WAVES IN THE SOLAR ATMOSPHERE. II. LARGE-AMPLITUDE ACOUSTIC PULSE PROPAGATION

ROBERT F. STEIN
Brandeis University, Waltham, Massachusetts

AND

ROBERT A. SCHWARTZ
New York University
Received 1971 December 20; revised 1972 May 24

ABSTRACT

Numerical experiments are performed with vertically propagating acoustic pulses by solving the nonlinear equations of fluid motion using a finite-difference technique. The pulse energy, dissipation, wake, and atmospheric heating are investigated, and the results compared with weak-shock theory. The ratio of pulse frequency to the acoustic cutoff frequency, \( N_{ae} = \gamma E/2c \), is found to be a crucial parameter. Weak-shock theory gives reasonable results for pulse widths less than 50 seconds (\( \omega > 2N_{ae} \)), but greatly overestimates the pulse energy and dissipation for longer pulses. Significant dissipation begins at the height where the crest of a simple wave overtakes its trough. For pulses with \( \omega > 2N_{ae} \) the minimum damping length is about 500 km and occurs at about 1000 km above \( \tau_{6000} = 1 \). For lower-frequency pulses the minimum damping length is about 1000 km and occurs higher up. Until hydrogen is nearly completely ionized, ionization and radiation keep the temperature rise small.

I. INTRODUCTION

This is the second in a series of papers examining in detail the generation and propagation of waves in the solar atmosphere. The first paper (Stein 1968) discussed the generation of acoustic waves by turbulence in the convection zone. In this paper we investigate the vertical propagation of acoustic pulses in the solar atmosphere. Experience with individual pulse propagation should provide insight into the problems of chromospheric and coronal heating and the 5-minute solar oscillation. Our calculations are numerical experiments and provide information on the dependence of the height variation of dissipation upon pulse amplitude and width, on effects of hydrogen ionization and radiation upon the heating produced by the shock dissipation, and on the wake produced by a pulse.

It is generally believed that the energy source for heating the chromosphere and corona is a mechanical energy flux of waves produced in the convection zone (Biermann 1946, 1948; Schwarzschild 1948). These waves are thought to be sound waves (Schatzman 1949; de Jager and Kuperus 1961; Kopp 1968; Kuperus 1969; Jordan 1970; Ulmschneider 1971), or fast-mode magnetohydrodynamic (MHD) waves (Osterbrock 1961; Uchida 1963; Milkey 1970), which as they propagate upward form shocks, dissipate their energy, and heat the gas. The shock propagation and dissipation was calculated by using weak-shock theory.

In our computer experiments the propagation of pulses in a stratified atmosphere is investigated by numerically integrating the full nonlinear equations of fluid motion in one-dimensional, Lagrangian, finite-difference form (Richtmyer and Morton 1967; Christy 1967). To facilitate the numerical calculations we considered idealized conditions that are very different from those actually found on the Sun, but think we have
retained the significant physical effects that influence the propagation of acoustic
wave in the absence of a magnetic field. If the idealized nature of the problem investigated
is borne in mind, our main results are the following.

1. The pulse energy depends on its initial width and amplitude as in weak-shock
theory for propagating waves with widths \( \tau < \tau_{ac} \) half the acoustic cutoff period

\[
\tau_{ac} = 2\pi/N_{ac} = 4\pi H/c = 4nc/yg
\]

(\( H \) is the scale height, \( c \) is the sound speed,
\( N_{ac} \) is the acoustic cutoff angular frequency, \( g \) is the gravitational acceleration, and
\( \gamma = c_p/c_v \) is the ratio of specific heats), and varies approximately as \( \tau^{-3} \) for non-
propagating waves with widths greater than half the acoustic cutoff period:

\[
E(\tau, u_0) \approx \frac{1}{2} \rho_0 u_0^2 c_0 \left[ \frac{1}{\tau} + \left( \frac{2\tau}{\tau_{ac}} \right)^{\frac{3}{2}} \right]^{-1},
\]

where \( E \) is the pulse energy per unit area, \( u_0 \) is the initial velocity amplitude, and \( \rho_0 \)
the mean density (figs. 3 and 4).

2. The distance for dissipation to become important is roughly the distance for a
wave crest to overtake its trough:

\[
z = 2H \ln \left[ 1 + \frac{\gamma}{2(\gamma + 1)} \frac{g\tau}{u_0} \right].
\]

In a uniform atmosphere this distance is \( \pi/2 \) the distance for a shock to initially form.

3. The nonlinear calculations and weak-shock theory give the same increase in
Mach number \((\mathcal{M})\) with height for small pulse widths \( \tau < \tau_{ac}/2 \), but weak-shock
theory gives too great a Mach-number increase for large-width pulses \( \tau \geq \tau_{ac}/2 \)
(fig. 2). For the same Mach number, the nonlinear calculations give a greater dissipation
than weak-shock theory—by 10 percent for \( \mathcal{M} = 1.13 \) and by \( \sim 50 \) percent for
\( \mathcal{M} = 2 \). For wide pulses, weak-shock theory's overestimate of the dissipate, to give dissipations much greater than
found in the nonlinear calculations (fig. 5).

4. The minimum damping length for propagating waves \( \tau < \tau_{ac}/2 \) is

\[
\xi \approx 500 \text{ km} \approx 4H,
\]

nearly independently of pulse amplitude and width. For nonpropagating waves
\( \tau > \tau_{ac}/2 \) the minimum damping distance is about twice the above value.

5. In the region where hydrogen is partially ionized, hydrogen ionization produced
by the pulse soaks up the energy dissipated and keeps the temperature rise small
\( \sim 500^\circ \text{K} \).

6. Radiation from the \( \text{H}^- \) ion and hydrogen recombination wipes out the small
temperature rise left by partially ionized hydrogen.

7. The pulse leaves behind it a standing wave wake at the acoustic cutoff frequency,

\[
\nu_{ac} = \frac{1}{\tau_{ac}} = \frac{N_{ac}}{2 \pi} = \frac{1}{2 \pi} \frac{y \rho}{2c}.
\]

These wakes become more important with respect to the initial pulse as the height
increases.

II. METHOD

The properties of waves in a stratified atmosphere most interesting to us here are
the steepening and speeding up of waves and their transformation into shocks as
they propagate through many scale heights of density, and the heating of the medium
by these shocks once formed. We therefore use the full nonlinear equations to calculate
the fluid motion. For radially propagating waves, the equations of motion in the
Lagrangian representation are
\[
\frac{\partial u}{\partial t} = -\frac{GM_0}{R^2} - 4\pi R^2 \frac{\partial P}{\partial \mathcal{M}},
\]
(1)
\[
\frac{\partial R}{\partial t} = u,
\]
(2)
\[
\rho(M, t) = \frac{1}{4\pi R^2} \frac{\partial R}{\partial \mathcal{M}},
\]
(3)
\[
\frac{\partial \epsilon}{\partial t} = -P \frac{\partial}{\partial t} \left(\frac{1}{\rho}\right) + Q.
\]
(4)

Here $R = \text{radius}$, $u = \text{fluid velocity}$, $\rho = \text{density}$, $P = \text{pressure}$, $\mathcal{M} = \text{mass interior}$
to given point in the atmosphere, $\epsilon = \text{internal energy (thermal plus ionization) per gram}$, and $Q = \text{net radiative absorption per gram per second}$. These equations were
solved numerically by a finite-difference technique similar to that which has proved
useful in many other astrophysical problems (Richtmyer and Morton 1967; Christy
1967; Colgate and White 1966; Schwartz 1967; Leibacher 1971; Hill 1971; Smith
1971.) Because of the complicated behavior of ionization and radiation, the energy
equation must be solved for $\epsilon(T)$, $P(T)$, and $Q(T)$ ($T = \text{temperature}$) simultaneously,
using a Newton-Raphson iteration procedure.

It is prohibitively time consuming to correctly evaluate the effects of radiation and
changes in the ionization equilibrium in these equations; this would involve constructing
a non-LTE atmosphere model every time step! We have, therefore, attempted to
include these effects in what seems to be a physically reasonable and numerically
tractable approximation.

Most of our calculations included variable hydrogen ionization, but no radiation.
To determine the degree of ionization of hydrogen, which is the main contributor to
the ionization energy, we used Saha's equation. Although strictly this is incorrect
because the atmosphere is not in LTE, it should provide an adequate estimate of
ionization effects on the wave propagation, but with hydrogen ionization occurring a
few hundred kilometers too low down. (We are not interested in the spectroscopic
state of the gas.)

When radiation was also included, the net radiative energy input (or loss) was
evaluated by using the optically thin approximation. We included emission from the
$\text{H}^-$ ion,
\[
\epsilon_{\text{H}^-} = 4\sigma_{\text{H}^-} T^4 \rho \text{ ergs g}^{-1} \text{ s}^{-1},
\]
\[
\sigma_{\text{H}^-} = 5.3 \times 10^{-11} n_{\text{H}^0} P e T^{-4} \text{ cm}^{-1}
\]
(Whitney 1963); recombination to all levels of hydrogen above the ground state,
\[
\epsilon_{\text{re}} = 9.1 \times 10^{-23} T^{-1/2} n_e n_{\text{H}^+} / \rho \text{ ergs g}^{-1} \text{ s}^{-1}
\]
(Kaplan and Pikelner 1970); and bremsstrahlung,
\[
\epsilon_{\text{br}} = 1.435 \times 10^{-27} n_e n_{\text{H}^+} T^{1/2} / \rho \text{ ergs g}^{-1} \text{ s}^{-1}
\]
(Kaplan and Pikelner 1970). Since we are interested in modeling the propagation of
waves in the real solar atmosphere, we must insure that our initial atmosphere is in
equilibrium. To achieve this we took
\[
Q = -\epsilon_{\text{TOTAL}} + \epsilon_{\text{TOTAL}}(\text{equilibrium atmosphere}),
\]
where \( \epsilon_{\text{TOTAL}} \) (equilibrium atmosphere) is the total emission rate, for a given zone, in the equilibrium atmosphere, and represents radiative absorption plus mechanical heating in the equilibrium atmosphere. This procedure balances the zeroth-order net radiative loss rate by the average heating rate (both of which are unknown) and eliminates the lowest-order effects of radiation from the calculation.

The set of finite-difference equations for the time-dependent behavior of the atmosphere is the "apparatus" for our numerical experiments on the response of the atmosphere to perturbations. The initial model atmosphere was taken from the Bilderberg model (Gingerich and de Jager 1968). The perturbations were acoustic pulses, produced by making the bottom of the model atmosphere, at \( r_{5000} = 1 \), move like a piston which started from rest, moved upward, and stopped, with a half sine wave in velocity (half cosine bell in position). After the piston stopped, the bottom of the model atmosphere was treated as a transmitting boundary — allowing any downward-propagating waves to pass through without reflection (Leibacher 1971). The upper boundary of the model atmosphere was treated as either a free (constant pressure) or a transmitting boundary. The results were insensitive to the boundary conditions used.

The conditions considered in our numerical experiments are idealized and very different from those actually found on the Sun. We considered only radially propagating waves; we ignored inhomogeneities; we assumed the magnetic field was zero; we considered only pulses propagating into a static atmosphere (rather than a dynamic one with waves already moving through it); we ignored departures from LTE. Yet for all the approximations, made to facilitate the numerical calculations, we think we have retained the significant physical effects. One should, however, bear in mind the idealized nature of our model when considering our results.

### III. Weak-Shock Theory

Many previous authors have calculated the heating of the chromosphere by using the theory of weak shock waves (Schatzman 1949; Osterbrock 1961; de Jager and Kuperus 1961; Uchida 1963; Kopp 1968; Jordan 1970; Ulmschneider 1971). In this approximation all quantities are calculated to lowest order in \( \alpha = \mathcal{M} - 1 \), where \( \mathcal{M} = (\text{shock front velocity})/(\text{sound speed}) \) is the Mach number. It is useful to summarize the results of weak-shock theory in order to compare them with our nonlinear calculations and to clarify some of the confusion in the literature. Although many of these results are to be found in the extensive literature on this problem, we believe the analytic solution for the propagation of weak shocks in an isothermal atmosphere to be new. The basis of weak-shock theory is the fact that the entropy increase across a weak shock is third order in the shock strength, so to second order all quantities are the same as for an adiabatic simple wave (Courant and Friedrichs 1948; Landau and Lifshitz 1959).

As a small-amplitude acoustic wave propagates, it steepens. The compressions overtake the rarefactions, changing a sinusoidal profile into a sawtooth profile, because the phase velocity

\[
u + c(u) = c_0 + \frac{1}{2}(\gamma + 1)u\]

is greatest where the forward velocity is greatest. A shock develops when some portion of the velocity profile becomes vertical. For an initially sinusoidal pulse the shock develops at the front of the compression. In a uniform density medium the shock develops in a distance

\[
z = \frac{2\lambda}{\pi(\gamma + 1)} \frac{c_0}{u}
\]

(5)
where \( \lambda \) is the pulse width and \( u \) its velocity amplitude. The crest overtakes the trough in a distance given by

\[
\int \frac{\gamma + 1}{2} v dt = \int \frac{\gamma + 1}{2} \frac{v}{c_0} dz = \frac{\lambda}{2},
\]

which is \( \pi/2 \) times greater. In a stratified atmosphere, where the density changes, the velocity amplitude also changes. A small-amplitude acoustic wave prior to shock formation does not dissipate any energy, so it has constant flux:

\[
F = (P - P_0)u = \rho_0 u^2 c_0 = \text{const}.
\]

Thus, for constant sound speed \( c_0 \), the velocity amplitude increases as the wave propagates into decreasing density regions:

\[
u \propto \rho_0^{-1/2}.
\]

If we evaluate the condition for the crest to overtake the trough in an exponential atmosphere, \( \rho = \rho_0 e^{-z/H} \), where \( u = u_0 e^{z/2H} \), we get a distance

\[
z = 2H \ln \left( \frac{\lambda c_0}{2H u_0 \gamma + 1} \right) + 1.
\]

(6)

The energy carried by a wave per unit area is

\[
E = \int_0^z (P - P_0)udt,
\]

where \( P_0 \) is the ambient pressure. In a weak shock the pressure and velocity are related as for a simple wave

\[
P = P_0 \left(1 + \frac{\gamma - 1}{2} \frac{u}{c_0}\right)^{2/(\gamma - 1)},
\]

so, since \( \Delta u/c_0 = 4\alpha/(\gamma + 1) \) is small (where \( \Delta u \) is the velocity change across the shock),

\[
P - P_0 = \gamma P_0 u/c_0
\]

to lowest order. Thus the energy per unit area of a weak wave is

\[
E = \frac{\gamma P_0}{c_0} \int u^2 dt = \rho_0 \int u^2 dz.
\]

(7)

In this paper we consider the propagation of pulses. Once a shock has formed from a pulse its velocity profile is nearly a single sawtooth (fig. 1):

\[
u = u_0 + \Delta u(1 - t/\tau) \quad (0 < t < \tau),
\]

where \( u_0 \) is the velocity immediately ahead and \( \Delta u \) the velocity change across the shock. The energy per unit area for such a weak sawtooth pulse is

\[
E = \frac{1}{3} \rho_0 (\Delta u)^2 \lambda = \frac{16\gamma}{3(\gamma + 1)^2} P_0 \alpha^2.
\]

(8)

This expression differs from most previous calculations of chromospheric heating which considered trains of waves, which develop into trains of sawtooth (N-wave) shocks with velocity profile

\[
u = u_0 + \frac{1}{2} \Delta u(1 - 2t/\tau) \quad (0 < t < \tau).
\]
The energy per unit area of an $N$-wave is one-quarter that for a pulse of equal velocity amplitude,
\[ E = \frac{1}{12} \rho_0 (\Delta u)^2 \lambda, \tag{9} \]
because the downward motion in the tail cancels most of the upward motion in the head of the wave.

The energy dissipated by a weak shock wave is third order in the shock strength, and is given by
\[ T_0(s_1 - s_0) = \frac{16}{3(y + 1)^2} c_0^2 \alpha^3, \tag{10} \]
where $s_1$ and $s_0$ are the entropy behind and ahead of the shock. The rate of mass flow through the shock is $\rho_0(U_s - u_0) \text{ g cm}^{-2} \text{ s}^{-1}$, so for zero mean flow ($\mu_0 = 0$) we get
\[ \frac{dE}{dz} = -\frac{16 \gamma}{3(y + 1)^2} \rho_0 c_0^2 \alpha^3. \tag{11} \]

The same expression for $dE/dz$ is obtained for both a pulse and an $N$-wave by calculating the decrease in wave energy due to the overtaking rarefaction wave. For comparison, the dissipation due to the artificial viscosity in the numerical program is
\[ \int Tds = \frac{\gamma + 1}{4} U_s^2 \left( 1 - \frac{\rho_0^2}{\rho_1^2} + 2 \frac{\rho_0}{\rho_1} \ln \frac{\rho_0}{\rho_1} \right), \tag{12} \]
where $U_s$ is the shock front velocity; $\rho_1$ is the density behind, and $\rho_0$ that ahead of, the shock. For weak shocks this reduces to the above result, and for strong shocks it becomes
\[ \int Tds \xrightarrow{\mu \to \infty} U_s^2 \left( \frac{\gamma}{\gamma + 1} - \frac{\gamma - 1}{2} \ln \frac{\gamma + 1}{\gamma - 1} \right). \]

The width $\lambda$ of a pulse (but not an $N$-wave) also changes as the wave propagates, since the leading edge of the pulse moves with speed $\mathcal{M}c_0$ and the trailing edge with speed $c_0$. Thus
\[ \frac{d\lambda}{dt} = c_0(M - 1), \quad \text{or} \quad \frac{d\lambda}{dz} = \alpha. \tag{13} \]

We can now derive an equation for the variation of Mach number with height, in an isothermal atmosphere from the above expressions for $E$, $dE/dz$, and $d\lambda/dz$. If we take the logarithmic derivative of the wave energy and solve for $dE/dz$, we get
\[ \frac{1}{\alpha} \frac{d\alpha}{dz} = \frac{1}{2} \left( \frac{1}{H} - \frac{1}{\lambda} \frac{d\lambda}{dz} + \frac{1}{E} \frac{dE}{dz} \right), \tag{14} \]
where
\[ H = \left| \frac{1}{P} \frac{dP}{dz} \right|^{-1} \]
is the scale height. For a pulse, from equations (8) and (11),
\[ \frac{1}{E} \frac{dE}{dz} = -\frac{\alpha}{\lambda}. \tag{15} \]
Thus the equations for weak pulse propagation are
\[ \frac{1}{\alpha} \frac{d\alpha}{dz} = \frac{1}{2H} - \frac{\alpha}{\lambda} \]  
(16)

and
\[ \frac{d\lambda}{dz} = \alpha . \]  
(13)

The $1/2H$ term gives the amplitude growth at constant energy due to decreasing density, and the $-\alpha/\lambda$ term gives the weakening due to dissipation and increase in pulse width. For an $N$-wave, from equations (9) and (11), we obtain
\[ \frac{1}{E} \frac{dE}{dz} = -\frac{4\alpha}{\lambda}, \]  
(17)

so the equation for weak $N$-wave propagation is
\[ \frac{1}{\alpha} \frac{d\alpha}{dz} = \frac{1}{2H} - \frac{2\alpha}{\lambda}, \]  
(18)

where $\lambda = \lambda_0 = \text{constant}$.

These propagation equations can be easily solved. In the case of a pulse, equation (16) divided by equation (13) has the solution
\[ \alpha = \frac{\lambda}{4H} - \frac{\kappa}{\lambda}, \]

Then
\[ \frac{d\lambda}{dz} = \alpha = \frac{\lambda}{4H} - \frac{\kappa}{\lambda}, \]

which has the solution
\[ \lambda^2 = a^2 e^{2\lambda/2H} + \kappa^2. \]

The constants are determined by the boundary condition $\alpha \to \alpha_0$ and $\lambda \to \lambda_0$ when $z \to 0$. Thus for a pulse,
\[ \alpha = \alpha_0 e^{2\lambda z/2H} \left[ 4 \frac{H}{\lambda_0} \alpha_0 (e^{2\lambda z/2H} - s) + 1 \right]^{1/2} = \frac{\alpha_0 \lambda_0}{\lambda} e^{2\lambda z/2H}, \]  
(19)

\[ \lambda = \lambda_0 \left[ 4 \frac{H}{\lambda_0} \alpha_0 (e^{2\lambda z/2H} - 1) + 1 \right]^{1/2}. \]  
(20)

For $z \gg H$,
\[ \alpha \sim e^{2\lambda z/4H} \sim \rho_0^{-1/4}, \]  
(21)

in contrast to the behavior of small-amplitude waves where $\alpha \sim \rho_0^{-1/2}$. In the case of $N$-waves, equation (18) can be rewritten as
\[ \frac{d\alpha^{-1}}{dz} + \frac{\alpha^{-1}}{2H} = \frac{2}{\lambda_0}, \]

which has the solution
\[ \alpha^{-1} = \frac{4H}{\lambda_0} + \frac{1 - (4H/\lambda_0) \alpha_0}{\alpha_0 e^{2\lambda z/2H}}. \]  
(22)

For $z \gg H$,
\[ \alpha \to \lambda_0/4H = \text{const} . \]  
(23)
for an $N$-wave, contrary to the exponential increase found above for pulses. The strength of an $N$-wave increases less than for a pulse because for a given velocity amplitude the damping is the same as for a pulse, but the energy is only one-quarter as great. From this asymptotic result, we see that weak-shock theory for $N$-wave trains should not be too bad for $\lambda < 4H$ or $\tau < 4H/c_0 \approx 70$ seconds.

The most serious defect in weak-shock theory is its assumption of infinite frequency and therefore its neglect of the dynamical interaction with the stratification, which reduces the wave energy for periods greater than the acoustic cutoff period.

IV. SHOCK PROPAGATION

The behavior of a pulse depends on whether its width is less or greater than half the acoustic cutoff period

$$\tau_{ac} = \frac{2\pi}{N_{ac}} = \frac{4\pi H \rho}{c} \approx 200 \text{ seconds} \quad (24)$$

(Lamb 1932; Moore and Spiegel 1964). Pulses of shorter duration propagate through the atmosphere as normal acoustic waves and steepen into shocks. Pulses of longer duration produce little compression, so the piston moves the atmosphere upward quasi-rigidly, and only a small-amplitude transient compressional wave propagates upward and forms a shock (fig. 1). Note that the atmosphere is extended above its initial height (of 2200 km) due to the energy and momentum carried by the waves.

The energy carried by a pulse depends on its amplitude and width. For pulse widths much less than the acoustic cutoff, the wave energy (per unit area) is approximately that given by weak-shock theory,

$$E = \frac{1}{2} \rho_0 u_{\text{max}}^2 c_0 \tau, \quad (25)$$

where $\tau$ is the pulse width and $u_{\text{max}}$ the maximum velocity in the pulse. For widths greater than the acoustic cutoff, most of the work done by the piston goes into raising the entire atmosphere without compressing the gas. Only that portion of the pulse power

$$P(\omega) = \frac{4\omega_{\text{pulse}}^2}{(\omega + \omega_p)^2(\omega - \omega_p)^2} \cos^2 \left(\frac{\pi \omega}{2 \omega_p}\right)$$

Fig. 1.—Velocity profiles of pulses. Each curve shows the velocity, scaled by $\rho_0^{-1/6}$, as a function of height. (a) Pulse width = 50 s. Interval between curves is 30 s. (b) Pulse width = 400 s. Interval between curves is 40 s.
Fig. 2—Growth of Mach number with height (in an isothermal atmosphere). Curves are results of weak-shock theory; points are numerical results.
at frequencies \( \omega > N_{ao} = 2\pi/\tau_{ao} \) contributes to the propagating transient; and for \( \omega_{ao} \gg \omega_{\text{pulse}} = 2\pi/\tau_s \),

\[
P(\omega) \approx 4\omega_{\text{pulse}}^2/\omega^4 \cos^2 \left( \frac{\tau}{2\tau_{ao}} \right) \approx \tau^{-2}.
\]

The pulse energy for widths greater than \( \tau_{ao}/2 \approx 100 \) s is approximately

\[
E \approx \frac{1}{2} \rho_0 u_{\text{max}}^2 c \tau_{ao} (2\tau/\tau_{ao})^{-3},
\]

in contrast to the result from weak-shock theory, which neglects gravity. This approximation is, of course, only a fit to the behavior observed in the numerical experiments, and not an analytic result.

The process of shock formation is a steepening of the wave profile until nearly discontinuous increases in velocity, pressure, density, and temperature occur. This process is shown in figure 1, which shows the velocity profile at different times for two pulses of widths 50 and 400 seconds. When the shock develops, the wave profile

![Graph showing velocity profile](image)

**Fig. 3.** RMS velocity as a function of height. (a) Results for pulses of different widths (\( \tau = 50, 100, 200, 400 \) s), but the same amplitude \( u_{\text{max}}(0) = 0.32 \) km s\(^{-1}\). (b) Results for pulses of different amplitude \( u_{\text{max}}(0) = 0.08, 0.32, 1.28 \) km s\(^{-1}\), but the same width (\( \tau = 50 \) s). (c) Effect of ionization and radiation, \( \tau = 50 \) s and \( u_{\text{max}}(0) = 0.32 \) km s\(^{-1}\). (d) Effect of ionization and radiation, \( \tau = 200 \) s and \( u_{\text{max}}(0) = 0.32 \) km s\(^{-1}\).
changes from a nearly sinusoidal velocity and pressure profile to a sawtooth \((N\text{-wave})\) profile in the velocity and a narrow peak in the pressure. Thus, in the high chromosphere, the radiative emission from behind the shock front will occupy only a narrow region behind the front and the emission will occur in a time much shorter than the period \(\tau\) of the wave.

As the shock propagates outward through the chromosphere, it becomes stronger and its velocity increases. The growth of the shock Mach number with the height is shown in figure 2. The solid curve shows the prediction from weak-shock theory, and the points are the results of our nonlinear calculations. For \(\tau \ll \tau_{\text{ad}}/2\) the agreement with weak-shock theory is good. As the pulse width increases, the nonlinear results show a slower growth of Mach number. For all widths

\[
    u_{\text{max}} \sim \rho_0^{-1/4} \quad \text{(approximately)},
\]

as expected from weak-shock theory. Because of this increase in velocity amplitude, the velocity in the pulse propagation (fig. 1) has been scaled by \(\rho_0^{1/5}\) so as to reduce the increase in amplitude to a small amount. The root mean square (rms) velocity of the pulse and its wakes is shown in figure 3. We find that

\[
    u_{\text{rms}} \sim \rho_0^{-0.20} \quad \text{(approximately)}
\]

in the absence of radiation, whose inclusion reduces the rate of growth slightly (figs. 3c and 3d). This rapid growth in the rms velocity is in contradiction to the observations of the 5-minute oscillation, for which Mein (1966) finds

\[
    u_{\text{rms}} \sim \rho_0^{-0.20}
\]

between the photosphere and Ca \(\Pi\) \(K\). The height at which shock dissipation becomes significant is shown in table 1. Defining a height of formation of a shock is somewhat ambiguous in our nonlinear

<table>
<thead>
<tr>
<th>TABLE 1</th>
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<tbody>
<tr>
<td><strong>ONSET OF DISSIPATION</strong></td>
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<tr>
<td><strong>A. PULSE WIDTH = 50 SECONDS</strong></td>
</tr>
<tr>
<td>Amplitude (km s(^{-1}))</td>
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<tr>
<td>0.01</td>
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<tr>
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<td>2.56</td>
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**B. AMPLITUDE = 0.32 KILOMETERS PER SECOND**

| Pulse Width (seconds) | \((z/H)_{\text{NL}}\) | \((z/H)_{\text{WS}}\) |
| 50 | 5.3 | 5.4 |
| 100 | 6.0 | 6.6 |
| 200 | 10.0 | 8.0 |
| 400 | 11.4 | 9.4 |

**NOTE.** — \((z/H)_{\text{NL}}\) = results of nonlinear numerical calculation; \((z/H)_{\text{WS}} = 2 \ln (1 + \eta_k \xi / u_0) \) (eq. [6]).
Fig. 4.—Dissipation per gram as a function of height. (a) Results for pulses of different widths, but the same amplitude [$u_{\text{max}}(0) = 0.32 \text{ km s}^{-1}$]. (b) Results for pulses of different amplitudes, but the same width ($\tau = 50$ s).

numerical calculations. We have chosen the height at which substantial dissipation begins (fig. 4), which coincides with the height at which the medium begins to be heated (fig. 8), in the absence of radiation. We see that the larger-width, lower-amplitude pulses propagate further before a shock forms. The distance a pulse must travel before significant dissipation begins very nearly coincides with the distance for the crest of a simple wave to overtake its trough (table 1).

V. HEATING

The temperature of the gas at any layer is determined by the balance between energy input from shocks (and, in the high chromosphere, by conduction) and loss via radiation.

We have calculated the dissipation, or irreversible work done on the gas, from the work done by the artificial viscosity. Where shock waves exist, the conservation laws, which are still satisfied by the modified equations, guarantee that this work is equal to the dissipation calculated from the Rankine-Hugoniot relations. Where the waves are not shocks, the dissipation depends sensitively on the form used for the artificial viscosity; but under such conditions the dissipation is always negligible in our calculations. The shock dissipation by pulses of various widths and amplitudes is shown in figure 4, and compared with weak shock theory results in figure 5 (for pulse widths
τ = 50 and 200 s). These plots show the heating rate per gram, so the steep increase is due to the decrease in density as well as the increase in shock strength. From figure 5 we see that the overestimate of α = M − 1 by weak-shock theory is serious, since the dissipation is proportional to α². Even for τ = 50 s, about a quarter of τ_{ac}, the weak-shock approximation gives twice the true dissipation for heights above 1400 km, and the error is more serious for larger pulse widths. We can also see from figure 5 that with the correct Mach number (found from the nonlinear calculations), the weak-shock formula (10) for the dissipation seriously underestimates the dissipation for wide pulses, because of its neglect of the acoustic cutoff.

How this dissipation affects the wave energy (in the absence of radiative losses) is shown in figure 6. The decrease in wave energy begins at the same height as significant dissipation (eq. [6]). The rate of energy loss per unit volume, dE/dz, quickly reaches a maximum and then slowly decreases until the shocks become strong and momentum transfer to the gas extends the atmosphere. When shock-wave extension of the atmosphere becomes important, the energy available for heating the gas,

\[ E = \int (P - P_0) u dt = \int p u dt - P_0 \Delta z \]
Fig. 6.—Logarithm of wave energy per unit area $E$ as a function of height. (a) $\tau = 50, 100, 200, 400$ s and $u_{\text{max}}(0) = 0.32$ km s$^{-1}$. (b) $\tau = 50$ s and $u_{\text{max}}(0) = 0.08, 0.32$, and 1.28 km s$^{-1}$.

(which is what is plotted here), drops sharply. Figure 6 shows that about 90 percent of the shock energy is deposited below 2000 km (the height at which the transition to the corona is thought to begin).

The damping length

$$\xi = E / \rho Q,$$

where $Q =$ dissipation per gram, is shown in figure 7. For pulses narrower than half the acoustic cutoff, the minimum damping length is about 500 km or four scale heights, and occurs at about 1000 km. For larger pulse widths, the minimum damping length is about 1300 km and occurs at greater heights. A damping length can also be defined in terms of the decrease in wave energy

$$\xi = \left| \frac{E}{dE} \right| .$$

This total damping length is about 20 percent smaller than the dissipation damping length, indicating that there is some weakening due to rarefactions overtaking the shock.

In our calculations the lower boundary condition and the coefficient in the artificial-viscosity expression have only a slight effect on the waves. The upper boundary con-
Fig. 7.—Damping length, $\xi = E/\rho Q$, where $Q =$ dissipation per gram, as a function of height; $u_{\text{max}}(0) = 0.32$ km s$^{-1}$.

Fig. 8.—Effects of ionization (ION) and radiation (RAD + ION) on temperature increase. Amplitude $u_{\text{max}}(0) = 0.32$ s. (a) $\tau = 50$ s; (b) $\tau = 200$ s.
dion has no effect on the waves in the lower atmosphere but a large effect near the upper boundary.

For a given shock dissipation, the temperature rise in the gas depends on the energy absorbed by ionization and lost by radiation. The effect of ionization on the temperature is quite dramatic. Figure 8 shows the temperature rise caused by a pulse. The temperature rise plotted here is the difference between the peak temperature and the initial temperature at a given mass zone. Calculation of the upper curve neglected changes in ionization when the gas is heated; the lower dashed curve takes variable hydrogen ionization into account. In the region of partial ionization, the bulk of the energy (> 90 percent) goes into ionizing hydrogen. The increase in gas temperature is several hundred degrees when hydrogen is nearly completely neutral, and then becomes small until hydrogen ionization is nearly complete, where the now strong shocks produce a large temperature rise (fig. 8). It is important to remember, however, that this calculation was done for only a single pulse; the height at which the temperature rise occurs will be different when a sequence of waves is considered.

The effects of radiation on the waves is shown in figures 8 and 9. Radiative losses are the same order of magnitude as the shock dissipation and wipe out the small temperature increase left over after hydrogen ionization, until the ionization is nearly complete. Non-LTE effects will, however, increase the ground-state population and so

![Diagram](image-url)

**Fig. 9.**—Effects of ionization and radiation on pulse energy. Amplitude \( u_{\text{max}}(0) = 0.32 \text{ km s}^{-1} \).
(a) \( \tau = 50 \text{ s} \); (b) \( \tau = 200 \text{ s} \).
reduce the ionization losses. Radiation also directly damps the waves (Stein and Spiegel 1967), but under solar conditions this damping is small except near $\tau = 1$ where our optically thin approximation breaks down and the radiative losses (about 50–90 percent of the wave energy, fig. 9) are overestimated. The dissipation can be found from figure 7.

VI. WAKES

An acoustic wave propagating through an atmosphere in a gravitational field leaves behind it a wave that oscillates at the acoustic cutoff frequency $1/\tau_{ac}$ (Lamb 1932; Schmidt and Zirker 1963; Kato 1966) (fig. 10). A pulse has a spectrum

$$P(\omega) = \frac{4\omega_0^2}{(\omega + \omega_0)^2(\omega - \omega_0)^2} \cos^2 \left( \frac{\pi \omega}{2 \omega_0} \right),$$

which has a small peak at the driving frequency $\omega_0$ and falls off as $1/\omega^4$ at large frequency with its first zero at $\omega = 3\omega_0$. Those components at the acoustic cutoff frequency, $N_{ac} = 2\pi/\tau_{ac}$, have zero group velocity and infinite phase velocity. Thus the spectral components near the critical frequency propagate slower than the main body of the pulse, and are left behind as a standing wave wake. Nonlinear coupling between different frequencies shifts the peak to shorter periods at greater heights.

Figure 11 shows spectra of the pulse and its wakes at heights of 100 and 500 km. For pulse widths less than half the cutoff period, the dominant contribution occurs at the cutoff period. The relative importance of oscillation at the cutoff period increases with height, reflecting the greater damping of the initial pulse which is stronger than the wake and dissipates more energy. For pulse widths greater than half the cutoff period, the atmosphere moves quasi-rigidly and the spectrum is peaked near zero frequency. However, here too the wake at the cutoff period dominates at large heights.

![Figure 10](image)

**Fig. 10.**—Change in shape as the pulse propagates through the atmosphere. Each line represents the velocity history of the mass element whose initial height is shown on the left. In order to fit all the curves on one graph, the quantity plotted is actually \((u/u_0)^{1/3}\). The formation of a shock front is evident, as is the increase in shock Mach number (the slope of the line drawn through the initial peak).
because the lower frequency components do not propagate through the atmosphere. Table 2 shows the change in period of the power spectrum peak with height, and table 3 shows the decrease with height of power \( \rho_0 u^2 \) at the peak. At greater heights the power decreases faster because of shock dissipation. The period shift of the spectrum peak cannot be a filtering effect, because the peak power at the greater height is larger than the off-peak power at the lower height. Note that the wake occurs at the cutoff period \( \sim 200 \) s and not at the observed period of 300 s. We have also seen that the rms velocity amplitude of these wakes increases as \( \rho_0^{-1/2} \) rather than the much slower observed increase. Thus, such wakes are probably not responsible for the solar 5-minute oscillation.

Many details of the wave structure can be seen from a phase diagram of pressure versus velocity as a function of time (fig. 12). Pressure and velocity are normalized to have equal amplitude. This phase plot has a spiral shape because the amplitude of the wake decreases with time. The major axis of the initial pulse lies in the first and third quadrants, so pressure and velocity are in phase, the wave energy

\[
E = \int (P - \rho_0) u dt > 0,
\]
Fig. 12.—Phase diagram of pressure versus velocity as a function of time for adiabatic pulses of widths \( \tau = 50 \) and 200 s at a height of 500 km. The spiral pattern corresponds to the initial pulse and its following wakes of decreasing amplitude. The wave energy per unit area is \( E = \frac{1}{2} (P - P_0) \frac{dP}{dt} \), so a spiral with its major axis in the first and third quadrants represents a wave transferring energy upward, a spiral with its major axis in the second and fourth quadrants represents a wave transferring energy downward, and a spiral with its major axis vertical represents a wave that is not transferring any energy. The blip near the end of the spiral pointing into the second quadrant is the downward-propagating reflected wave.
and the initial pulse transfers energy upward. The major axis of the wakes is vertical, so pressure and velocity are 90° out of phase and no energy is propagated. In the initial pulse, we see that the pressure decreases faster than the velocity. For the 50-s pulse, this distortion is due to shock formation. For the 200-s pulse, the pressure decreases very rapidly because only the transient produces compression. Radiation increases the wake amplitude decay and changes the pressure-velocity phase from 90° to 110°.

VII. DISCUSSION

The only motions observed on the Sun with substantial amplitude are the so-called 5-minute oscillations, with periods between 24 and 7 minutes. At present it is not clear whether there are oscillations of shorter periods. Since the observed Doppler shift in a line is an average over its height of formation, oscillations with wavelengths shorter than this height range are unobservable. Satellite observations of lines formed in the narrow transition region may ultimately yield information on periods as short as 20 s, and the compression behind strong shocks will also decrease the depth of formation from that obtained for a static atmosphere. Evidence for the existence of shocks on the Sun comes from observations by Harvey (1969) of 5-minute oscillations in prominences which show a sawtooth velocity profile and observations by Beckers and Tallant (1969) and Liu, Sheeley, and Smith (1972) of Ca II K2 violet flashes which show 50-s intensity flashes separated on the average by 200 s.

From the above calculations we see that such long-period waves only start dissipating energy 700 km above optical depth $\tau_{5000} = 1$. Locating the source of waves deeper in the convection zone does not affect the height of dissipation significantly, since the density there is nearly constant, so the velocity amplitude is nearly constant, and the uniform-medium shock-formation relation applies. In a uniform medium the steepening distance for periods greater than 50 s is so large that such waves do not develop into shocks until they reach heights where the density gradient increases the velocity amplitude. Thus individual pulses with widths greater than 200 s only produce significant dissipation higher than 1200 km above $\tau_{5000} = 1$, or 700 km (two scale heights) above the temperature minimum. To obtain shock dissipation at the level of the temperature minimum, pulses with widths less than 25 s are needed. However, radiative dissipation of waves of all periods can be significant at and below the level of the temperature minimum (fig. 9).

The transition to the corona begins when hydrogen becomes ionized and the Lyman continuum and Lα become optically thin. To radiate a given rate of energy input per unit volume, the temperature must increase as the density decreases. However, the temperature sensitivity of the radiation rate decreases as hydrogen becomes ionized, so the temperature gradient must steepen; the transition region starts at about 9000° K and 1800 km (Thomas and Athay 1961; Defouw 1970; Noyes and Kalkofen 1970).

At 2000 km dissipation has reduced the wave flux by about 10. However, these one-dimensional calculations ignore the effects of refraction. If the index of refraction in the corona is $n$ relative to the chromosphere, refraction diminishes the intensity by a factor of $n^2$. In addition, reflection reduces the transmitted intensity by another factor of $n$, so the intensity is proportional to $n^3 = \left( T_{\text{chrom}}/T_{\text{corona}} \right)^{3/2} \approx 10^{-3}$ (Kopp 1968). Since the initial flux $F = \rho u^2 c \approx 10^8$ ergs cm$^{-2}$ s$^{-1}$ in the photosphere, only $10^4$ ergs cm$^{-2}$ s$^{-1}$ will reach the corona, which is much less than the amount, about $10^6$ ergs cm$^{-2}$ s$^{-1}$ (Athay 1966), observed to be lost from the corona in radiation, thermal conduction back down, and the solar wind. Thus it appears that pure acoustic waves (in the absence of a magnetic field) cannot provide sufficient energy to heat the corona. Gravity waves cannot help, because they are not transmitted through the photosphere (Uchida 1967).
We next turn to a qualitative consideration of the role played by magnetic fields. In the presence of a magnetic field the acoustic waves become fast-mode MHD waves. These propagate with the sound speed \( c \) for \( c \gg a = B_0/\sqrt{(4\pi p_0)} \) = Alfvén speed, and with the Alfvén speed for \( a \gg c \). The first effect of the magnetic field is to increase the refraction of the fast-mode waves. Because of the outward decreasing density the Alfvén speed increases upward. The refraction of the fast mode is thus increased with respect to the acoustic waves in the regions \( a > c \). The magnetic field is also horizontally inhomogeneous, and the fast mode is refracted away from the high field regions.

The second effect of the magnetic field is to couple the different MHD modes because of changing Alfvén speed, magnetic field orientation, and nonlinearities. In the WKB limit only rotation of the magnetic field couples the modes (Frisch 1964); in the full linearized wave approximation, changing phase velocity produces coupling; and additional wave-wave interactions occur in the nonlinear regime. The most important coupling transfers energy from the fast to the Alfvén mode in the regions where \( a > c \). The significance of putting energy into the Alfvén mode is that (i) Alfvén wave ray paths follow the field lines, (ii) 45 percent of the Alfvén mode flux is transmitted through the transition zone to the corona, and (iii) the Alfvén mode is non-compressive and does not dissipate energy in the chromosphere.

Two major problems remain in this discussion of heating: (i) How is the Alfvén mode damped in the corona where the plasma is essentially collisionless? (ii) What produces the enhanced emission in high field regions in the photosphere below the temperature minimum?

We gratefully acknowledge the hospitality and support of the Sacramento Peak Observatory (where most of the numerical calculations were performed), the National Center for Atmospheric Research, supported by the National Science Foundation (where the calculations were concluded), the Joint Institute for Laboratory Astrophysics (where this paper was written), and the Smithsonian Astrophysical Observatory where one of us (R. F. S.) is a consultant and where the code was developed and the illustrations were produced. We had fruitful discussions with many people during the course of this work. In particular, we wish to thank Stuart Jordan and John Leibacher.

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