A characteristic function equation for two-interval forced-choice experiments

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The percent correct responses in a two-interval forced-choice detection experiment can be calculated from the probability density functions for the excitation on the two intervals. This letter introduces an equation which relates the percent correct responses to the characteristic functions of the probability densities. The equation can result in considerable mathematical simplification in the prediction of detection performance in cases of multiple sources of signal or noise.

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INTRODUCTION

In a two-interval forced-choice (2IFC) detection experiment there are two time intervals. During one interval only noise is presented to a detector, during the other interval both noise and signal are presented. The noise may be either part of the experimental environment or an irreducible detector noise.

Signal detection theory relates choices between the intervals to the distributions of a one-dimensional random variable, generally the logarithm of the likelihood ratio (Green and Swets, 1966). We refer to this random variable as the decision variable and represent it by symbol x. The probability that the decision variable has a value between x and $x+dx$ is given by the density $f_n(x)dx$ during the noise-alone interval and by $f_{sn}(x)dx$ during the interval with both signal and noise.

A standard result of signal detection theory is that the percent correct responses, P_c, in a 2IFC experiment, is a function of the probability densities, as given in Eq. (1):

$$P_c=\int_{-\infty}^{\infty} dx f_{sn}(x)\int_{-\infty}^{x} dx' f_n(x') . \tag{1}$$

In psychoacoustics it is common to assume that densities f_n and f_{sn} are Gaussian, and in that event it is easy to evaluate the integrals in Eq. (1). The Gaussian assumption is a plausible one if variable x is itself the sum of very many random variables. In other cases, as discussed below, the densities may not be Gaussian. The following analysis of Eq. (1) in the general case may prove useful in these situations.

In mathematical statistics it is often useful to deal with densities in terms of their characteristic functions, which are the Fourier transforms of the probability densities. The characteristic function for the noise density is $\phi_n(k)$ where

$$\phi_n(k)=\int_{-\infty}^{\infty} \exp(ikx) f_n(x)dx . \tag{2}$$

(Parzen, 1962). The inverse transform is

$$f_n(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp(-ikx)\phi_n(k) . \tag{3}$$

Similar transformations relate f_{sn} to its characteristic function ϕ_{sn}.

In order to use characteristic functions in the analysis of a 2IFC experiment one needs to be able to relate the percent correct responses to the characteristic functions for the noise and the signal-plus-noise densities. Using standard methods of complex variable analysis one can show that the relationship is given by the following equation,[1]

$$P_c=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\frac{dk}{k}\mathrm{Im}[\phi_{sn}(k)\phi_n(-k)] , \tag{4}$$

where Im denotes the imaginary part.

I. APPLICATIONS

The formula of Eq. (4) may be usefully applied when the decision variable x is a function of a single physical dimension of the stimulus and when there is additivity on that dimension from multiple sources of signal and noise. This is the case, for example, when the decision variable is based upon energy incident on the detector. Formally we continue to use symbol x for the physical dimension.

Suppose that there are various sources of signal and noise with uncorrelated densities which are individually known. Because of additivity the densities f_n and f_{sn} are *convolution integrals* involving the individual signal and noise densities. Therefore, the evaluation of the percent correct responses from Eq. (1) may be a formidable computational problem. On the other hand, the characteristic functions ϕ_n and ϕ_{sn} are simply *products* of the individual signal and noise characteristic functions. Therefore, with Eq. (4) the final calculation of the percent correct responses may be reduced to a single integral.

II. EXAMPLE

Suppose that a signal excitation is exponentially distributed and that the noise consists of two constituents, one of them Gaussian, the other oscillating with a sinusoidal excitation strength. For the signal alone,

$$f_s(x)=\lambda e^{-\lambda x}, \quad x\geq 0 \qquad (5)$$
$$=0, \quad x<0,$$

and

$$\phi_s(k)=(1-ik/\lambda)^{-1}\,. \tag{6}$$

For the Gaussian noise source,

$$f_{n1}(x)=(1/2\pi\sqrt{\sigma})\exp(-\tfrac{1}{2}x^2/\sigma^2)\,, \tag{7}$$

and

$$\phi_{n1}(k)=\exp(-\tfrac{1}{2}k^2\sigma^2)\,. \tag{8}$$

For the periodic noise source with sinusoidal amplitude x_m

$$\begin{aligned} f_{n2}(x)&=\pi^{-1}(x_m^2-x^2)^{-1/2}\,, && -x_m<x<x_m \\ &=0\,, && |x|>x_m \end{aligned} \tag{9}$$

and

$$\phi_{n2}=J_0(kx_m)\,, \tag{10}$$

where J_0 is a Bessel function of the first kind.[2]

Using the characteristic functions one can immediately write down the integral from Eq. (4) for the percent correct responses in a 2IFC experiment.

$$P_c=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty\frac{dk}{k}e^{-k^2\sigma^2}J_0^2(kx_m)\left(\frac{k/\lambda}{1+k^2/\lambda^2}\right)\,. \tag{11}$$

This integral can be readily done numerically by computer for various values of the signal and noise parameters. The simplicity of this integral can be compared with the complexity of the five nested integrals required if one calculates the percent correct in terms of probability densities.

III. VARIANTS

Several interesting variants of Eq. (4) may be noted. Let ϕ_s be the characteristic function of the density caused by the signal in the ideal case of no noise. Then, with signal and noise uncorrelated,

$$\phi_{sn}=\phi_n\phi_s\,. \tag{12}$$

$$P_c=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty\frac{dk}{k}\,|\phi_n(k)|^2\,\mathrm{Im}\phi_s(k)\,. \tag{13}$$

This form for P_c emphasizes that the average value of the noise decision variable is of no significance. If the noise density is shifted by some constant value X, the effect is to multiply the characteristic function $\phi_n(k)$ by $\exp(ikX)$, which leaves its absolute value unchanged.

Alternatively let function g be the probability density that the decision variable on the signal interval is greater than that on the noise-alone interval by an amount x, i.e.,

$$g(x)=\int_{-\infty}^{\infty}dx' f_{sn}(x')f_n(x'-x)\,. \tag{14}$$

Let $\psi(k)$ be the Fourier transform of $g(x)$. Then because $\psi(k)=\phi_{sn}(k)\phi_n(-k)$, Eq. (4) becomes,

$$P_c=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty\frac{dk}{k}\mathrm{Im}\psi(k)\,. \tag{15}$$

There are several additional points which seem worth mentioning. First, this letter has focused on a detection experiment, but the mathematical results can be applied equally well to a discrimination experiment. Densities f and the corresponding characteristic functions then represent the excitations caused by the two signals to be discriminated, and Eq. (4) predicts the percent correct. Second, it is natural to wonder whether the use of characteristic functions can produce a similarly simple expression for the prediction of the results of an mIFC experiment, where m is some integer greater than 2. Unfortunately there is no correspondingly simple form. The reason is that the analysis of the mIFC experiment involves products of integrals which are not in the convolution form. The use of characteristic functions may simplify the analysis in individual cases, but there is no particularly simple form which applies in general. Finally it is important to note that the above expressions for P_c are correct only if the constituent noise sources are mutually uncorrelated and are uncorrelated with the signal.

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[1]A derivation of Eq. (4) starting with Eq. (1) is available from the author. Please request report number 27P.

[2]If a noise source consists of a number of sinusoidal components with different periods and amplitudes then the characteristic function for the composite is a product of Bessel functions, one for each component with the corresponding value of x_m. This statement is true only if the components are not phase locked, i.e., only if the component frequencies are not harmonics of a common fundamental.

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