frac{p^2}{q^2} \mu^2 (1 - \mu^2)
\]

\[
+ \frac{1}{2} k^4 \frac{\omega_4^2}{\omega_2^2} \cos^2 \theta_k \left[ \cos^2 \theta_k \mu^2 + \frac{p^2}{q^2} (1 - \mu^2) \right]
\]

\[
\left\{ 1 - \frac{1}{2} \sin^2 \theta_k + (1 - 3 \cos^2 \theta_k) \mu^2 \right\} - \frac{1}{2} \frac{p k \mu}{q^2} \sin^2 \theta_k (1 - \mu^2)
\]

\[
+ \frac{1}{2} k^2 \omega_1 \left( \frac{2}{\gamma} - \frac{\omega_2^2}{\omega^2} \right)^2 \left[ \cos^2 \theta_k \mu^2 + \frac{p^2}{q^2} (1 - \mu^2) \right]
\]

\[
\left\{ 1 - \frac{1}{2} \sin^2 \theta_k + (1 - 3 \cos^2 \theta_k) \mu^2 \right\} - \frac{1}{2} \frac{p k \mu}{q^2} \sin^2 \theta_k (1 - \mu^2)
\]

\[
- 2 k^2 \omega_1 \frac{\omega_2^2}{\omega^2} \left( \frac{2}{\gamma} - \frac{\omega_2^2}{\omega^2} \right) \cos^2 \theta_k \left[ \sin^2 \theta_k \mu^2 - \frac{p^2}{2 q^2} \mu^2 (1 - \mu^2) \right]
\]

\[
(3 - 5 \cos^2 \theta_k) + \frac{p k \mu}{q^2} \sin^2 \theta_k (1 - \mu^2)
\]

\[
+ k^2 \omega_1^2 \frac{\omega_2^2}{\omega^2} \cos^2 \theta_k \left[ \frac{1}{2} \sin^2 \theta_k \{ 1 + \mu^2 + \cos^2 \theta_k (1 - 3 \mu^2) \}
\]

\[
+ \frac{1}{2} \frac{p^2}{q^2} (1 - \mu^2) \left\{ \cos^2 \theta_k (1 + \mu^2 + \cos^2 \theta_k (1 - 3 \mu^2))
\right\}
\]

\[
- \sin^2 \theta_k \left( 1 - 5 \cos^2 \theta_k \mu^2 - \frac{3}{4} \sin^2 \theta_k (1 - \mu^2) \right)
\]

\[
- \frac{2 p k \mu}{q^2} \cos^2 \theta_k \sin^2 \theta_k (1 - \mu^2)
\]
\[-2 \left( \frac{2 - \gamma}{\gamma} \right) k^2 \omega_1^2 \left( \frac{\omega^2 - \omega_2^2}{\omega_2^2} \right) \left[ \cos^2 \theta_k \mu^2 + \frac{1}{2} \frac{p^2}{q^2} \mu^2 (1 - \mu^2) \right] (1 - 3 \cos^2 \theta_k) - \frac{p k \mu}{2 q^2} \sin^2 \theta_k (1 - \mu^2) \]\[+ 2 \left( \frac{2 - \gamma}{\gamma} \right) k^2 \omega_1^4 \omega_2^2 \cos^2 \theta_k \left[ \sin^2 \theta_k \mu^2 - \frac{1}{2} \frac{p^2}{q^2} \mu^2 (1 - \mu^2) \right] (3 - 5 \cos^2 \theta_k) + \frac{p k \mu}{q^2} \sin^2 \theta_k (1 - \mu^2) \]\[+ \left( \frac{2 - \gamma}{\gamma} \right)^2 \omega_1^4 \left[ \frac{1}{2} \sin^2 \theta \{ 1 + \mu^2 + \cos^2 \theta_k (1 - 3 \mu^2) \} \right] + \frac{1}{2} \frac{p^2}{q^2} (1 - \mu^2) \left\{ \cos^2 \theta_k (1 + \mu^2 + \cos^2 \theta_k (1 - 3 \mu^2)) \right\} - \sin^2 \theta_k \left( 1 - 5 \cos^2 \theta_k \mu^2 - \frac{3}{4} \sin^2 \theta_k (1 - \mu^2) \right) \left\{ - \frac{2 p k \mu}{q^2} \cos^2 \theta_k \sin^2 \theta_k (1 - \mu^2) \right\}. \tag{34} \]

Here
\[q = |k - p| = [k^2 + p^2 - 2 k p \mu]^{\frac{1}{2}}, \tag{35} \]

where \(\mu = \cos \theta_{pk}\) and the wave vector of the emitted waves is (Equation 72)
\[k^2 = \frac{\omega^2 (\omega^4 - 2 \omega^2 \omega_2^2 \cos^2 \theta + \omega_2^4 \cos^2 \theta)}{c^2} (\omega^2 - \omega_2^2) (\omega^2 - \omega_2^2 \cos^2 \theta). \tag{36} \]

Note that \(k^2\) is negative (waves are attenuated) for \(\omega_1 > \omega > \omega_2\) and for \(\omega^2 < \omega_2^2 \cos^2 \theta\). Also, \(k^2\) becomes infinite at the critical angle \(\theta = \cos^{-1} \omega_1/\omega_2\) and for \(\omega = \omega_2\), while \(k^2\) becomes zero at \(\omega = \omega_1\). Thus the differentiations in the source function accentuate the effects of stratification by introducing powers of the wave vector into the emission expression. The angle between the wave vector \(k\) and the vertical is (Equation 73)
\[\cos^2 \theta_k = \frac{(\omega^2 - \omega_2^2)^2 \cos^2 \theta}{\omega^4 - 2 \omega^2 \omega_2^2 \cos^2 \theta + \omega_2^4 \cos^2 \theta} \]
and
\[\sin^2 \theta_k = \frac{\omega^4 \sin^2 \theta}{\omega^4 - 2 \omega^2 \omega_2^2 \cos^2 \theta + \omega_2^4 \cos^2 \theta}. \]

If we separate the dimensional factors, according to
\[[k] = 1/l_0, \quad [u_k] = u_0, \quad [\omega] = u_0/l_0, \quad [E(p)] = u_0^2 l_0, \tag{37} \]
where $u_0$ is the turbulence-velocity scale and $l_0$ the turbulence-length scale, we have

$$F(x, \omega) = \frac{x}{|x|^2} \rho_0 u_0^2 V M^5 F(\omega)_{N,D},$$

$$F(\omega)_{N,D} = \frac{1}{4 \omega} \int_0^\infty \int_{-1}^1 \frac{E(p) E(q)}{q^2} g(p, q, \omega) f(\omega, \theta, k, p, \mu),$$

(38)

where $p, q, \omega, u_p, E(p)$ etc. are dimensionless, and the dimensionless critical frequency is

$$\omega_1 = \left(\frac{1}{2} M \right) (l_0/H).$$

(39)

Since we use Lagrangian correlations, the frequency of the waves from the smaller eddies is shifted by the Doppler effect

$$\omega' = \omega \left(1 + \frac{\nabla}{c} \cdot \hat{x} \right),$$

where $\nabla$ is the mean velocity of the larger eddies. The energy flux is also modified by a factor

$$\left(1 - \frac{\nabla}{c} \cdot \hat{x} \right)^{-5},$$

which arises from the change in the volume elements in the integration over the source region and from a change in the emission time in the moving system (Frowcs Williams, 1963). Both of these effects are small at very small Mach number.

Equations (32) and (38) are our main result – the expressions for the energy flux of acoustic and gravity waves. The emission depends on the stratification through the dispersion relation or slowness surface and on the spectrum of the turbulence emitting the waves through the source function. Our expression is the lowest-order term in a Mach-number expansion of the source function, and is valid only when a negligible fraction of the turbulent energy is radiated and for small Mach number turbulent flows.

3. Discussion of Acoustic and Gravity Wave Emission

A. THE ACOUSTIC EMISSION

The total acoustic power emitted per unit volume (in units of $(\rho_0 u_0^3/l_0)M^5$) is shown as a function of Mach number in Figure 2, and its spectrum is shown in Figure 3. The acoustic power is roughly proportional to $M^5$ at intermediate Mach numbers, but falls off at very small Mach numbers and at Mach numbers approaching one (Figure 2). From Figure 3 we see that the acoustic power spectrum is proportional to $\omega^{3.3}$ at low
frequencies (but not too near $\omega_1$) and falls off sharply at very high frequencies. Figure 4 shows that the energy emission is nearly isotropic at high frequencies, but is sharply peaked in the vertical direction near $\omega_1$.

Many of these properties of the acoustic emission (those not depending essentially on the stratification) may be understood from a simple dimensional analysis of the wave equation (10) and the expression for the energy flux (17). The ($n$th) term in the multipole expansion of the source function (21) has the dimensions

$$\gamma^2 c S^{(n)} \propto P_0^\frac{1}{4} \left( \frac{\partial}{\partial x} \right)^n \frac{u^2}{l^2} l^n$$

$$\propto P_0^\frac{1}{4} k^n l^n \frac{u^2}{l^2} \propto P_0^\frac{1}{4} c_0^2 (k l) \frac{\rho_0 u^2}{l^2},$$

where $u$ is a typical turbulent velocity and $l$ the turbulence-length scale. The contribu-
Fig. 3. Non-dimensional acoustic power spectrum (in units of $\rho_0 u_0^2 M^2$) as a function of the dimensionless frequency ($\omega$ in units of $u_0/l_0$). The solid lines are for an exponential spectrum with exponential frequency spectrum and the dashed lines for $\omega^2$ times a Gaussian frequency spectrum.

The relation to the scaled pressure fluctuation, $P'$, from the ($n$th) multipole source term is then

$$P^{(n)} \propto \frac{1}{c^3} \frac{cS}{l^3} \propto P_0 \left( \frac{u}{c} \right)^2 k^n l^n \left( \frac{l}{r} \right).$$

(40)

The wave vector is, at high frequencies, approximately

$$k \approx \frac{\omega}{c} = \frac{\omega'}{c} \frac{u}{l},$$

where $\omega'$ is the dimensionless frequency and $l/u$ the time scale of the turbulence, so

$$P^{(n)} \propto P_0 \left( \frac{u}{c} \right)^{n+2} \omega'^n \frac{l}{r}.$$

Thus, the energy flux (17) arising from the ($n$th) multipole term is

$$F^{(n)} \propto c^2 P^{(n)} \left( \frac{\partial}{\partial t} \right)^{-1} \frac{\partial}{\partial x} P^{(n)}$$

$$\propto \rho_0 u^3 \left( \frac{u}{c} \right)^{2n+1} \omega'^{2n} \left( \frac{l}{r} \right)^2,$$

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The $M^5\omega^{3.3}$ dependence of the emission must therefore occur because the quadrupole source term is dominant. This sensitivity of the energy emission to the Mach number of the turbulence and the frequency of the waves arises because the opposite motions from opposite sides of the eddy, producing the sound waves, cancel at large distances. The net (unbalanced) contribution is proportional to the product of the frequency and retarded time difference, or the ratio of the eddy size to the wavelength,

$$\frac{l}{\omega} = \frac{l}{\lambda} = M\omega'$$
Fig. 5. Spectrum of non-dimensional upward flux showing contribution of different multipole source terms, for an exponential turbulence spectrum, at Mach number 0.1.

(Lighthill, 1955). The dominance of the quadrupole source term occurs because at high frequencies, from (41), the highest multipole term present (quadrupole) dominates, while as ω approaches the critical frequency ω₁ all multipole orders contribute equally if the turbulence-length scale is the order of the scale height (Figure 5). And, the largest contribution to the power comes from the high-frequency regions of the spectrum, since the quadrupole emission goes as ω⁴ to near the cutoff frequency.

We now turn to those effects produced by stratification. The first of these is the decrease in the power output at very small Mach numbers below that given by the $M^5$ factor. It occurs because there is a maximum frequency emitted by the turbulence, while the minimum frequency ω₁ increases as $M^{-1}$, cutting off the emission on the low side (Figure 3) (also see Unno, 1964). The decrease in the power output at Mach numbers approaching one, on the other hand, is due to interference. It does not occur when the wavelength of the waves is set equal infinity and the retardation differences within the source region neglected (Figure 6). As the turbulent velocity approaches the sound speed, the wavelength of the high-frequency waves becomes much less than the size of the dominant eddies:

$$k_{\text{wave}} \approx M\omega'k_{\text{turb}},$$
and the eddy, which by definition is a coherent source (incoherent motions belong to smaller eddies), will interfere with the waves. This decrease of the dimensionless frequency of the emission maximum with increasing Mach number has been observed experimentally in noise emission from jets (Lighthill, 1954). A word of caution is, however, in order at this point: In the source-function expansion that we have used, the lowest-order neglected term (trilinear in the turbulent velocities) is only of order $M$ smaller than the terms retained, so results for $M>0.1$ are uncertain.

Generally, stratification effects are small for $\omega \gg \omega_1$. Thus at high frequencies the emission is isotropic, and, since most of the energy is generated at these frequencies, the total flux is nearly isotropic. At frequencies near $\omega_2$, however, the inverse powers of $(\omega^2 - \omega_2^2 \cos^2 \theta)$ produce strong peaking of the vertical direction (Kato, 1966), and, as we have seen, the dipole and monopole-source terms become important. There is also a peak in the emission for $\omega \approx \omega_1$ due to the inverse factor $(\omega^2 - \omega_2^2)$ which becomes small if $\omega_1$ is close to $\omega_2$ (Kato, 1966).

We postpone quantitative discussion of the acoustic emission until our consideration of the effects of the turbulence spectra.
B. THE GRAVITY WAVE EMISSION

The first fact to note about the gravity-wave emission is that it occurs close to the critical angle $\theta = \cos^{-1} \omega/\omega_2$. The second is a low-frequency divergence in the gravity-wave power output, with $P(\omega) \propto \omega^{-1}$, so the total power output is logarithmically infinite. The vertical energy flux is, however, insensitive to the divergence, since the low-frequency waves are emitted in the horizontal direction. Third, gravity-wave emission is proportional to $(l/H)^5$, rather than $M^5$ as for the acoustic emission, so the emission from large eddies is very great.

Physically, gravity waves are emitted predominantly at the critical angle because they propagate in that direction, where the gravitational restoring force ($\propto \omega_2^2 \cos^2 \theta$) balances the acceleration ($\propto \omega^2$). The additional compressional restoring force enables the acceleration to be balanced at angles greater than the critical angle. Mathematically, gravity waves are emitted predominantly at the critical angle, first, because the slowness surface (Figure 1) is generated there by straight lines, so arcs rather than points on the surface contribute to emission in the same direction (see discussion following Equation 20), and second, because the wave vectors introduced by the differentiations in the source function become infinite at the critical angle. Both these effects are represented by factors of $(\omega^2 - \omega_2^2 \cos^2 \theta)$ in the denominator of the expression (32) for the emitted energy flux.

The singularity at the critical angle (due to the vanishing curvature of the slowness surface) is responsible for the great efficiency of gravity wave generation. The flux does not, however, actually become infinite, because as the critical angle is approached the wave number $|k|$ becomes very large, so either $|k - p|$ or $|p|$ or both must get large, and the emission (which is proportional to $E(p)E(|k - p|)$) will be reduced, since the turbulence spectrum goes to zero at large wave numbers. Thus the maximum emission of gravity waves occurs at angles very close to, but slightly greater than the critical angle.

Consider the limiting form of the expression (32) for the gravity-wave energy flux near the critical angle:

$$\cos \theta = \omega/\omega_2.$$  

The wave number is

$$k^2 \approx -\frac{\omega^2}{c^2} \frac{\omega^2}{\omega_2^2 \cos^2 \theta}$$  \hspace{1cm} (43)

and gets very large. As a result, the highest multipole source term (quadrupole) dominates the emission. If in addition, we assume that $k \gg p$, and neglect terms of order $p/k$, then $q = |k - p| \approx k$, and $\mu = \cos \theta_{pk}$ appears only in the angular factor $f(\omega, \theta, k, p, \mu)$ (34), the dominant term of which is the third quadrupole term

$$\frac{1}{2} \omega_2^4 \omega^4 \cos^4 \theta_{kk} \approx \frac{1}{2} \omega^4 \left[ \frac{\omega_2^2 - \omega^2}{\omega^2 - \omega_2^2 \cos^2 \theta} \right]^2 \mu^2.$$
Thus the \( \mu \) integration in the energy-flux expression may be performed analytically, with the result that

\[
F(x, \omega) = \frac{x}{12|x|^2 \omega} \left[ \frac{\omega^4}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2 \cos^2 \theta)} \right]^{\frac{1}{2}} \\
\times \omega^4 \left[ \frac{\omega^2 - \omega_2^2}{\omega^2 - \omega_3^2 \cos^2 \theta} \right] \frac{E(k)}{k^2} \int_0^\infty \int_0^\infty d^3x \rho_0(x) \int_0^\infty dp E(p) g(p, k, \omega). \quad (44)
\]

The emitted flux will go to zero as the critical angle is approached and \( k^2 \) gets very large, if the turbulence-energy spectrum decreases faster than \( k^{-5} \) for large \( k \).

Although the gravity-wave power has a low-frequency divergence, the vertical gravity-wave flux,

\[
F_z(\omega) = \int_0^{\pi/2} |F(x, \omega)| \|x\|^2 \sin \theta \cos \theta \, d\theta,
\]

is well defined, since the low-frequency waves are emitted in the horizontal direction and do not contribute to the vertical flux. The vertical flux spectrum is nearly independ-
ent of frequency (Figure 7) but depends on the Mach number and ratio of length scale to scale height as

$$F_z(\omega) \propto M \left( \frac{l}{H} \right)^4. \quad (45)$$

The total vertical flux, which is the integral of $F_z(\omega)$ from $\omega = 0$ to $\omega = \omega_2$, therefore goes as

$$F_z(\omega) \omega_2 \propto \left( \frac{l}{H} \right)^5, \quad (46)$$

(Figure 8). This behavior of the gravity-wave flux is obtainable by dimensional analysis. In the expression for the pressure fluctuation (40), the wave number must now be evaluated at low frequencies for which it is approximately $k \approx \omega_1/c \approx 1/H$, so that the contribution to the scaled pressure fluctuation from the $(n)$th multipole source term is

$$P^{(n)} \propto P_0^3 \left( \frac{u}{c} \right)^2 \left( \frac{l}{H} \right)^n \frac{l}{r},$$

and the energy flux (17) is

$$F^{(n)} \propto \rho_0 u^3 \left( \frac{u}{c} \right) \left( \frac{l}{H} \right)^{2n} \left( \frac{l}{r} \right)^2, \quad (47)$$

Fig. 8. Total upward dimensionless gravity-wave flux (in units of $\rho_0 u_0^3/l_0$) for $l_0 = H$. 

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which for the dominant quadrupole source term \((n=2)\) has the same dependence on the Mach number and \((l/H)\) as is observed (45).

C. EFFECT OF THE TURBULENCE SPECTRUM

The qualitative properties of acoustic and gravity-wave emission by turbulent motions are independent of the specific form of the turbulence spectrum, depending only on the fact it goes to zero at sufficiently high frequencies and large wave numbers. The numerical value of the power output or upward flux is, however, sensitive to certain features in the turbulence spectrum. The acoustic emission is proportional to the height of the maximum in the spectrum at \(k=2\pi/l\) and \(\omega=ku\), where \(l\) is the turbulence-length scale, and to the extent of the high-frequency tail. The gravity emission is proportional to the extent of the large wave-number tail of the spectrum, and diverges for spectra which fall off slower than \(k^{-5}\) for large \(k\) (44). Both types of emission are insensitive to the small wave number behavior of the spectra, which is fortunate since it is the spectrum of the large-size eddies that depends on the driving mechanism.

The shape of the turbulence spectrum for stratified atmospheres is not known. However, certain properties may be deduced from general considerations. For the special case of isotropic, homogeneous, incompressible turbulence, dimensional

![Diagram](https://example.com/diagram.png)

Fig. 9. The turbulence-energy spectrum, in units of \(u_0^2l_0\) vs. wave-number in units of \(l_0^{-1}\), where \(u_0\) is the turbulence-velocity scale and \(l_0\) is the turbulence-length scale. Curves are shown for the exponential spectrum with \(k_0 = \pi/2l_0\), the Spiegel spectrum with \(Ra/2\pi^4 = 10\) and 1000, and for the Kolmogorov spectrum with \(k_0 = 2\pi/l_0\).
analysis (Kolmogorov, 1941; Landau and Lifshitz, 1959) shows that the spectrum must have the Kolmogorov form, \( E(k) \propto k^{-5/3} \), for intermediate-size eddies. For very large eddies (since the Fourier transform (26) must be finite as \( k \to 0 \)) the spectrum \( E(k) \) must decrease at least as fast as \( k^2 \) as \( k \to 0 \). On the other hand, for very small eddies (since it is generally assumed that all-order derivatives of the velocity correlation must be defined so all-order moments of the spectrum must exist) \( E(k) \) must decrease at least exponentially as \( k \to \infty \). Finally, the spectrum is normalized by the requirement that \( \int_0^\infty E(k) \, dk = \frac{2}{3} u_0^2 \). These are all the general restrictions we can impose on the turbulence spectrum.

We have considered three spectral forms (Figure 9). The first is an exponential spectrum

\[
E(k) = \frac{u_0^2}{k_0} \frac{1}{16 \left( \frac{k}{k_0} \right)^4} e^{-k/k_0},
\]

(48)

which is large at very small wave number and decreases exponentially at large wave numbers. The second is a spectrum derived by Spiegel (1962), from Boussinesq theory at very low Prandtl number using Kovasznay’s expression for the non-linear interaction between turbulent modes. The result is

\[
E(k) = \frac{u_0^2}{k_0} A \left( \frac{k}{k_0} \right)^{-5/3} \left[ 1 - \left( \frac{k}{k_0} \right)^{-8/3} - \frac{4}{7} \left( \frac{\pi}{k_0 l_0} \right) \left( 1 - \left( \frac{k}{k_0} \right)^{-14/3} \right) - \frac{2(k_0 l_0)^4}{Ra} \left( \left( \frac{k}{k_0} \right)^{4/3} - 1 \right) \right]^2.
\]

(49)

For large Rayleigh number \( (Ra = gL^4 (d\gamma/dz)/\nu k) = \) buoyancy force/viscous force the normalization constant is \( A = 9/\pi \). The spectrum has the Kolmogorov form \( k^{-5/3} \) at intermediate wave number, and goes to zero quadratically at \( k_0 = \pi / l_0 \) and \( k^*_0 = k_0 (\frac{3}{4}(Ra/2\pi^4))^{1/2} \). The third is a pure Kolmogorov spectrum

\[
E(k) = \frac{u_0^2}{k_0} \left( \frac{k}{k_0} \right)^{-5/3} (k \geq k_0),
\]

(50)

which falls off very slowly at large wave numbers. \( k_0 \) is chosen so the maxima of all the spectra occur close to \( k = 2\pi / l_0 \), hence \( k_0 = \pi / 2l_0 \) for the exponential spectrum, \( k_0 \approx \pi / l_0 \) for Spiegel’s spectrum, and \( k_0 = 2\pi / l_0 \) for the Kolmogorov spectrum. The turbulent velocities \( u_k \) (31) for these spectra are shown in Figure 10. We have also considered three forms for the frequency part of the spectrum \( \Delta (\omega / ku_k) \) (Equations (28) to (30)): an exponential, a Gaussian, and \( \omega^2 \) times a Gaussian function of the frequency.

The gravity-wave emission is sensitive to the large wave-number behavior of the turbulence spectrum, as can be seen from Figure 8 and Equation (44), because the emission occurs predominantly near the critical angle where the wave number becomes infinite. The greater the extent of the large wave-number tail of the spectrum,
the nearer the singularity from \((\omega^2 - \omega_0^2 \cos^2 \theta)^{-1}\) the emission peak will occur, and the greater the resulting emission. The acoustic power output and its spectrum is shown for various turbulence spectra in Figures 2 and 11. It is sensitive to the maximum of the turbulence spectra, because it is proportional to \(E(|\mathbf{k} - \mathbf{p}|)E(\rho)\) and the wave numbers of acoustic waves are small. As the Mach number decreases and the low frequency cutoff \(\omega_1\) increases, the total acoustic emission is sensitive to the rapidity with which its spectrum falls off at high frequency. This high-frequency falloff depends on the form of the frequency part of the turbulence spectrum \(\Delta(\omega/k\mathbf{u}_0)\) at large \(\omega\), and on the form of the spatial part of the spectrum \(E(k)\) at large wave number, since as the frequency increases so does the wave number \(k\).

D. THE NUMERICAL VALUE OF THE EMISSION

The acoustic output, in units of \((\rho_0 u_0^3/l_0) M^5\), is shown in Figure 2, and the vertical gravity wave flux, in units of \((\rho_0 u_0^3/l_0)(l_0/H)^5\), is shown in Figure 8. Because the acoustic emission is very nearly isotropic, its vertical flux is \(\frac{1}{4}\) its power output.
The acoustic power output per unit volume is approximately

\[ P \approx 10^3 \frac{\rho \mu^3}{l_0} M^5 \text{ ergs/cm}^3 \text{ sec}. \]

The range of values of the power output is shown in Table I, and is about two orders of magnitude at \( M = 0.1 \), smaller as the Mach number approaches one and larger at very small Mach numbers where the emission is very sensitive to the form of the turbulence spectrum.

We see that at intermediate Mach numbers (~0.1) the acoustic power is about 10 times that found by Proudman (1952) \( [38(\rho \mu^3/l_0)M^5] \) for the Kolmogorov spectrum. This is due to the factor \( \omega^4 \) in the emission expression, which increases its value over the simple dimensional formula.

The upward gravity wave flux emission per unit volume is greater than about

\[ F_z \approx 10^2 \frac{\rho \mu^3}{l_0} \left( \frac{l_0}{H} \right)^5 \text{ ergs/cm}^2 \text{ sec}, \]
TABLE I

Coefficient for Acoustic Emission ($\alpha_{ac}$)

\[ P_{ac} = \alpha_{ac} \frac{\rho_0 u_0^3}{l_0} M^5 \]

<table>
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<th>Mach 0.01</th>
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<td>Exponential Frequency Spectrum</td>
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<td>316</td>
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<td>Spiegel Spectrum, $Ra/2\pi^4 = 10$</td>
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<tr>
<td>Exponential Spectrum</td>
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<td>$10^{-18}$</td>
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<tr>
<td>Exponential Spectrum</td>
<td>174</td>
<td>470</td>
<td>$10^{-10}$</td>
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</tbody>
</table>

TABLE II

Coefficient for Gravity Wave Emission ($\alpha_g$)

\[ F_{2g} = \alpha_g \frac{\rho_0 u_0^3}{l_0} \left( \frac{l_0}{H} \right)^5 \]

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<tr>
<td>Exponential Frequency Spectrum</td>
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<td>(ii) Gaussian Frequency Spectrum</td>
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</tr>
<tr>
<td>Exponential Spectrum</td>
<td>270</td>
<td>220</td>
<td>57</td>
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</tbody>
</table>

depending on the large wave-number tail of the turbulence spectrum. The precise values are given in Table II. In the sun where the turbulence-length scale is the scale height, the gravity-wave emission from turbulence in stable layers is much larger than the acoustic, so large in fact that the present weak-emission theory does not apply. We can only say that the emission will be a non-negligible fraction of the turbulent energy; to calculate it quantitatively the non-linear interaction between the waves and the turbulence must be considered.

The expression for the energy flux emitted by turbulence in a stratified atmosphere (32) derived in this paper, will, in a subsequent paper, be used to calculate the acoustic energy flux emitted from the solar convection zone.
Appendix A: Derivation of the Wave Equation

In Section 1 we stated the inhomogeneous wave equation for the pressure fluctuation. Here we present its derivation. We separate the mean values (indicated by subscript 0), which are functions only of height \( z \), and the fluctuating quantities (indicated by subscript 1):

\[
P(\mathbf{r}, t) = P_0(z) + P_1(\mathbf{r}, t) \\
\rho(\mathbf{r}, t) = \rho_0(z) + \rho_1(\mathbf{r}, t).
\]

We assume that the unperturbed state is static, \( \mathbf{u}_0 = 0 \), and is in hydrostatic equilibrium

\[
\nabla P_0 = -\rho_0 \mathbf{g}
\]

where \( \mathbf{g} = (0, 0, -g) \) is the gravitational acceleration.

Since we wish to derive the inhomogeneous wave equation for the pressure fluctuation \( P_1 \), we must express the velocity and density fluctuations in terms of the pressure fluctuation. From the momentum conservation and continuity equations ((1) and (3)),

\[
\mathbf{u} = -\frac{1}{\rho_0} \left( \frac{\partial}{\partial t} \right)^{-1} \left[ \nabla P_1 - \rho_1 \mathbf{g} - \mathbf{f} \right] \tag{51}
\]

and

\[
\rho_1 = \left( \frac{\partial}{\partial t} \right)^{-1} \left[ -\nabla \cdot \rho_0 \mathbf{u} + q \right]. \tag{52}
\]

On substituting for \( \rho_1 \), from (52), in the expression (51) for \( \mathbf{u} \), we get

\[
\mathbf{u} = -\frac{1}{\rho_0} \left( \frac{\partial}{\partial t} \right)^{-1} \left[ \nabla P_1 + \mathbf{g} \left( \frac{\partial}{\partial t} \right)^{-1} \nabla \cdot \rho_0 \mathbf{u} - \mathbf{g} \left( \frac{\partial}{\partial t} \right)^{-1} q \right].
\]

Using the adiabatic relation (5), we find that

\[
\nabla \cdot (\rho_0 \mathbf{u}) = \frac{1}{c_0^2} \left[ -\frac{\partial P_1}{\partial t} + (\gamma - 1) \rho_0 \mathbf{g} \cdot \mathbf{u} + h \right]. \tag{53}
\]

Substituting this equation in the relation (51) and rearranging terms, we have an expression for the velocity as a linear function of the pressure plus non-linear terms:

\[
\mathbf{u}_1 = -\frac{1}{\rho_0} \left( \frac{\partial}{\partial t} \right)^{-1} \left[ \nabla_1 P_1 - \mathbf{f}_1 \right] \tag{54}
\]

\[
w = -\frac{1}{\rho_0} \left( \frac{\partial}{\partial t} \right)^{-1} \left( 1 + \left( \frac{\partial}{\partial t} \right)^{-2} \omega_2^2 \right)^{-1}
\times \left[ \frac{\partial P_1}{\partial z} + \frac{g}{c_0^2} P_1 - f_z + \frac{g}{c_0^2} \left( \frac{\partial}{\partial t} \right)^{-1} (c_0^2 q - h) \right]. \tag{55}
\]
Here $u_1$ and $w$ are the horizontal and vertical components of the velocity respectively, and $V_1$ is horizontal component of the gradient.

Turning now to the density fluctuation, we have, on substituting (53) for $\nabla \rho_0 u$ in (52) for $\rho_1$

$$\rho_1 = \frac{P_1}{c_0^2} - \rho_0 \left( \frac{\gamma - 1}{c_0^2} \right) \left( \frac{\partial}{\partial t} \right)^{-1} g \cdot u + \frac{1}{c_0^2} \left( \frac{\partial}{\partial t} \right)^{-1} (c_0^2 q - h).$$

The density fluctuation can be expressed in terms of the pressure fluctuations plus non-linear terms by eliminating $u$ with (54) and (55):

$$\rho_1 = \left(1 + \left( \frac{\partial}{\partial t} \right)^{-2} \omega_2^2 \right)^{-1} \left[ \frac{P_1}{c_0^2} + \left( \frac{\partial}{\partial t} \right)^{-2} \left\{ (\gamma - 1) \frac{g}{c_0^2} \cdot (\nabla P_1 - f) \right\} \right. \right.$$

$$\left. + \frac{1}{c_0^2} \left( \frac{\partial}{\partial t} \right)^{-1} (c_0^2 q - h) \right]. \tag{56}$$

We derive the inhomogeneous wave equation by eliminating the density fluctuation and velocity from the basic equations, (1), (3) and (5), using the relations (54), (55), and (56). We start by rearranging the adiabatic motion equation (53) as

$$\frac{\partial P_1}{\partial t} - (\gamma - 1) g \cdot \rho_0 u + c_0^2 \nabla \cdot \rho_0 u = h.$$

Differentiate with respect to time, and eliminate $\rho_0 (\partial u/\partial t)$ using the momentum equation (1) with the density fluctuation eliminated by (56). This gives

$$\frac{\partial^2 P_1}{\partial t^2} + (\gamma - 1) g \cdot (\nabla P_1 - f) - \frac{\omega_2^2}{1 + \left( \frac{\partial}{\partial t} \right)^{-2} \omega_2^2} \times \left[ P_1 + (\gamma - 1) \left( \frac{\partial}{\partial t} \right)^{-2} g \cdot (\nabla P_1 - f) + \left( \frac{\partial}{\partial t} \right)^{-1} (c_0^2 q - h) \right] - c_0^2 \nabla^2 P_1 + c_0^2 \nabla \cdot f$$

$$+ \frac{1}{1 + \left( \frac{\partial}{\partial t} \right)^{-2} \omega_2^2} g \cdot \nabla \left[ P_1 + (\gamma - 1) \left( \frac{\partial}{\partial t} \right)^{-2} \right.$$

$$\left. \times g \cdot (\nabla P_1 - f) + \left( \frac{\partial}{\partial t} \right)^{-1} (c_0^2 q - h) \right] = \frac{\partial h}{\partial t}.$$

Carrying out the differentiation in the last terms on the left-hand side, rearranging
terms, and multiplying by \((1 + (\partial / \partial t)^{-2} \omega_2^2)\), gives

\[
\frac{\partial^2 P_1}{\partial t^2} - \left(1 + \left(\frac{\partial}{\partial t}\right)^{-2} \omega_2^2\right) c_0^2 \nabla^2 P_1 + (\gamma - 1) \mathbf{g} \cdot \nabla P_1 \\
+ \mathbf{g} \cdot \nabla \left[ P_1 + (\gamma - 1) \left(\frac{\partial}{\partial t}\right)^{-2} \mathbf{g} \cdot \nabla P_1 \right] \\
= c_0^2 \left(1 + \left(\frac{\partial}{\partial t}\right)^{-2} \omega_2^2\right) \left(\nabla \cdot \mathbf{f} + \frac{\partial q}{\partial t}\right) + (\gamma - 1) \mathbf{g} \cdot \mathbf{f} \\
+ \mathbf{g} \cdot \nabla \left[ (\gamma - 1) \left(\frac{\partial}{\partial t}\right)^{-2} \mathbf{g} \cdot \mathbf{f} - \left(\frac{\partial}{\partial t}\right)^{-1} \left(c_0^2 q - h\right) \right] \\
- \frac{\partial}{\partial t} \left(c_0^2 q - h\right).
\]

Collecting derivatives of \(P_1\) and rearranging terms we get finally

\[
\frac{\partial^2 P_1}{\partial t^2} - c_0^2 \nabla^2 P_1 + \gamma \mathbf{g} \cdot \nabla P_1 - \left(\frac{\partial}{\partial t}\right)^{-2} c_0^2 \omega_2^2 \nabla^2 \nabla P_1 = S, \tag{57}
\]

where the source function is

\[
S = c_0^2 \left(1 + \left(\frac{\partial}{\partial t}\right)^{-2} \omega_2^2\right) \left(\nabla \cdot \mathbf{f} + \frac{\partial q}{\partial t}\right) + \left(\frac{\partial}{\partial t}\right)^{-1} \left(\frac{\partial^2}{\partial t^2} + \mathbf{g} \cdot \nabla\right) \\
\times \left[\left(\frac{\partial}{\partial t}\right)^{-1} (\gamma - 1) \mathbf{g} \cdot \mathbf{f} + h - c_0^2 q\right]. \tag{58}
\]

**Appendix B: Calculation of the Energy Flux Emited by Turbulence**

In Section 2 we outlined the derivation of the expression for the energy flux emitted, in the form of small amplitude compressional and gravity waves, by turbulence in an isothermal atmosphere. Here we present the details of the derivation. The mean energy flux in an isothermal atmosphere is (17)

\[
\mathbf{F} = \langle P_1 \mathbf{u} \rangle = -\frac{c_0^2}{\gamma} \left\langle \frac{\partial}{\partial t} \mathbf{P}' \right\rangle^{-1} \\
\times \left\{ \nabla \mathbf{P}' + \left(1 + \left(\frac{\partial}{\partial t}\right)^{-2} \omega_2^2\right)^{-1} \left(\nabla_z - \frac{2 - \gamma}{2 c_0^2} \mathbf{g}\right) \mathbf{P}' \right\}, \tag{59}
\]

where the pressure fluctuation, \(\mathbf{P}'\), is the solution of the inhomogeneous wave equation in an isothermal atmosphere (\((10)\) and \((13)\)).
1. **Solution of the Wave Equation**

To solve the wave equation, make a space-time Fourier transform. Define $P(k, \omega)$ as the Fourier transform of $P'(x, t)$:

$$P'(x, t) = \int P(k, \omega) e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}} \, d^3k \, d\omega$$

$$P(k, \omega) = \frac{1}{(2\pi)^4} \int P'(x', t') e^{-i\omega t' - i\mathbf{k} \cdot \mathbf{x}'} \, d^3x' \, dt'.$$

(60)

Then the wave equation becomes

$$\int \left\{ -\omega^2 + c_0^2 k^2 + \omega_1^2 - c_0^2 \frac{\omega^2}{\omega_1^2} (l^2 + m^2) \right\} P(k, \omega) e^{i\omega t - i\mathbf{k} \cdot \mathbf{x}} \, d^3k \, d\omega = \gamma^2 c_0 S(x, t),$$

where $k = (l, m, n)$, and $n$ is the vertical ($z$) component of the wave vector. Multiplying by $e^{-i\omega t' + i\mathbf{k} \cdot \mathbf{x}'}$ and integrating over $x, t$ space, gives

$$\left[ c_0^2 \left\{ n^2 + \left(1 - \frac{\omega_2^2}{\omega_1^2}\right)(l^2 + m^2) \right\} - \omega^2 + \omega_1^2 \right] P(k, \omega) = \gamma^2 c_0 S(k, \omega),$$

where $S(k, \omega)$ is the Fourier transform of the source function:

$$S(k, \omega) = \frac{1}{(2\pi)^4} \int S(x', t') e^{-i\omega t' + i\mathbf{k} \cdot \mathbf{x}'} \, d^3x' \, dt'.$$

(61)

Thus the Fourier transform of the pressure fluctuations emitted by the turbulent motions is

$$P(k, \omega) = \frac{\gamma^2}{c_0} \frac{S(k, \omega)}{n^2 + \left(1 - \frac{\omega_2^2}{\omega_1^2}\right)(l^2 + m^2) - \frac{\omega^2}{c_0^2} + i\varepsilon},$$

(62)

where we introduce the notation $\delta^2 \equiv \omega^2 - \omega_1^2$.

2. **The Energy Flux**

We can now evaluate the mean energy flux (59) by expressing $P'(x, t)$ in terms of its Fourier transform (60). Taking the time average, we obtain

$$F(x) = \frac{c_0^2}{\gamma} \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \int \int \int P(k', \omega') \times \left\{ \frac{1}{i\omega''} \left\{ i l'' + \frac{\omega''}{\omega'' - \omega_2^2} \left( \frac{in''}{2} + \frac{\gamma g}{c_0^2} \right) \right\} \right\}

\times P^*(k'', \omega'') e^{i(\omega'' - \omega')t - i(k' - k'') \cdot x} \, d^3k' \, d\omega' \, d^3k'' \, d\omega''.$$
The time integration may be performed, and gives
\[ \int_{-\infty}^{\infty} e^{i(\omega' - \omega)t} dt = 2\pi \delta(\omega' - \omega'). \]

Then we can perform one \( \omega \) integration, and on substituting the solution (62) for \( P(k, \omega) \) we find that the energy flux is
\[ F(x) = \int_{-\infty}^{\infty} F(x, \omega) d\omega \]
\[ F(x, \omega) = \lim_{T \to \infty} \frac{2\pi}{T} \int_{-\infty}^{\infty} \left( \frac{1}{\omega} \right) \left( \frac{\omega^2}{\omega^2 - \omega_2} \right) \left( l'' + m'' \right) \frac{\frac{2 - \gamma}{\omega}}{\frac{\omega}{c_0}} \]
\[ \times \frac{S(k', \omega)}{n^2 + \left( \frac{\omega^2 - \omega_2^2}{\omega^2} \right) (l'^2 + m'^2) - \frac{\partial^2}{c_0^2} + i\epsilon} \]
\[ \times \frac{S(-k'', -\omega'')}{n''^2 + \left( \frac{\omega^2 - \omega_2^2}{\omega^2} \right) (l''^2 + m''^2) - \frac{\partial^2}{c_0^2} + i\epsilon} \]
\[ \times e^{-i(k' - k'') \cdot x} d^3k' d^3k'' \quad (63) \]

where \( F(x, \omega) \) is the spectral resolution of the energy flux.

3. ASYMPTOTIC VALUE OF THE FOURIER TRANSFORM

To evaluate this expression we use Lighthill's (1960) expression for the asymptotic Fourier transform at large \( |x| \):
\[ \int \frac{F(k)}{G(k)} e^{ik \cdot x} d^3k = \frac{4\pi^2}{|x|} \sum \frac{F(k) e^{ik \cdot x}}{|\nabla_k G|^2}, \quad (64) \]

where the sum is over the set \( k \) on the surface \( G = 0 \) where the normal (which is in the direction of propagation, since \( u_\parallel = -\nabla_k G(\partial G/\partial \omega)^{-1} \)) is in the direction \( x \). This expression includes the radiation condition. \( K \) is the Gaussian curvature which is the product of the three principle curvatures:
\[ K = \sum_{G_{\alpha \beta} (G_{\gamma \gamma} G_{\beta \gamma} - G_{\beta \gamma}) + 2 \sum_{G_{\alpha \gamma} (G_{\alpha \gamma} G_{\beta \gamma} - G_{\alpha \gamma} G_{\beta \gamma})}{(\sum G_{\alpha \beta})^2}, \]

where the sums are over all cyclic permutations of \( \alpha, \beta, \gamma \). The denominator in our integral (63) is
\[ G = n^2 + \left( 1 - \frac{\omega_2^2}{\omega^2} \right) (l^2 + m^2) - \frac{\partial^2}{c_0^2}. \quad (65) \]
Its derivative is

$$\nabla_k G = 2 \left( \begin{array}{c} 1 - \frac{\omega^2}{\omega^2} \\
\frac{\omega^2}{\omega^2} 
\end{array} \right) l, \left[ \begin{array}{c} 1 - \frac{\omega^2}{\omega^2} \\
\frac{\omega^2}{\omega^2} 
\end{array} \right] m, n \right),$$ (66)

and the curvature is

$$K = \frac{\sum G^a G_{\theta \theta} G_{\theta \theta}}{\left( \sum G^a \right)^2} = \frac{\left( 1 - \frac{\omega^2}{\omega^2} \right)^2 \left[ \left( 1 - \frac{\omega^2}{\omega^2} \right) (l^2 + m^2) + n^2 \right]}{\left[ n^2 + \left( 1 - \frac{\omega^2}{\omega^2} \right) (l^2 + m^2) \right]^2}. $$ (67)

We wish to pass from expressions in terms of the components of the wave vector to those in terms of the frequency and direction of propagation. The latter is the direction of the group velocity, so

$$\hat{x} = \frac{\nabla G}{|\nabla G|} = \left( \begin{array}{c} 1 - \frac{\omega^2}{\omega^2} \\
\frac{\omega^2}{\omega^2} 
\end{array} \right) l, \left[ \begin{array}{c} 1 - \frac{\omega^2}{\omega^2} \\
\frac{\omega^2}{\omega^2} 
\end{array} \right] m, n \right),$$ (68)

and the cosine of the angle between the propagation direction and the vertical is

$$\cos \theta = \hat{z} \cdot \hat{x} = \frac{n}{\left[ \left( 1 - \frac{\omega^2}{\omega^2} \right) (l^2 + m^2) + n^2 \right]^2}. $$ (69)

The components of the wave vector are obtained by solving equation (69) for $n^2$ and using the resulting expression to eliminate $n^2$ from the dispersion relation $G=0$ (65), which can then be solved for the horizontal component of the wave vector

$$l^2 + m^2 = \frac{\omega^2}{c^2_0} \frac{\omega^4 (1 - \cos^2 \theta)}{(\omega^2 - \omega^2) (\omega^2 - \omega^2 \cos^2 \theta)}. $$ (70)

With this expression for the horizontal component, Equation (69) gives the vertical component of the wave vector

$$n^2 = \frac{\omega^2}{c^2_0} \frac{\omega^2 - \omega^2}{\omega^2 - \omega^2 \cos^2 \theta}. $$ (71)

The square of the magnitude of the wave vector is

$$k^2 = \frac{\omega^2}{c^2_0} \left[ \frac{\omega^4 - 2 \omega^2 \omega^2 \cos^2 \theta + \omega^4 \cos^2 \theta}{(\omega^2 - \omega^2) (\omega^2 - \omega^2 \cos^2 \theta)} \right]. $$ (72)

The direction of the wave vector is not the same as the direction of propagation. The angle between $\hat{k}$ and the vertical is

$$\cos^2 \theta_k = \frac{n^2}{k^2} = \frac{(\omega^2 - \omega^2)^2 \cos^2 \theta}{\omega^4 - 2 \omega^2 \omega^2 \cos^2 \theta + \omega^4 \cos^2 \theta}, $$ (73)
or
\[
\tan \theta_k = \left[ \frac{l^2 + m^2}{n^2} \right]^{\frac{1}{2}} = \frac{\omega^2}{\omega^2 - \omega_2^2} \tan \theta.
\tag{74}
\]

Using Lighthill's expression (64) for the asymptotic value of the Fourier integral (63), with the values (66) and (67) for VG and K respectively, we find that the energy flux is
\[
F(\mathbf{x}, \omega) = \lim_{T \to \infty} \frac{8\pi^5}{T|x|^2 \omega^2 - \omega_2^2} \left[ \frac{\omega^2 - \omega_2^2}{\omega^2} l, \frac{\omega^2 - \omega_2^2}{\omega^2} m, \frac{n - \frac{2 - \gamma}{\gamma} \omega_1}{c_0} \right] \left[ \left( 1 - \frac{\omega_2^2}{\omega^2} \right)^2 (l^2 + m^2) + n^2 \right]^{\frac{1}{2}} \times \frac{\omega_2^2}{c_0} \left( \frac{\omega^2 - \omega_2^2}{\omega^2} \right)^2 S(\mathbf{k}, \omega) S^*(\mathbf{k}, \omega).
\]

Using the expressions (68) for the direction of propagation and (71) for \(n^2\), we can reduce this formula to
\[
F(\mathbf{x}, \omega) = \lim_{T \to \infty} \frac{8\pi^5}{T|x|^2 c_0} \left\{ \frac{\omega}{\omega} \left[ \left( \omega^4 \frac{\omega}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_2^2 \cos^2 \theta)} \right]^{\frac{1}{2}} - i \frac{2 - \gamma}{\gamma} \left( \frac{\omega \omega_1}{\omega^2 - \omega_2^2} \right) \right\} \left[ \left( \omega^4 \frac{\omega}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_2^2 \cos^2 \theta)} \right]^{\frac{1}{2}} S(\mathbf{k}, \omega) S^*(\mathbf{k}, \omega). \tag{75}
\]

The second term is always imaginary, and further, it vanishes upon integration over \(\omega\), since it is an odd function of \(\omega\), so we can neglect it. Thus the energy flux is
\[
F(\mathbf{x}, \omega) = \lim_{T \to \infty} \frac{8\pi^5}{T|x|^2 \omega} \left[ \frac{\omega}{c_0} \right]^{\frac{1}{2}} \left( \omega^4 \frac{\omega}{\omega^2 - \omega_2^2 \cos^2 \theta} \right) \left( \omega^2 - \omega_2^2 \right) S(\mathbf{k}, \omega) S^*(\mathbf{k}, \omega). \tag{76}
\]

KRAICHSNAN (1953) has derived the corresponding expression for a homogeneous medium.

4. THE MULTIPOLAR EXPANSION OF THE SOURCE FUNCTION

We must now consider the explicit expression for the source function (13):
\[
\gamma^4 c_0 S = P_0^4 \left[ \frac{\omega^2 - \omega_2^2}{\omega^2} \right] V \cdot \nabla \rho_0 u u + (\gamma - 1) g \left( 1 + \frac{g}{\omega^2} \frac{\partial}{\partial z} \right) V \cdot \rho_0 u w.
\]

We expand the source function in multipoles (UNNO, 1964). The first term is
\[
P_0^4 \frac{\partial^2 (\rho u \mu)}{\partial x_i \partial x_j} = \frac{P_0^4}{P_0^3} \frac{\partial^2 (\rho u \mu)}{\partial x_i \partial x_j} + P_0^4 \left[ 2 \frac{\partial \rho_0}{\partial z} \frac{\partial}{\partial x_j} (\rho_0 u w) + \frac{\partial^2 \rho_0}{\partial z^2} (\rho_0 u w) \right].
\]
In an isothermal atmosphere
\[
\frac{d \rho^\pm_0}{dz} = -\frac{\gamma g}{2c_0^2} \rho^\pm_0
\]
and
\[
\frac{d^2 \rho^\pm_0}{dz^2} = \frac{\gamma^2 g^2}{4c_0^4} \rho^\pm_0.
\]

Thus
\[
P_0^{-\frac{1}{2}} \frac{\partial^2 (\rho^0_0 u_j u_j)}{\partial x_i \partial x_j} = \frac{\gamma^2}{c_0} \left[ \frac{\partial^2 (\rho^0_0 u_j u_j)}{\partial x_i \partial x_j} - \frac{\gamma g}{2c_0^2} \frac{\partial}{\partial x_j} (\rho^0_0 u_j w) + \frac{\gamma^2 g^2}{4c_0^4} (\rho^0_0 w w) \right].
\] (77)

The second term is
\[
P_0^{-\frac{1}{2}} \frac{\partial}{\partial z} \frac{\partial}{\partial x_j} (\rho^0_0 u_j w) = \frac{\gamma^2}{c_0} \left[ \frac{\partial^2 (\rho^0_0 u_j w)}{\partial z \partial x_j} - \frac{\gamma g}{2c_0^2} \frac{\partial}{\partial x_j} (\rho^0_0 w w) \right] \times \left\{ \frac{\partial}{\partial x_j} (\rho^0_0 u_j w) + \frac{\partial}{\partial z} (\rho^0_0 w w) \right\} + \frac{\gamma^2 g^2}{4c_0^4} (\rho^0_0 w w).
\] (78)

The last term is
\[
P_0^{-\frac{1}{2}} \frac{\partial}{\partial x_j} (\rho^0_0 u_j w) = \frac{\gamma^2}{c_0} \left[ \frac{\partial}{\partial x_j} (\rho^0_0 u_j w) - \frac{\gamma g}{2c_0^2} (\rho^0_0 w w) \right].
\] (79)

Using the results (77) to (79), we find that the multipole expansion of the source function is
\[
S = \left[ \left( 1 - \frac{\omega^2}{\omega^2} + \frac{\omega^2}{\omega^2} \delta_{i3} \right) \frac{\partial^2 (\rho^0_0 u_i u_j)}{\partial x_i \partial x_j} \right]
\left[ \frac{1}{\gamma} - \frac{1}{2 \omega^2} + \frac{1}{2 \omega^2} \delta_{i3} \right] \frac{\partial (\rho^0_0 u_j w)}{\partial x_j} + \omega^2 \left( \frac{2 - \gamma}{\gamma} \right) (\rho^0_0 w w).
\] (80)

To compress notation, we write the source function (80) as
\[
S(x, t) = \alpha \frac{\partial^2 (\rho^0_0 u_i u_j)}{\partial x_i \partial x_j} - 2 \frac{\omega^2}{\omega^2} \beta \frac{\partial (\rho^0_0 u_j w)}{\partial x_j} + \omega^2 \frac{\delta (\rho^0_0 w w)},
\] (81)
where
\[
\alpha = \left( 1 - \frac{\omega^2}{\omega^2} + \frac{1}{2 \omega^2} \delta_{i3} + \delta_{j3} \right)
\]
\[
\beta = \left( \frac{1}{\gamma} - \frac{1}{2 \omega^2} + \frac{1}{2 \omega^2} \delta_{i3} \right)
\]
\[
\gamma = \left( \frac{2 - \gamma}{\gamma} \right).
\] (82)

We can now evaluate $|S(k, \omega)|^2$, which appears in Equation (76) for the energy
flux. By definition of the Fourier transform of the source function (61)

$$|S(k, \omega)|^2 = \frac{1}{(2\pi)^8} \int \int S(x', t') e^{-i\omega t' + ik \cdot x'} d^3x' dt' \times \int \int S(x'', t'') e^{i\omega t'' - ik \cdot x''} d^3x'' dt''. \tag{83}$$

First, integrate by parts in each of the transforms of the source function in (83), according to

$$\int_v \nabla \cdot u \, d^3x = - \int_v \nabla f \cdot u \, d^3x + \int_S f u \cdot dS,$$

where the surface integral vanishes, since the turbulence is confined to a finite region. This eliminates the derivatives in the multipole expansion of the source function (81), which becomes

$$S(x', t') = \rho_0^2 \left[ - \alpha k_i k_j u_j' u_i' + 2i \frac{\omega_1}{\epsilon_0} \beta k_j u_j' w' + \frac{\omega_1^2}{\epsilon_0^2} \delta w' w' \right]. \tag{84}$$

Next, transform the coordinates in expression (83) for $|S(k, \omega)|^2$ to the average position and time

$$x_0 = \frac{x' + x''}{2}, \quad t_0 = \frac{t' + t''}{2},$$

and the relative separation and time interval

$$r = x'' - x', \quad \tau = t'' - t',$$

of the turbulent elements interacting to produce the waves. Then

$$|S(k, \omega)|^2 = \frac{1}{(2\pi)^8} \iiint S \left( x_0 - \frac{r}{2}, t_0 - \frac{\tau}{2} \right) \times S \left( x_0 + \frac{r}{2}, t_0 + \frac{\tau}{2} \right) e^{i\omega t - ik \cdot r} d^3x_0 dt_0 d^3r \, d\tau. \tag{85}$$

Performing the integration over $t_0$ in (85) gives a time average of the product $S' S''$:

$$|S(k, \omega)|^2 = \frac{T}{(2\pi)^8} \iiint \left\langle S \left( x_0 - \frac{r}{2}, t_0 - \frac{\tau}{2} \right) S \left( x_0 + \frac{r}{2}, t_0 + \frac{\tau}{2} \right) \right\rangle e^{i\omega t - ik \cdot r} d^3x_0 d^3r \, d\tau$$

$$= \frac{T}{(2\pi)^8} \iiint \rho_0(x_0) \zeta(x_0, r, \tau) e^{i\omega t - ik \cdot r} d^3x_0 d^3r \, d\tau, \tag{86}$$
where $\zeta$ is the correlation

$$
\zeta(x_0, r, \tau) = \frac{1}{\rho_0(x_0)} S\left(\frac{x_0 - r}{2}, t_0 - \frac{\tau}{2}\right) S\left(\frac{x_0 + r}{2}, t_0 + \frac{\tau}{2}\right) \left( 1 - \alpha k_j k_j u_j u_j' + 2i \frac{\omega_1}{c_0} \beta k_j u_j' w' + \frac{\omega_1^2}{c_0^2} \delta w' w' \right) \times \left( 1 - \alpha k_j k_m u_j u_m' - 2i \frac{\omega_1}{c_0} \beta k_m u_m'' w'' + \frac{\omega_1^2}{c_0^2} \delta w'' w'' \right) \right). \tag{87}
$$

With the result (86) for $|S(k, \omega)|^2$, with the correlation $\zeta$ given by (87), the expression (76) for the energy flux becomes

$$
F(x, \omega) = \frac{\hat{\kappa} \pi}{2|x|^2 c_0} \frac{\omega}{\omega^4 - \omega^2 \cos^2 \theta} \left( \frac{\omega^4}{\omega^2 - \omega^2 \cos^2 \theta} \right) \times \frac{1}{(2\pi)^4} \int \rho_0(x_0) \zeta(x_0, r, \tau) e^{i\omega r - ik \cdot r} d^3 r d\tau d^3 x_0. \tag{88}
$$

5. TURBULENCE VELOCITY CORRELATION

We now evaluate the correlation $\zeta$ (87). When the indicated multiplication is performed $\zeta$ can be separated into real and imaginary parts, which are, respectively, symmetric and antisymmetric in $k$ and hence in $\omega$ (since the radiation condition requires $k \rightarrow -k$ when $\omega \rightarrow -\omega$).

$$
\zeta = \alpha^2 k_j k_j k_m u_j u_j u_m'' + 4 \beta^2 \frac{\omega_1}{c_0^2} k_j k_m u_j' w' u_m''
$$

$$
- \alpha \delta \frac{\omega_1^2}{c_0^2} \left\{ k_j k_j \left( u_j u_j' w'' + k_i k_m \left( w' w' u_m'' \right) \right) + \delta \frac{\omega_1^4}{c_0^4} \left( w' w' w'' \right) \right\}
$$

$$
+ 2i \beta \alpha \frac{\omega_1}{c_0} \left( \alpha k_j k_m + \delta \frac{\omega_1^2}{c_0^2} \delta j_3 \delta m_3 \right) \left\{ k_i \left( u_i u_j u_m'' \right) - k_i \left( u_j' w' u_m'' \right) \right\}. \tag{89}
$$

The imaginary last term is odd in $k$ and so antisymmetric in $\omega$, and thus gives no contribution to the flux upon integration over $\omega$. Furthermore, reducing the fourth-order velocity correlations to second-order correlations and assuming isotropic homogeneous turbulence, as we shall do, this term vanishes identically. We thus retain only the real terms. Substituting the expressions (82) for $\alpha, \beta, \delta$ in the real part of the expression (89) for $\zeta$, using the expressions (72) and (71) for the wave vector and its vertical component, introducing the notation $v = \hat{k} \cdot \mathbf{u}$ and $w = \tilde{z} \cdot \mathbf{u}$ for the components of the turbulent velocity in the direction of the wave vector and the vertical direction respectively, and reducing the fourth-order turbulent velocity correlations to second
order by (24), the correlation $\zeta$ becomes

$$\zeta(x_0, r, \tau) = 2 \frac{\omega^4}{c_0^4} \left[ \frac{\omega^4}{\omega^2 (\omega^2 - \omega_2^2 \cos^2 \theta)} \right] \left( \frac{\omega^4 - 2\omega^2 \omega_2^2 \cos^2 \theta + \omega_2^4 \cos^2 \theta}{\omega^2 (\omega^2 - \omega_2^2 \cos^2 \theta)^2} \right) \langle v' v'' \rangle^2$$

$$+ 4 \frac{\omega^4}{c_0^4} \left[ \frac{\omega^2 (\omega^4 - 2\omega^2 \omega_2^2 \cos^2 \theta + \omega_2^4 \cos^2 \theta)}{(\omega^2 - \omega_2^2 \cos^2 \theta)^2} \right] \cos \theta \langle v' v'' \rangle \langle v' w'' \rangle$$

$$+ \frac{\omega^4}{c_0^4} \left[ \frac{\omega^4}{(\omega^2 - \omega_2^2 \cos^2 \theta)^2} \right] \langle v' v'' \rangle \langle w' w'' \rangle + \langle v' w'' \rangle^2$$

$$+ \left( \frac{2}{\gamma} - \frac{\omega_2^2}{\omega^2} \right)^2 \frac{\omega_1^2 \omega_2^2 (\omega^4 - 2\omega^2 \omega_2^2 \cos^2 \theta + \omega_2^4 \cos^2 \theta)}{\omega_2^2 (\omega^2 - \omega_2^2 \cos^2 \theta)} \langle w' w'' \rangle^2$$

$$+ 4 \left( \frac{2}{\gamma} - \frac{\omega_2^2}{\omega^2} \right) \frac{\omega_1^2 \omega_2^2 (\omega^4 - 2\omega^2 \omega_2^2 \cos^2 \theta + \omega_2^4 \cos^2 \theta)}{\omega^2 - \omega_2^2 \cos^2 \theta} \cos \theta \langle v' w'' \rangle \langle w' w'' \rangle$$

$$+ 2 \frac{\omega_1^2 \omega_2^2 (\omega^4 - \omega_2^4 \cos^2 \theta)}{\omega^2 - \omega_2^2 \cos^2 \theta} \langle w' w'' \rangle^2$$

$$- 4 \left( \frac{2}{\gamma} - \frac{\omega_2^2}{\omega^2} \right) \frac{\omega_1^2 \omega_2^2 (\omega^4 - 2\omega^2 \omega_2^2 \cos^2 \theta + \omega_2^4 \cos^2 \theta)}{\omega^2 (\omega^2 - \omega_2^2 \cos^2 \theta)} \langle v' v'' \rangle^2$$

$$+ 4 \left( \frac{2}{\gamma} - \frac{\omega_2^2}{\omega^2} \right) \frac{\omega_1^2 \omega_2^2 (\omega^4 - 2\omega^2 \omega_2^2 \cos^2 \theta + \omega_2^4 \cos^2 \theta)}{\omega^2 - \omega_2^2 \cos^2 \theta} \cos \theta \langle v' w'' \rangle \langle w' w'' \rangle + 2 \left( \frac{2}{\gamma} - \frac{\omega_2^2}{\omega^2} \right) \frac{\omega_1^4}{c_0^4} \langle w' w'' \rangle^2.$$ 

(90)

6. CONVOLUTION OF THE TURBULENCE SPECTRA

The expression (88) for the emitted energy flux contains a Fourier transform of the correlation $\zeta$ and thus of the product of the second-order velocity correlations. The Fourier transform of a product of functions is a convolution (25):

$$\frac{1}{(2\pi)^4} \int \frac{d^3 r}{r} \int_{-\infty}^{\infty} d\tau \, e^{-i(\omega r - k \cdot r)} \langle v'_1 v''_2 \rangle \langle v'_3 v''_4 \rangle$$

$$= \int \Phi_{12}(k - p, \omega - \sigma) \Phi_{34}(p, \sigma) \frac{d^3 p}{p} \frac{d\sigma}{\sigma} = J_{1234},$$

(91)

where $\Phi$ is the Fourier transform of the velocity correlation

$$\Phi_{ij}(k, \omega) = \frac{1}{(2\pi)^4} \int \langle u_i(x, t) u_j(x + r, t + \tau) \rangle e^{i(\omega \tau - k \cdot r)} d^3 r d\tau.$$
For isotropic, homogeneous turbulence (26),

$$\Phi_{ij}(k, \omega) = \frac{E(k, \omega)}{4\pi k^2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right), \quad (92)$$

where $E(k, \omega)$ is the turbulence-energy spectrum, which we factor into an equal time-energy spectrum $E(k)$ and a frequency factor $\Lambda(\omega, k)$, (27):

$$E(k, \omega) = E(k) \Lambda(\omega, k). \quad (93)$$

First, consider one integration over $\sigma$ in the convolutions (91). This involves only the frequency factors and is

$$g(p, q = |k - p|, \omega) = \int_{-\infty}^{\infty} \Lambda(\omega - \sigma, q) \Lambda(\sigma, p) d\sigma. \quad (94)$$

We consider three types of frequency spectra. The first is an exponential spectrum (28), for which the $\sigma$ integral is

$$g(p, q, \omega) = \int_{-\infty}^{\infty} \frac{1}{qu_p qu_q} e^{-|\omega - \sigma|/qu_p} e^{-|\sigma|/qu_q} d\sigma. \quad (95)$$

Performing the $\sigma$ integration gives

$$g(p, q, \omega) = 2qu_p qu_q \frac{(pu_p e^{-\omega/pu_p} - qu_q e^{-\omega/qu_q})}{(pu_p + qu_q)(pu_p - qu_q)}, \quad (95)$$

where $q = |k - p|$. This term appears to become indefinite as $|k - p|$ approaches $p$, so we investigate this limit in detail. Let

$$|k - p| = q = p + \delta$$

and expand $u_q$ in powers of $(\delta/p)$. From (31)

$$u_q - u_p = \left[ \int_{p+\delta}^{2p+2\delta} E(k) \, dk \right] - \left[ \int_{p}^{2p} E(k) \, dk \right] = -u_p \left[ \frac{1}{2} \alpha + \frac{1}{8} \frac{(\delta/p)^2}{p} \right],$$

where

$$\alpha = \frac{E(p) p}{u^2_p} - \frac{E(2p) 2p}{u^2_p} \approx O(1).$$
Using this result, we can expand $g(p, q, \omega)$

$$g(p, q, \omega) = \frac{2e^{-\omega/pu_p}}{pu_p} \left[ 1 - \exp \left( \frac{\omega}{pu_p} \left( \frac{1 - \frac{\alpha}{2}}{p} + \cdots \right) \right) \right] \left[ 1 + \left( \frac{1 - \frac{\alpha}{2}}{p} + \cdots \right) \right].$$

For $\omega/pu_p$ large, $e^{-\omega/pu_p} \to 0$, while for $(\omega/pu_p)(\delta/p)$ small, we can expand the exponential giving the well-defined limiting expression

$$g(p, q, \omega) = \frac{e^{-\omega/pu_p}}{pu_p} \left( 1 + \frac{\omega}{pu_p} \right) \left( 1 + O \left( \frac{\omega}{pu_p} \sqrt{\frac{\delta}{p}} \right) \right).$$

This expansion is good for $(\omega/pu_p)\sqrt{\delta/p} \ll 1$, and we take as our condition that $(\omega/pu_p)\sqrt{\delta/p} < 10^{-2}$. The entire term is significant only for $\omega/pu_p < 10$, so our condition is

$$\frac{\delta}{p} \leq 10^{-6}$$

for the limiting expression to be valid.

The second type of frequency spectrum is the Gaussian spectrum (29), for which the $\sigma$ integral is

$$g(p, q, \omega) = \frac{4}{\pi pu_p q u_q} \int_{-\infty}^{\infty} e^{-\left(\frac{\omega}{pu_p} - \sigma\right)^2} \left(\frac{\sigma}{pu_p}\right)^2 d\sigma.$$

Performing the $\sigma$ integration gives

$$g(p, q, \omega) = \frac{4}{\left[\pi (p^2 u_p^2 + q^2 u_q^2)\right]^{3/2}} e^{-\omega^2/(p^2 u_p^2 + q^2 u_q^2)}.$$  \hspace{1cm} (96)

The final type of frequency spectrum is $\omega^2$ times a Gaussian (30), for which the $\sigma$ integral is

$$g(p, q, \omega) = \frac{16}{\pi (qu_q)^3 (pu_p)^3} \int_{-\infty}^{\infty} (\omega - \sigma)^2 \sigma^2 e^{-\left(\frac{\omega}{qu_q} - \sigma\right)^2} \left(\frac{\sigma}{pu_p}\right)^2 d\sigma.$$

Performing the $\sigma$ integration, by completing the square in the exponential and transforming to new coordinates so the integrand is reduced to a polynomial times a Gaussian, gives

$$g(p, q, \omega) = \frac{4}{\sqrt{\pi}} \frac{e^{-\omega^2/(p^2 u_p^2 + q^2 u_q^2)}}{\left(p^2 u_p^2 + q^2 u_q^2\right)^{3/4}} \left[ \frac{3 \left(\frac{pu_p qu_q}{p^2 u_p^2 + q^2 u_q^2}\right)^2}{\left(p^2 u_p^2 + q^2 u_q^2\right)^3} - 2\omega^2 \frac{4 p^2 u_p^2 q u_q}{\left(p^2 u_p^2 + q^2 u_q^2\right)^3} + 4 \frac{\omega^4 p^2 u_p^2 q^2 u_q^4}{\left(p^2 u_p^2 + q^2 u_q^2\right)^4} \right].$$  \hspace{1cm} (97)
Now consider the integration over $d^3p$ in the convolution (91). There are three types of correlations, depending on the angular factors appearing in (92). We take $k$ as the axis of the spherical coordinate system. Then,

$$q = |k - p| = [k^2 + p^2 - 2kp\mu]^\frac{1}{2},$$  

(98)

where $\mu = \cos\theta_{pk}$, and $\theta_{pk}$ is the angle between $p$ and $k$. This choice of coordinate axis is necessary so that the azimuthal angle $\phi$ does not appear in the expression for $|k - p|$, but only in the angular factors and so can be integrated over analytically. The angular factors in the convolutions involve (see Figure 12)

$$\frac{\hat{n} \cdot p}{|p|} = \cos \theta_p = \cos \theta_k \cos \theta_{pk} + \sin \theta_k \sin \theta_{pk} \cos \phi$$  

(99)

$$\frac{\hat{n} \cdot (k - p)}{|k - p|} = \cos \theta_q = \frac{k}{q} \cos \theta_k - \frac{p}{q} \cos \theta_p. \quad (100)$$

Here $\phi$ is the azimuthal angle of $p$ about $k$ with respect to $\phi$, and $\cos \theta_{pk} = \mu$.

We can analytically perform the $\phi$ integration in the three types of convolutions. The first convolution type is

$$J_{1122} = \int \frac{E(q, \omega - \sigma)}{4\pi q^2} \left(1 - \left(\frac{\hat{n} \cdot q}{q^2}\right)^2\right) \frac{E(p, \sigma)}{4\pi p^2} \left(1 - \left(\frac{\hat{n} \cdot (p)}{p^2}\right)^2\right) d^3p \, d\sigma$$

$$= \frac{1}{16\pi^2} \int \frac{E(q) E(p)}{q^2} g(p, q, \omega) (1 - \cos^2 \theta_{1q}) (1 - \cos^2 \theta_{2q}) \, dp \, d\mu \, d\phi,$$  

(101)
where \( q \) is given by Equation (98) and \( g \) by Equations (95) to (97). Use Equations (99) and (100) to express \( \cos \theta_{1q} \) and \( \cos \theta_{2q} \) in terms of \( \cos \theta_{1k}, \sin \theta_{1k}, \cos \theta_{2k}, \sin \theta_{2k}, \mu \) and \( \cos \phi \). Since \( \phi \) appears only as a polynomial in \( \cos \phi \) in the numerator of the convolution (101), it may be integrated over with the result

\[
J_{1122} = \frac{1}{2\pi} \int_0^\infty dp \int_{-1}^1 d\mu \frac{E(q) E(p)}{q^2} g(p, q, \omega)
\]

\[
\left\{1 - \left[\frac{1}{2} (1 - \mu^2) + \frac{1}{2} \cos^2 \theta_{2k} (3\mu^2 - 1) + \frac{k^2}{q^2} \cos^2 \theta_{1k} + \frac{p^2}{2q^2} \left\{1 - \mu^2 + \cos^2 \theta_{1k} (3\mu^2 - 1)\right\} - \frac{2pk}{q^2} \cos^2 \theta_{1k}\mu\right]\right.
\]

\[
+ \frac{k^2}{q^2} \cos^2 \theta_{1k} \cos^2 \theta_{2k}\mu^2 + \frac{1}{2} \cos^2 \theta_{1k} \sin^2 \theta_{2k} (1 - \mu^2)\right\}
\]

\[
+ \frac{p^2}{q^2} \left[\frac{1}{2} (\cos^2 \theta_{1k} + \cos^2 \theta_{2k}) \mu^2 (1 - \mu^2) + \cos^2 \theta_{1k} \cos^2 \theta_{2k}\mu^2 (2\mu^2 - 1) + 2\cos \theta_{1k} \sin \theta_{1k} \cos \theta_{2k}\sin \theta_{2k}\mu^2 (1 - \mu^2) \cos (\phi_1 - \phi_2)\right]
\]

\[
+ \frac{1}{2} \sin^2 \theta_{1k} \sin^2 \theta_{2k} (1 - \mu^2)^2 \left\{1 + 2 \cos^2 (\phi_1 - \phi_2)\right\}\right]
\]

\[
- \frac{p^2 k \mu}{q^2} \left[\cos^2 \theta_{1k} \left\{1 - \mu^2 + \cos^2 \theta_{2k} (3\mu^2 - 1)\right\}\right]
\]

\[
+ \cos \theta_{1k} \sin \theta_{1k} \cos \theta_{2k} \sin \theta_{2k} (1 - \mu^2) \cos (\phi_1 - \phi_2)\right\}\right\}.
\]

The second convolution type is

\[
J_{1212} = \int \frac{E(q, \omega - \sigma) (\hat{n}_1 \cdot q) (\hat{n}_1 \cdot q)}{4\pi q^2} \frac{E(p, \sigma) (\hat{n}_1 \cdot p) (\hat{n}_2 \cdot p)}{4\pi p^2} \frac{d^2 p}{p^2} d\sigma
\]

\[
= \frac{1}{16\pi^2} \int \frac{E(q) E(p)}{q^2} g(p, q, \omega) \cos \theta_{1q} \cos \theta_{2q} \cos \theta_{1p} \cos \theta_{2p} \frac{d^2 p}{d\mu} d\varphi.
\]

Again use Equations (99) and (100) to expand the angular factor in terms of the direction cosines of \( \mathbf{k} \) (with respect to \( \hat{n}_1 \) and \( \hat{n}_2 \)), \( \mu \) and \( \phi \). Then the \( \phi \) integration can again be performed, with the result

\[
J_{1212} = \frac{1}{8\pi} \int_0^\infty dp \int_{-1}^1 d\mu \frac{E(q) E(p)}{q^2} g(p, q, \omega) \left[\frac{k^2}{q^2} \left\{\cos^2 \theta_{1k} \cos^2 \theta_{2k}\mu^2\right\}\right.
\]

\[
+ \frac{p^2}{q^2} \left\{\frac{1}{2} \left(\cos^2 \theta_{1k} + \cos^2 \theta_{2k}\right) \mu^2 (1 - \mu^2) + \cos^2 \theta_{1k} \cos^2 \theta_{2k}\mu^2 (2\mu^2 - 1)\right\}\right].
\]
\[ + 2 \cos \theta_{1k} \sin \theta_{1k} \cos \theta_{2k} \sin \theta_{2k} \mu^2 (1 - \mu^2) \cos (\varphi_1 - \varphi_2) \]
\[ + \frac{1}{8} \sin^2 \theta_{1k} \cos^2 \theta_{2k} (1 - \mu^2)^2 (1 + 2 \cos^2 (\varphi_1 - \varphi_2)) \]
\[ - \frac{pn\mu}{q^2} \left\{ 2 \cos^2 \theta_{1k} \cos^2 \theta_{2k} \mu^2 + 2 \cos \theta_{1k} \sin \theta_{1k} \cos \theta_{2k} \sin \theta_{2k} (1 - \mu^2) \right\} \]
\[ + \frac{1}{8} \left( \cos^2 \theta_{1k} \sin^2 \theta_{2k} + \sin^2 \theta_{1k} \cos^2 \theta_{2k} \right) (1 - \mu^2) \right\}. \quad (104) \]

The third convolution type is
\[ J_{1112} = - \int \frac{E(q, \omega - \sigma)}{4\pi q^2} \left( 1 - \frac{(\hat{n}_1 \cdot \mathbf{q})^2}{q^2} \right) \frac{E(p, \sigma) (\hat{n}_1 \cdot \mathbf{p}) (\hat{n}_2 \cdot \mathbf{p})}{p^2} \, d^3 \mathbf{p} \, d\sigma \]
\[ = - \frac{1}{16\pi^2} \int \frac{E(q) E(p)}{q^2} g(p, q, \omega) (1 - \cos^2 \theta_{1q}) \cos \theta_{1p} \cos \theta_{2p} \, dp \, d\mu \, d\varphi. \quad (105) \]

As before expand \( \cos \theta_{1q}, \cos \theta_{1p}, \) and \( \cos \theta_{2p}, \) after which perform the \( \varphi \) integration giving
\[ J_{1112} = - \frac{1}{8\pi} \int \limits_0^\infty \, dp \int \limits_{-1}^1 \, d\mu \frac{E(q) E(p)}{q^2} g(p, q, \omega) \]
\[ \times \left\{ \cos \theta_{1k} \cos \theta_{2k} \mu^2 + \frac{1}{8} \sin \theta_{1k} \sin \theta_{2k} (1 - \mu^2) \cos (\varphi_1 - \varphi_2) \right\} \]
\[ - \frac{k^2}{q^2} \left\{ \cos^3 \theta_{1k} \cos \theta_{2k} \mu^2 + \frac{1}{8} \cos^2 \theta_{1k} \sin \theta_{1k} \sin \theta_{2k} (1 - \mu^2) \cos (\varphi_1 - \varphi_2) \right\} \]
\[ - \frac{p^2}{q^2} \left\{ \cos^3 \theta_{1k} \cos \theta_{2k} \mu^4 + \frac{1}{8} \cos^2 \theta_{1k} \sin \theta_{1k} \sin \theta_{2k} \mu^2 (1 - \mu^2) \cos (\varphi_1 - \varphi_2) \right\} \]
\[ + \frac{3}{8} \cos \theta_{1k} \sin^2 \theta_{1k} \cos \theta_{2k} \mu^2 (1 - \mu^2) \]
\[ + \frac{3}{8} \sin^3 \theta_{1k} \sin \theta_{2k} \mu^2 (1 - \mu^2) \cos (\varphi_1 - \varphi_2) \]
\[ + \frac{2pk}{q^2} \left\{ \cos^3 \theta_{1k} \cos \theta_{2k} \mu^2 + \cos \theta_{1k} \sin \theta_{1k} \sin \theta_{2k} \mu (1 - \mu^2) \cos (\varphi_1 - \varphi_2) \right\} \]
\[ + \frac{1}{8} \left( \cos^2 \theta_{1k} \sin^2 \theta_{2k} + \sin^2 \theta_{1k} \cos^2 \theta_{2k} \right) \right\}. \quad (106) \]

The Fourier transform of the velocity correlations (90) appearing in the energy flux (88) contains six convolutions which may be evaluated using the results (95)–(97) and (101)–(106). Note that since \( \mathbf{v} = \hat{\mathbf{k}} \cdot \mathbf{u} \) and \( \mathbf{w} = \hat{\mathbf{z}} \cdot \mathbf{u} \), we have \( \theta_{v2} = 0 \) and \( \theta_{v3} = \theta_{k} \), the angle between \( \mathbf{k} \) and the vertical. The six convolutions are
\[ J_{kkkk} = \frac{1}{8\pi} \int \limits_0^\infty \, dp \int \limits_{-1}^1 \, d\mu \frac{E(q) E(p)}{q^2} g(p, q, \omega) \frac{p^2}{q^2} (1 - \mu^2)^2 \quad (107) \]
\[ J_{kkzz} = \frac{1}{8\pi} \int_0^\infty dp \int_{-1}^1 d\mu \frac{E(q) E(p)}{q^2} g(p, q, \omega) \frac{p^2}{q^2} (1 - \mu^2) \times \left\{ 1 - \cos^2 \theta_k \mu^2 - \frac{1}{2} \sin^2 \theta_k (1 - \mu^2) \right\} \]

\[ J_{kkzz} = \frac{1}{8\pi} \int_0^\infty dp \int_{-1}^1 d\mu \frac{E(q) E(p)}{z^2} g(p, q, \omega) \left[ \cos^2 \theta_k \mu^2 \left( 1 - \frac{p^2}{q^2} (1 - \mu^2) \right) + \left( \frac{p^2}{q^2} - \frac{pk}{q^2} \right) \frac{1}{2} \sin^2 \theta_k (1 - \mu^2) \right] \]

\[ J_{kkzz} = -\frac{1}{8\pi} \int_0^\infty dp \int_{-1}^1 d\mu \frac{E(q) E(p)}{q^2} g(p, q, \omega) \cos \theta_k \mu^2 \left( 1 - \mu^2 \right) \]

\[ J_{kkzz} = -\frac{1}{8\pi} \int_0^\infty dp \int_{-1}^1 d\mu \frac{E(q) E(p)}{q^2} g(p, q, \omega) \sin \theta_k \mu^2 \left( 1 - \mu^2 \right) \]

\[ J_{zzzz} = \frac{1}{8\pi} \int_0^\infty dp \int_{-1}^1 d\mu \frac{E(q) E(p)}{q^2} g(p, q, \omega) \]

\[ \times \left[ 1 - \cos^2 \theta_k \mu^2 - \frac{1}{2} \sin^2 \theta_k (1 - \mu^2) \right. \]

\[ + \frac{k^2}{q^2} \left\{ - \cos^2 \theta_k + \cos^4 \theta_k \mu^2 + \frac{1}{2} \cos^2 \theta_k \sin^2 \theta_k (1 - \mu^2) \right\} \]

\[ + \frac{p^2}{q^2} \left\{ - \cos^2 \theta_k \mu^2 + \cos^4 \theta_k \mu^4 + 3 \cos^2 \theta_k \sin^2 \theta_k \mu^2 (1 - \mu^2) \right. \]

\[ - \frac{1}{2} \sin^2 \theta_k (1 - \mu^2) + \frac{3}{8} \sin^4 \theta_k (1 - \mu^2)^2 \]

\[ + \frac{2pk}{q^2} \left\{ \cos^2 \theta_k - \cos^4 \theta_k \mu^2 - \frac{1}{2} \cos^2 \theta_k \sin^2 \theta_k (1 - \mu^2) \right\} \right\}. \]

7. Expression for the emitted energy flux

The expression for the energy flux is finally obtained by substituting the convolutions (107)–(112) in the factor \( \zeta(x, r, \tau) \) (90) which appears in the expression for the energy.
flux (88), and using the fact that the spectrum is symmetric in \( \omega \). The energy flux is

\[
F(x, \omega) = \frac{\dot{\varphi}}{4\pi |x|^2 c_0 \omega} \left[ \frac{\omega^4}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_2^2 \cos^2 \theta)} \right] \times \int d^3 x_0 \rho_0(x_0) \int dp \int d\mu \frac{E(q) E(p)}{q^2} \times g(p, q, \omega) f(\omega, \theta, k, p, \mu).
\]

The factor \( g(p, q, \omega) \) is given by (95)–(97) depending on the form of the frequency spectrum and the angular factor is

\[
f(\omega, \theta, k, p, \mu) = \varphi^4 \left[ \frac{\omega^4 - 2\omega^2 \omega_2^2 \cos^2 \theta + \omega_2^4 \cos^2 \theta}{\omega^2 (\omega^2 - \omega_2^2 \cos^2 \theta)} \right]^{\frac{p^2}{q^2}} (1 - \mu^2)^2 \times \cos^2 \theta \frac{p^2}{q^2} \mu^2 (1 - \mu^2)
\]

\[
+ \frac{\varphi^4 \omega_2^4}{\omega^4} \left[ \frac{\omega^4 - 2\omega^2 \omega_2^2 \cos^2 \theta + \omega_2^4 \cos^2 \theta}{(\omega^2 - \omega_2^2 \cos^2 \theta)^2} \right] \cos^2 \theta
\]

\[
\times \left[ \cos^2 \theta_k \mu^2 + \frac{p^2}{q^2} (1 - \mu^2) \left\{ 1 - \frac{1}{2} \sin^2 \theta_k \right\} \right]
\]

\[
+ (1 - 3 \cos^2 \theta_k) \mu^2 \right] - \frac{1}{2} \frac{p k \mu}{q^2} \sin^2 \theta_k (1 - \mu^2)
\]

\[
+ \frac{\varphi^2 \omega_1^2}{2 \gamma} \left[ \frac{2}{\omega_2} \right] \left[ \frac{\omega^4 - 2\omega^2 \omega_2^2 \cos^2 \theta + \omega_2^4 \cos^2 \theta}{(\omega^2 - \omega_2^2 \cos^2 \theta)^2} \right] \times \left[ \cos^2 \theta_k \mu^2 + \frac{p^2}{q^2} (1 - \mu^2) \left\{ 1 - \frac{1}{2} \sin^2 \theta_k \right\} \right]
\]

\[
+ (1 - 3 \cos^2 \theta_k) \mu^2 \right] - \frac{1}{2} \frac{p k \mu}{q^2} \sin^2 \theta_k (1 - \mu^2)
\]

\[
- 2\varphi^2 \omega_1^2 \omega_2^2 \left[ \frac{2}{\gamma} - \frac{\omega_2^2}{\omega_2} \right] \frac{\omega^2 - \omega_2^2}{\omega^2 - \omega_2^2 \cos^2 \theta} \cos^2 \theta
\]

\[
\times \left[ \sin^2 \theta_k \mu^2 - \frac{1}{2} \frac{p^2}{q^2} \mu^2 (1 - \mu^2) (3 - 5 \cos^2 \theta_k) \right]
\]
\[
\begin{align*}
+ \frac{p k \mu}{q^2} \sin^2 \theta_k (1 - \mu^2) \\
+ \omega^2 \omega_1^2 \omega_4 \frac{\omega^2 - \omega_2^2}{\omega^2 - \omega_2^2 \cos^2 \theta} \cos^2 \theta \\
\times \left( \frac{1}{2} \sin^2 \theta_k \{1 + \mu^2 + \cos^2 \theta_k (1 - 3 \mu^2)\} \\
+ \frac{1}{2} \frac{p^2}{q^2} (1 - \mu^2) \{\cos^2 \theta_k (1 + \mu^2 + \cos^2 \theta_k (1 - 3 \mu^2)) \\
- \sin^2 \theta_k (1 - 5 \cos^2 \theta_k \mu^2 - \frac{1}{3} \sin^2 \theta_k (1 - \mu^2))\} \\
- \frac{2p k \mu}{q^2} \cos^2 \theta_k \sin^2 \theta_k (1 - \mu^2) \right) \\
- 2 \left( \frac{2 - \gamma}{\gamma} \right) \omega^2 \omega_1^2 \omega_4 \left[ \frac{\omega^4 - 2 \omega^2 \omega_2 \cos^2 \theta + \omega_4 \cos^2 \theta}{\omega^2 (\omega^2 - \omega_2^2 \cos^2 \theta)} \right] \\
\times \left( \cos^2 \theta_k \mu^2 + \frac{1}{2} \frac{p^2}{q^2} \mu^2 (1 - \mu^2) (1 - 3 \cos^2 \theta_k) \\
- \frac{pk \mu}{q^2} \frac{1}{2} \sin^2 \theta_k (1 - \mu^2) \right) \\
+ \frac{2}{\gamma} \left( \frac{2 - \gamma}{\gamma} \right) \omega^2 \omega_1^2 \omega_4 \frac{\omega^2 - \omega_2^2}{\omega^2 - \omega_2^2 \cos^2 \theta} \cos^2 \theta \\
\times \left( \sin^2 \theta_k \mu^2 - \frac{1}{2} \frac{p^2}{q^2} \mu^2 (1 - \mu^2) (1 - 5 \cos^2 \theta_k) \\
+ \frac{p k \mu}{q^2} (1 - \mu^2) \sin^2 \theta_k \right) \\
+ \left( \frac{2 - \gamma}{\gamma} \right)^2 \omega^4 \left[ \frac{1}{2} \sin^2 \theta_k \{1 + \mu^2 + \cos^2 \theta_k (1 - 3 \mu^2)\} \\
+ \frac{1}{2} \frac{p^2}{q^2} (1 - \mu^2) \{\cos^2 \theta_k (1 + \mu^2 + \cos^2 \theta_k (1 - 3 \mu^2)) \\
- \sin^2 \theta_k (1 - 5 \cos^2 \theta_k \mu^2 - \frac{1}{3} \sin^2 \theta_k (1 - \mu^2))\} \\
- \frac{2p k \mu}{q^2} \cos^2 \theta_k \sin^2 \theta_k (1 - \mu^2) \right]. \tag{114}
\end{align*}
\]

Acknowledgements

This work is based in part on a thesis submitted in partial fulfillment of the requirements for a Ph.D. in Physics, Columbia University, 1966. I would like to thank Drs. E. A. Spiegel and A. G. W. Cameron for guidance during the course of this work.
I am thankful to Columbia University for financial support under a NASA grant and to Dr. R. Jastrow for the hospitality of the Institute for Space Studies and the use of their computing facilities. Part of this work was also supported under NASA grant NGR-05-002-034.

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