WAVES IN THE SOLAR ATMOSPHERE. III. THE PROPAGATION OF PERIODIC WAVE TRAINS IN A GRAVITATIONAL ATMOSPHERE

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ABSTRACT

The validity of weak shock theory for the propagation of waves in a gravitational atmosphere is examined by comparing its results with those from numerical integration of the exact equations of motion. The weak-shock approximation is not valid for periods longer than half the acoustic cutoff. In addition, the relation between the 300-s solar oscillation and chromospheric and coronal heating is described.

Subject headings: atmospheres, solar — shock waves — solar atmospheric motions

Many authors have treated the propagation of periodic waves in a gravitational atmosphere by weak-shock theory (Schatzman 1949; Osterbrock 1961; de Jager and Kuperus 1961; Uchida 1963; Köpp 1968; Jordan 1968; Ulmschneider 1971). This approximation involves the assumption that the propagation of the waves is unaffected by gravity and the assumption of the existence of an infinitely sharp shock front which moves with a Mach number \( M \) slightly greater than 1. The dissipation across the shock is then proportional to \( (M - 1)^3 \), and higher-order terms are dropped. In this paper, we compare the results of the weak-shock theory with numerical integrations of the exact equations of motion in a one-dimensional, isothermal atmosphere. Weak-shock theory and the numerical solutions agree for small Mach numbers and periods shorter than the acoustic cutoff period, but are qualitatively different for waves with periods longer than the acoustic cutoff.

The results of weak-shock theory have been conveniently summarized in our previous paper (Stein and Schwartz 1972). It was shown there that the Mach number of a weak sawtooth wave propagating through an isothermal atmosphere is

\[
M - 1 = \frac{(M_0 - 1)e^{\gamma/2H}}{1 + 4H\lambda^{-1}(M_0 - 1)(e^{\gamma/2H} - 1)},
\]

where \( H \) is the scale height of the atmosphere. The energy dissipated is given by

\[
T\Delta S = \frac{16}{3(y + 1)^3} c_s^2(M - 1)^3.
\]

In order to compare this approximation with the exact nonlinear calculations which were done with the numerical hydrodynamic code described by Stein and Schwartz (1972), the weak-shock approximation was first checked in the regime in

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which it was expected to be very good—low Mach numbers and high frequencies. Our atmosphere had a temperature of 5700° K, and sound speed $c_s$ of $7 \text{ km s}^{-1}$, giving an acoustic cutoff period $\tau_{ac} = 4\pi H/c_s = 200 \text{ s}$. The atmosphere was excited by sawtooth waves of amplitudes 0.03 and 0.3 km s$^{-1}$ and a 50-s period. For simplicity, we constrained the waves to be isothermal (i.e., $\gamma = 1$) in all our numerical calculations. The results of this calculation are shown in figure 1. The Mach number computed from weak-shock theory agrees with the exact value for low Mach numbers. When $M - 1$ becomes of the order of 0.1, a discrepancy may be seen in the figure. Since this is due to the neglect of terms of the order of $(M - 1)^2$, the relative error is expected to be of the order of $(M - 1)$. This behavior may be verified from inspection of figure 1.

To investigate the effects of wave period, we next considered the atmosphere, initially at rest, to be excited by a lower boundary which moved with velocity $u = 0.32 \sin(t/\tau) \text{ km s}^{-1}$, as opposed to the sawtooth driving used above. The qualitative behavior of the solutions is different for $\tau > \tau_{ao}$ and $\tau < \tau_{ac}$. We chose periods of 100 and 400 s as representative of these two regimes. In some cases the effects of the transients, caused by the initially static conditions, are very large. The comparison with the weak-shock results was therefore made after the calculation had run for 20 to 40 periods and the effects of transients had become small.

Figure 2 shows the steady-state velocity profiles attained by the waves. Figure 2a, for a period of 100 s, shows the development of a sharp shock front, while figure 2b,
Fig. 2.—Steady-state velocity profile for sinusoidal driving with amplitude 0.32 km s$^{-1}$ and periods of (a) 100 s and (b) 400 s. Figures 2 and 3 refer to a mass element whose initial height was 1000 km.

for waves of 400-s period, has no indication of the presence of shocks. Not only is the velocity amplitude about one-quarter as great as for the short period (the initial amplitude was the same for both), but the velocity profile appears nearly sinusoidal. The small interwaves are beats between the imposed 400-s oscillation and the 200-s natural period of atmosphere and show that a complete steady state has not yet been achieved. From these figures, we expect the weak-shock theory to give a fairly good representation of the propagation of the 100-s wave.

Figure 3 shows the velocity and pressure profile at a given height (1000 km). We see that, for the 400-s wave (fig. 3b) the pressure and velocity are nearly 90° out of phase, while they are in phase (fig. 3a) for the 100-s wave. The long-period wave is nearly a standing wave with velocity nearly in phase at all heights and transfers little energy, since $E = \frac{1}{2}(P - P_0)udt$. This is the key to the failure of weak-shock theory for the long-period waves: although the amplitude of the wave does increase with height, the fact that little energy is being transferred retards shock formation.

We must, however, add a note of caution about the use of weak-shock theory even for the waves with periods much less than $\tau_\infty$. In figure 4, the dissipation (i.e., $\int TdS$) per period is plotted for the 100-s wave. The solid curve is the result of our nonlinear calculations after a steady state had been reached (approximately 40 periods after the start of the calculation). The dashed curve is the result of weak-shock theory. In
Fig. 3.—Pressure and velocity in the steady state for (a) 100-s and (b) 400-s periods
qualitative agreement with the analytical solution (2), the dissipation becomes nearly constant above about 1500 km but the value found by the numerical calculation is nearly 10 times greater than that predicted by the weak-shock theory, because the shock is no longer weak \( (M \approx 2) \). Note that the driving amplitude is 0.32 km s\(^{-1}\), which corresponds to the upper curve in figure 1. Even though the wave is weak at the piston, it becomes rapidly stronger with increasing height. Below 500 km, however, the weak-shock predications are higher than the true dissipation because this theory assumes the existence of a shock where none has yet formed. The caution which must be exercised in the use of weak-shock theory for models of chromospheric heating is self-evident.

The breakdown of weak-shock theory for long-period waves is dramatically illustrated by figure 5. In the steady state, the dissipation is very small (about \( 10^4 \) times less than the weak-shock value) at all heights. The solid line indicates the total dissipation after 19 periods of the motion, nearly all of which was due to the first
Fig. 5.—Dissipation for waves of 400-s period. The dissipation per period after 19 waves, as well as the dissipation by the initial pulse in the static atmosphere, and the total dissipation for 19 waves are shown.

transient at the beginning of the calculation (crosses). At heights above 1300 km this initial transient has formed a shock, and the dissipation is nearly equal to the weak-shock value. At heights greater than 1800 km a steady state has not been attained even after 19 periods, and shocks form and dissipate energy.

The present results, combined with those of Leibacher (1971) and Leibacher and Stein (1973), have led Leibacher and ourselves to the following description of the relation between the 300-s oscillation and chromospheric heating. The 300-s oscillations are produced by thermal instability in the solar convection zone. There are two instabilities: the Eddington valve (as in pulsating stars), and the buoyant over-stability described by Spiegel (1964) and Moore and Spiegel (1964) for nonradially propagating acoustic waves. These instabilities build up the energy of waves in the convection zone. Waves are trapped below by the rising temperature which reflects
nonradially propagating waves where their horizontal phase velocity equals the sound speed. Waves are trapped above by the sharp temperature drop in the thin, very superadiabatic region at the top of the convection zone, which reduces the acoustic cutoff period and reflects waves with longer periods. Because these instabilities are weak, only well-trapped waves will build up an appreciable amplitude. The observed 300-s oscillation is the result of a leakage from the top of this resonant cavity. Because the 300-s oscillation has a period longer than the natural period of the photosphere, it is nonpropagating there and does not form shocks until the high chromosphere (∼1800 km). These long-period waves dominate the heating of the upper chromosphere and corona. Below the height where shocks form, dissipation occurs by transfer of radiation from the compressed to the rarefied portion of the wave. In the low chromosphere, additional dissipation may be produced by high-frequency noise produced by turbulence in the convection zone (Lighthill mechanism). The spectrum of this noise has a maximum in the range 30–60 s. Because these are propagating waves in the photosphere, they steepen, form shocks, and begin dissipating energy after traveling only a few scale heights.

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