III. IN THREE-DIMENSIONAL SPACE

We have an intuitive understanding of why air is distributed uniformly throughout a room. We may imagine the room divided into two halves and show that the probability of finding half the molecules in one side and half in the other is much greater than finding $\frac{1}{3}$ of the molecules in one side and $\frac{2}{3}$ in the other. One may draw the analogy with a collection of Avogadro's number of coins tossed in the air. On intuitive grounds we expect to find $\frac{1}{2}$ heads and $\frac{1}{2}$ tails. It would be reassuring if we could also visualize the predominance of the reverse-kinetic states over the other three classes. To this end, imagine a one-dimensional box divided into two equal halves. The left half contains 40 particles and the right 60 particles. The most probable spatial distribution is that in which the particles are uniformly distributed over their respective sides. For simplicity, assume all particles have the same speed. The most probable velocity distribution is that in which there are as many particles directed to the right as to the left in each half of the box. It is not difficult to see that this is a reverse-kinetic state. It is clear that the number of particles in the right side decreases and the number in the left side increases in both temporal directions. In a similar way it is not difficult to see that reverse-antikinetic states require a very unlikely particle and velocity configuration.

IV. SUMMARY

In conclusion, we may say that macroscopic systems *persist* in an equilibrium state because there is one macrostate which has many more associated microstates than any other. Macroscopic systems *approach* this equilibrium state because (1) there are many more trajectories leading toward the equilibrium state and (2) the probability of persisting in a kinetic trajectory is high while the probability of persisting in an antikinetic trajectory is low.

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The dynamically shifted oscillator

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The dynamically shifted oscillator is described by an equation of motion which includes both a Hookian restoring force and a nonlinear force which depends only upon the sign of the displacement. Exact expressions are obtained for the amplitude-dependent frequency and for the displacement waveform. The theory is applied to the study of the vibrations of the two-point librator and of the spring doorstop.

I. INTRODUCTION

Nonlinear mechanical and electrical oscillators abound in the natural world. In most cases the equations which describe them have no exact solution. This paper concerns oscillators described by a simple nonlinear differential equation for which an exact solution is possible.

We consider a one-dimensional oscillator with displacement x and mass m, with the Hamiltonian

$$H = \tfrac{1}{2}m\dot{x}^2 + V(x), \tag{1}$$

where the potential is given by

$$\begin{aligned} V(x) &= \tfrac{1}{2}k(x + x_0)^2 \qquad (x \geqslant 0), \\ V(x) &= \tfrac{1}{2}k(x - x_0)^2 \qquad (x < 0). \end{aligned} \tag{2}$$

Parameter x_0 is the *dynamic shift*. If it is zero then Eq. (1) describes a simple harmonic oscillator with force constant k. If the dynamic shift is not zero, Eq. (1) describes an oscillator with an apparent equilibrium position which depends upon the sign of the displacement. We call this a *dynamically shifted oscillator.*

The potential is a continuous function of x, but its first derivative is not. Therefore Newton's law for the oscillator includes a discontinuity at the origin,

$$m\ddot{x} = -kx - s\,\mathrm{sign}(x), \tag{3}$$

where $s = kx_0$ and $\mathrm{sign}(x) = x/|x|$. Parameter s is the stiffness parameter (finite restoring force for zero displacement). The stiffness is positive when the dynamic shift x_0 is positive. Potentials and stress–strain relationships for the cases of $x_0 > 0$ and $x_0 < 0$ are shown in Fig. 1.

II. OSCILLATOR FREQUENCY

Because of the nonlinearity in Eq. (3) the oscillator frequency is a function of amplitude a. For very large amplitudes (i.e., $ka \gg s$) the linear term kx in Eq. (3) has a large magnitude and dominates the right-hand side of the equation for most of the time. In this limit the oscillator frequen-

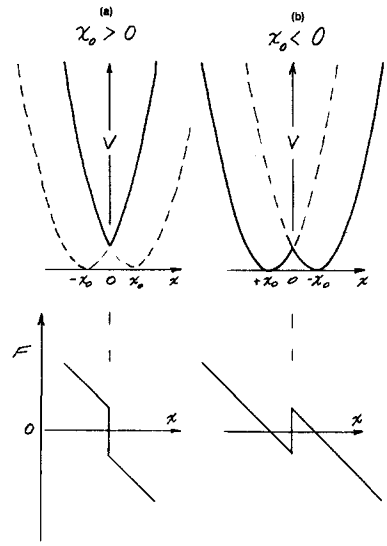

Fig. 1. The top figures show the potential (solid curve) for the dynamically displaced oscillator, made from parabolas (dashed). The lower figures show the restoring force F versus displacement x. Positive and negative stiffness cases are shown in (a) and (b).

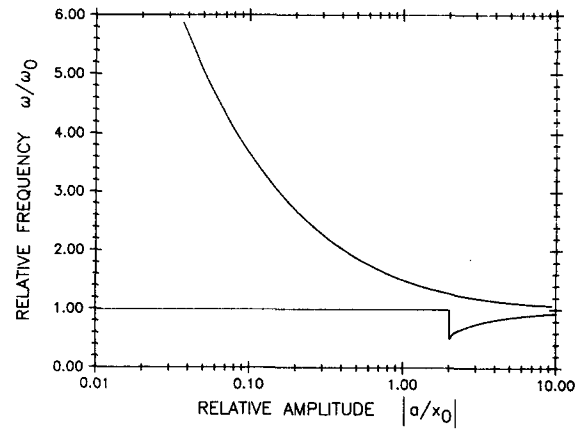

Fig. 2. Angular frequency relative to the frequency in the harmonic limit, versus the amplitude relative to the dynamic shift. The upper curve applies to positive stiffness $x_0 > 0$, the lower curve to negative stiffness, $x_0 < 0$.

cy is simply

$$\omega_0 = \sqrt{k/m}. \tag{4}$$

If the amplitude is not very large and the stiffness is positive, the effect of the nonlinearity is to increase the restoring force, and hence to increase the oscillator frequency. If the stiffness is negative, both the restoring force and the oscillator frequency are decreased from their large-amplitude values.

The frequency of the oscillator can be calculated exactly. The first step is to solve Eq. (1) for the velocity,

$$\frac{dx}{dt} = \sqrt{\frac{2(E - V)}{m}}, \tag{5}$$

or

$$dt = dx[2(E - V)/m]^{-1/2}. \tag{6}$$

Then, integrating both sides of Eq. (6) over a single cycle of the motion, using conservation of energy $E = k(a + x_0)^2/2$, gives an expression for the period of oscillation,

$$T(a) = 2\pi\sqrt{\frac{m}{k}}\left[1 - \frac{2}{\pi}\sin^{-1}\left(\frac{x_0}{a + x_0}\right)\right]. \tag{7}$$

Therefore, the angular frequency is given by

$$\omega(a) = \omega_0\left[1 - \frac{2}{\pi}\sin^{-1}\left(\frac{x_0}{a + x_0}\right)\right]^{-1}. \tag{8}$$

The frequency depends upon the amplitude only through its ratio to the dynamical shift.

Equation (8) has a solution for positive stiffness ($x_0 > 0$) for all values of a ($a \neq 0$). For negative stiffness a solution of Eq. (8) exists only for $a > 2|x_0|$. Only then is the total energy adequate to overcome the potential barrier at the origin. Otherwise, the oscillator remains confined to one of the two wells shown in Fig. 1(b), and the angular frequency is ω_0. Figure 2 shows the amplitude-dependent oscillator frequency relative to the harmonic limit, computed from Eq. (8).

III. THE DISPLACEMENT WAVEFORM

It is possible to find an exact solution for the displacement function $x(t)$. The reason is that the time dependence of the nonlinearity, the last term on the right side in Eq. (3), is completely determined by the simple fact that the solution must be symmetrical about the value $x = 0$. The dynamically shifted oscillator is almost the only nonlinear system for which such a statement can be made.

[The other system is the harmonic oscillator with Coulomb friction discussed by Minorski.[1] That system is described by our Eq. (3) with sign(x) replaced by sign($\dot{x}$). Such a system oscillates with the harmonic oscillator frequency ω_0, but the oscillations are damped. Therefore, the waveform is not periodic and cannot be written in the form of Eq. (10).]

The nonlinear term is a square-wave function of time, with values $\pm s$. It is given by the Fourier series,

$$s\,\mathrm{sign}(x) = \frac{4s}{\pi}\sum_{n\ \mathrm{odd}}(-1)^{(n-1)/2}\frac{\cos(n\omega t)}{n}. \tag{9}$$

A. Series solution

The displacement x can be expanded in a Fourier series,

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \exp(in\omega t). \tag{10}$$

Substituting into Eq. (3) provides an equation for the coefficients a_n:

$$a_n(n^2\omega^2 - \omega_0^2) - (2x_0\omega_0^2/\pi|n|)(-1)^{(n-1)/2} = 0 \quad (n\ \text{odd}), \tag{11a}$$

$$a_n = 0 \quad (n\ \text{even}). \tag{11b}$$

Equation (11a) can be solved for the coefficients a_n by using Eq. (8) for ω, and Eq. (4) for ω_0. These coefficients are the amplitudes of the harmonics of the vibration.

B. Quasi-harmonic approximation

If one keeps only the first term in the Fourier series in Eq. (10) one obtains the quasi-harmonic approximation.

Then $a = a_1$, and the oscillation frequency is

$$\omega = \omega_0[1 + (4/\pi)(x_0/a)]^{1/2}. \qquad (12)$$

As expected, the frequency is equal to the harmonic value ω_0 for large amplitude a.

C. Cosine plus parabola approximation

For harmonic number n equal to 3 or greater the factor $(n^2\omega^2 - \omega_0^2)$ in Eq. (11a) is approximately equal to $n^2\omega^2$. (This is especially true for positive stiffness where $\omega > \omega_0$.) With this approximation the coefficients a_n are given by b_n, where

$$b_n = \frac{4}{\pi}(-1)^{(n-1)/2} x_0 \frac{\omega_0^2}{\omega^2} \frac{1}{|n^3|}, \qquad (n \geqslant 3). \qquad (13)$$

Then adding and subtracting the term with coefficient b_1, one obtains an approximation for the displacement given by

$$x(t) = [a_1 - (4/\pi)x_0(\omega_0/\omega)^2]\cos\omega t + \sum_{n=1,3,5,\ldots} b_n \cos(n\omega t), \qquad (14)$$

where the coefficients b_n are given by Eq. (13) for *all* the odd values of n, including $n = 1$. The coefficients are inversely proportional to the cube of the harmonic number n. This can be compared to the coefficients in the expansion of the triangle wave, which are inversely proportional to the square of the harmonic number. Therefore, the sum in Eq. (14) represents the integral of a triangle wave, namely a parabola wave.

The parabola wave is a sequence of upward and downward parabolas linked together at zero crossings to make a waveform which looks like a distorted cosine wave. The parabola wave of unit amplitude (± 1) has the Fourier representation:

$$P(t) = \frac{32}{\pi^3} \sum_{n=1,3,5} (-1)^{(n-1)/2} \frac{1}{n^3} \cos(n\omega t). \qquad (15)$$

Therefore, the approximate solution in Eq. (14) becomes a linear combination of a cosine wave and a parabola wave,

$$x(t) = [a_1 - (4/\pi)x_0(\omega_0/\omega)^2]\cos\omega t + x_0(\omega_0/\omega)^2(\pi^2/8)P(t). \qquad (16)$$

The approximation becomes an exact solution in two limits. In the harmonic limit, when $x_0 = 0$, the solution is the cosine wave. The other limit is the first application in Sec. IV.

IV. APPLICATIONS

A. The symmetrical bouncing ball

In the extreme inharmonic limit the force constant k is zero. The restoring force is then independent of displacement, except that it is $-s$ for positive displacements and $+s$ for negative displacements.

A physical representation of this situation involves an infinite sheet of electrical charge which has a hole in the middle. A ball, charged opposite to the sheet, oscillates through the hole as described by Eq. (3) with $k = 0$. (Note that opposite charges on sheet and ball correspond to the case of positive stiffness. Similarly, charged sheet and ball is a case with no equilibrium and is of no interest.) The angular frequency of oscillation is given by

$$\omega = (\pi/2)\sqrt{s/2am}. \qquad (17)$$

The physical situation in this simple limiting case, namely constant acceleration on each side of the charged sheet, clearly leads to a parabola wave solution for the displacement function. The parabola wave is also the correct limiting solution to Eq. (16), a fact which can be shown if the limit $k = 0$ is properly understood: Because the stiffness is given by $s = kx_0$, a finite stiffness is obtained in the limit $k = 0$ only if x_0 diverges. In that limit the ratio (ω_0/ω), as computed from Eq. (8), vanishes, but the product $x_0(\omega_0/\omega)^2$, which appears in Eq. (16) for the waveform, remains finite and has the limiting value of $8a/\pi^2$. Then, with a_1 given by its value for a parabola wave with amplitude a, $a_1 = 32a/\pi^3$, the first term in Eq. (16) vanishes, leaving as a final solution the displacement waveform $aP(t)$, the parabola wave.

B. The spring doorstop

The spring doorstop is a coil of stiff wire in static compression so that a force is required to make a gap between adjacent turns. Because the coil can be bent the spring doorstop minimizes the effects of stubbing one's toes in the dark. It is also a favorite musical instrument for two-year olds. Efficiently coupled to a large sound board (the wall), the spring doorstop produces a splendid rattling sound which can be heard throughout the house.

Experimental observations[2] provide the following information about the system.

(1) As the doorstop (bending) vibrations decay their frequency increases by more than an octave.

(2) Waves propagate on the spring only briefly after impact at zero crossings. Therefore the system might be considered to be a rigid vibrator.

(3) Vibration is generally in two dimensions, but there is a preferred polarization. If the initial displacement is not parallel to the preferred polarization the system ultimately rotates into the preferred polarization. If the initial displacement is parallel to the preferred polarization the displacement may remain in that direction throughout the motion. The system then is one-dimensional.

(4) The frequency of vibration (inverse of the period) increases linearly with the number of cycles n, i.e.,

$$\omega(n) = \omega(0) + pn, \qquad (18)$$

where p is constant. This means that the frequency increases exponentially with time,

$$\omega(t) = \omega(0)\exp(pt). \qquad (19)$$

Rules (1) and (4) were discovered by digitally recording the vibrations of doorstops vibrating with the preferred polarization. A piezoelectric transducer mounted close to the doorstop provided the input voltage to an A–D converter. The digital recordings were graphed and the periods were determined by eye. Figure 3 shows the frequencies for the stiffest and the least-stiff doorstops available in local hardware stores. The slopes are, respectively, $p/2\pi = 0.6$ and 0.5 Hz/cycle.

Like Chinese opera gongs of the "softening" type[3] the spring doorstop is a vibrating system under a static stress, in which the frequency increases with decreasing displacement. We model the doorstop by the dynamically shifted oscillator equation, with positive stiffness and with added damping.

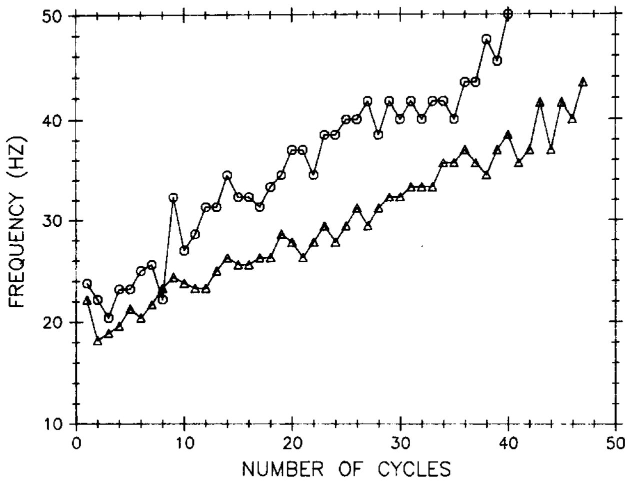

Fig. 3. Experimental values of the inverse periods for two spring doorstops; O, stiff and △, less stiff.

A reasonable first choice for the damping is one in which energy is lost only at zero crossings, to brief lossy vibrations of the coils of the spring. Then, the amplitude for each half-cycle can be expressed in terms of the frequency by inverting Eq. (8). Although neither x_0 nor ω_0 is known for this system, we can find a function for the amplitude by noting that for the stiff doorstop the amplitude decayed to about 10% of its initial value after 40 cycles. The data from the $n = 6$th to the $n = 40$th cycle are well fitted if the energy of the system is proportional to $1/n$. That means that the energy loss per zero crossing is proportional to the square of the energy.

C. The two-point librator

The two-point librator is realized in chrome-plated wire figures ubiquitously sold in novelty shops. The figure rests upon two pointed feet, as shown in Fig. 4(a), and rocks from side to side. (It also rotates slightly about a vertical axis, but that is neglected here.) The essential geometry of the two-point librator is shown in Fig. 4(b). Because the center of mass lies below the feet the librator is stable.

When the librator pivots about the left foot, as shown in Fig. 4(b), the torque tending to restore it to equilibrium is

$$\tau_1 = -Mg(h \sin\theta + b\cos\theta). \tag{20}$$

For the other half-cycle the librator pivots about the right foot; the angular displacement θ is negative, and the torque is

$$\tau_2 = -Mg(h \sin\theta - b\cos\theta). \tag{21}$$

By symmetry, the moment of inertia I is the same about each foot, and the dynamical equation for angular displacement θ becomes

$$I\ddot{\theta} = -Mg[h\sin\theta + b\,\mathrm{sign}(\theta)\cos\theta]. \tag{22}$$

In the limit of small displacements, the equation becomes

$$I\ddot{\theta} = -Mgh\theta - Mgb\,\mathrm{sign}(\theta), \tag{23}$$

which is of the same form as Eq. (3), with positive s. Therefore, the two-point librator bears the same relation to the dynamically shifted oscillator as a pendulum bears to a harmonic oscillator.

Because the "stiffness' is positive the frequency of the librator increases as the amplitude decreases due to energy loss to friction.

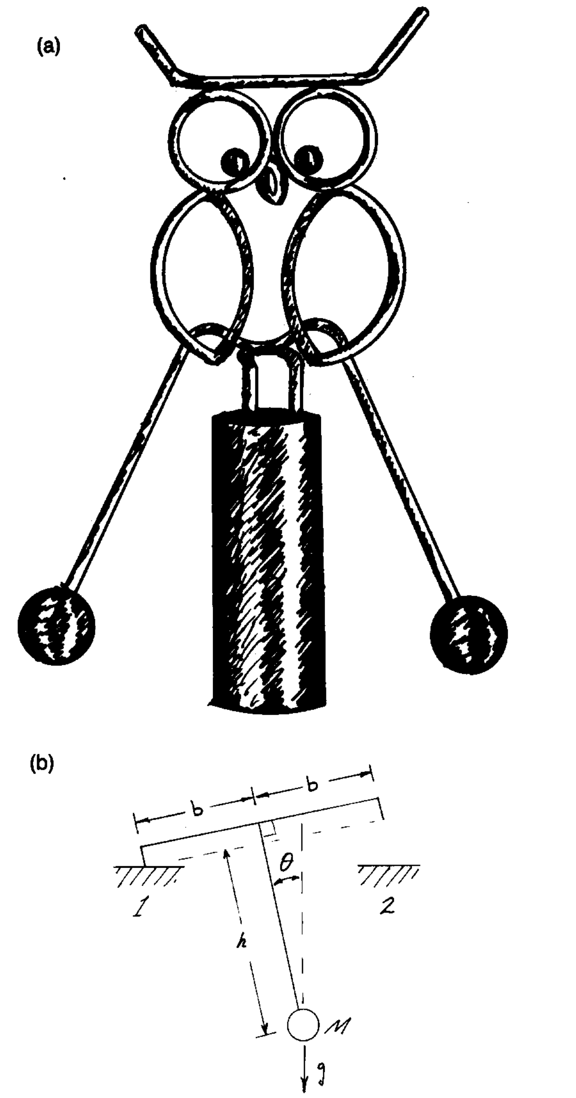

Fig. 4. (a) A typical two-point librator and (b) its essential geometry.

The period of the librator shown in Fig. 4(a) was measured by using it as an electrical switch. A pedestal was made from two blocks of metal, mutually insulated by a piece of paper in the vertical plane so that the two feet of the librator touched different blocks. A voltage between the two blocks was momentarily shorted to zero by the conducting librator at zero crossings, when both feet were in contact with the blocks. The voltage was read by an analog to digital converter, and a computer calculated the zero crossing times.

The times between successive zero crossings, measured for an initial displacement of $a(0) = 0.26$ rads, are shown in Fig. 5. Because the librator is not actually symmetrical the durations for one direction of motion are longer then the durations for the other. Therefore, successive points in Fig. 5 lie on two different curves, but the two curves are related by a simple constant factor.

Unlike the case of the spring doorstop, the dynamic displacement parameter for the librator x_0 can be known. It is b/h, and finding it requires only that one measure the separation of the two feet and the depth of the center of gravity,

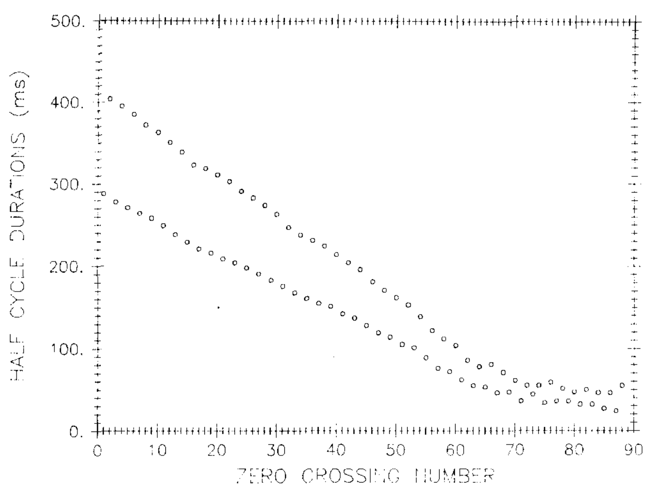

Fig. 5. Measured time intervals between successive zero crossings as a function of zero crossing number for the two-point librator.

in our case, $2b = 9$ mm and $h = 5$ mm. Using the initial value of $a(0)/x_0$, and the data for the durations of the longer half-cycles we were able to calculate the amplitude, and the kinetic energy, as a function of cycle number from the inverse of Eq. (7). A graph of the log of the kinetic energy versus cycle number was fitted rather well by two straight lines. From cycle 1 to 21 (zero crossings 2–42) the slope corresponded to a loss of 8% of the kinetic energy per cycle. For cycles 21 to 37 the slope corresponded to a loss of 16% per cycle. We attribute the increased rate of energy loss after the 21st cycle to the onset of a small rotational motion about the vertical axis, which could not be avoided with our experimental design. The rotational motion extracts additional energy from the librating motion.

D. A system with negative stiffness

For negative stiffness, unlike the cases described in Sec. IV A–C, the expression for the potential energy includes a *negative* term proportional to the absolute value of the displacement. This situation occurs for a parallel plate capacitor (plate separation d) with a dielectric slab (thickness also equal to d and dielectric constant κ) which is free to slide along one direction parallel to the plates. The plates and the slab both have length l and width w, and the plates are connected to a battery with voltage V_B. When the dielectric is centered the potential energy is $V_0 = C_0 V_B^2/2$, where $C_0 = \epsilon_0 \kappa wl/d$. When the dielectric is displaced from center, by x along dimension l, the potential energy decreases to

$$V = V_0[1 - (\kappa - 1)/\kappa]|x/l|. \quad (25)$$

Thus the centered position is one of metastable equilibrium.

The system becomes bistable if springs connected to the dielectric slab tend to restore it to the center. Then the motion of the slab obeys Eq. (3) with a stiffness parameter s equal to $V_0(1 - \kappa)/(\kappa l)$.

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Classical and semiclassical diamagnetism: A critique of treatment in elementary texts

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Diamagnetism does not exist as a classical phenomenon. Yet many elementary physics texts attempt to give "insight" into this subject using purely classical physics. We believe that these approaches mislead rather than help the students. On the other hand, just as semiclassical methods are employed in the chapters on quantum mechanics, these methods may be used to give an acceptable "explanation" of diamagnetism.

I. INTRODUCTION

When a nonmagnetic material is placed in an external magnetic field **B**, the material typically develops an induced magnetic moment. If this induced moment is opposite to **B**, the system is called "diamagnetic." Being a many-body system generally held at nonzero temperatures, a detailed understanding of diamagnetism must entail statistical mechanics and thermodynamics. It is well known[1] that within this framework, *classical* diamagnetism does