The frequency-domain grating

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(Received 18 March 1985; accepted for publication 8 July 1985)

It is shown that a complex tone can be analyzed into its harmonics by adding to it a series of $N-1$ similar complex tones with uniformly spaced fundamental frequencies. The harmonics appear in a time-ordered sequence, and this is the basis of a computer music effect employed by composer Jean-Claude Risset. The mathematical derivation involves the summing of a geometric progression to form a slowly varying amplitude factor. The effect is directly analogous to the analysis of white light into its constituent colors by a diffraction grating with N slits. The temporal ordering of the sequence does not depend upon the relative phases of the harmonics of the complex tone, nor upon the number of tones which is summed, nor is it sensitive to amplitude changes. The order can be changed by a progressive delay of any harmonic in the summed complex tones. Calculations are made to investigate the stability of the effect against randomness in the tones to explain why the effect does not normally appear in vocal or instrumental choruses.

PACS numbers: 43.20.Wd, 43.75.Wx, 43.85.Kr

INTRODUCTION

At the 11th International Conference on Acoustics, composer Jean-Claude Risset[1] demonstrated a fascinating computer-generated sound. He created seven periodic complex tones which were identical except for a small shift in fundamental frequency. The first tone was unshifted, the second was shifted by $\delta\omega$, the third by $2\delta\omega$, the fourth by $3\delta\omega$, and so on. When the seven tones were added together the listener heard a sequence of the individual harmonics of the complex tone, as though a narrow-band filter had been moved haphazardly across the spectrum.

The purpose of this report is to explain the effect mathematically and to show the correspondence between this acoustical effect and the optics of a diffraction grating. The conclusion is that the superposition of a complex tone with its frequency shifted versions has the effect of analyzing the complex tone into harmonics in the same way that a diffraction grating analyzes white light into its constituent colors. The acoustical effect will therefore be called "the frequency-domain grating."

I. THE ACOUSTICAL SIGNAL

A periodic complex tone is represented as a sum of its harmonics,

$$c(t) = \sum_{h=1}^{M} a_h \cos[h\omega_1 t + \phi_h]. \qquad (1)$$

There are M harmonics, with index h, amplitude a_h, angular frequency $h\omega$, and phase ϕ_h. Frequency ω_1 is the angular frequency of the fundamental component. Throughout this paper the term "tone" will refer to a single sum of harmonic components as in Eq. (1).

The sum of N complex tones is the signal,

$$x(t) = \sum_{l=1}^{N} \sum_{h=1}^{M} a(l,h) \cos[h\omega_l t + \phi(l,h)], \qquad (2)$$

where the amplitude and phase of the h th harmonic of the l th tone (with fundamental component frequency ω_l) are completely general.

To create Risset's effect, the fundamental frequencies are chosen according to the rule

$$\omega_l = \omega_1 + (l-1)\delta\omega. \qquad (3)$$

The shift $\delta\omega$ for successive tones must be small; for example, $\delta f = \delta\omega/2\pi = 0.1$ Hz.

The object of the present mathematical development is simply to separate the signal $x(t)$ into harmonic components (rapidly varying) and amplitude factors A_h which are slowly varying because $\delta\omega$ is small, i.e.,

$$x(t) = \sum_{h=1}^{M} A_h(t) \cos[h\bar{\omega}t + \bar{\phi}_h], \qquad (4)$$

where $\bar{\omega}$ and $\bar{\phi}_h$ are the average frequency and phase, averaged over the N tones.

If the complex tones are identical, except for their frequencies, then amplitudes and phases are independent of l,

$$a(l,h) = a_h \qquad (5)$$

and

$$\phi(l,h) = \phi_h. \qquad (6)$$

In that case, the separation is easy to do because the sum over the N tones is a geometric series. For any harmonic h the contribution to the sum in Eq. (2) is the real part of

$$a_h e^{i\phi_h} \sum_{l=1}^{N} e^{ih\omega_l t}.$$

The average phase angle $\bar{\phi}_h$ is ϕ_h; the average fundamental frequency for Eq. (4) is

$$\bar{\omega} = \omega_1 + [(N-1)/2]\delta\omega, \qquad (7)$$

and the amplitude of the h th harmonic of the signal becomes

$$A_h = a_h[\sin N\gamma(h)/\sin\gamma(h)], \qquad (8)$$

where

$$\gamma(h) = h\,\delta\omega\, t/2. \qquad (9)$$

In the power spectrum, the power of the h th harmonic is the square of the amplitude,

$$P_h = a_h^2\ [\sin^2 N\gamma(h)/\sin^2\gamma(h)]. \quad (10)$$

The amplitude A_h is a slowly varying function of time because t is multiplied by the small shift $\delta\omega$. A peak occurs in the amplitude of the h th harmonic at times when both the numerator and the denominator of Eq. (8) are zero, i.e., when $\gamma(h)$ is an integral multiple of π, viz.,

$$h\,\delta\omega\,t = 2\pi m, \quad (11)$$

where m is an integer. Peaks for different harmonics occur at different times, and this is the origin of Risset's effect.

II. AN ACOUSTICAL EXAMPLE

We generated a digital signal consisting of $N = 7$ identical tones, each with $M = 7$ harmonics, with $a_h = 1$, and all in cosine phase, i.e., $\phi(h) = 0$. The file was 8000 samples long, successive fundamentals differed by one cycle in the file, and there were 501 cycles of the average fundamental. Therefore, the signal at time point i was

$$X(i) = \sum_{h=1}^{7}\sum_{l=1}^{7} \cos\frac{2\pi h}{8000}(497 + l)i \quad (0 < i < \infty). \quad (12)$$

The seventh harmonic of the highest fundamental was $7(504) = 3528$, which is less than 8000/2 so that the sampling theorem was satisfied.

Because the values of h and l are integers $X(i + 8000) = X(i)$; successive playings of the file lead to a continuous sound with no transients. Because the values of h and l are *successive* integers there is no periodicity in the sound shorter than the file itself.

If the time between successive samples is Δt then the total file duration is 8000 Δt, the average fundamental frequency is $\bar{f} = 501/(8000\Delta t)$, and the frequency spacing is $\delta f = 1/(8000\Delta t)$. If the time between successive samples is 1250 μs then the file is 10 s long, the average fundamental frequency is 50.1 Hz, and the frequency spacing is 0.1 Hz.

The amplitude, from Eqs. (8) and (9), for the h th harmonic at time point i, is

$$A_h(i) = \sin(7\pi h i/8000)/\sin(\pi h i/8000). \quad (13)$$

Figure 1 shows the amplitudes as a function of time, computed from Eq. (13) for the entire file. Figure 2 shows how the file sounds. There is a time inversion symmetry about the center of the file $X(i) = X(8000 - i)$. Therefore, the melody of harmonics consists of a statement and its retrograde for each playing of the file. Notes are given in Fig. 2 for the first 4000 time points; the serpentine arrow of time shows the retrograde nature and the continuity.

III. THE OPTICAL DIFFRACTION GRATING

Fraunhofer diffraction by a grating of N slits, uniformly separated by distance d, is described in all texts on optics, e.g., Jenkins and White.[2] The light to be analyzed is passed through the grating and projected onto a screen. Light waves which come from different slits travel paths of different lengths. Their interference leads to the diffraction pattern on the screen with intensity maxima and minima. The positions of maxima are different for different frequencies of light, and this is how the grating serves as a frequency analyzer.

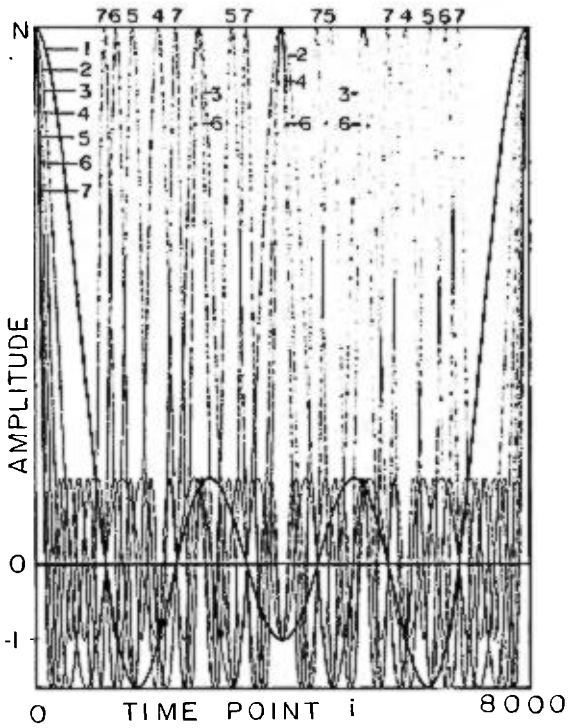

FIG. 1. Amplitude as a function of time point i, from Eq. (13), for each of seven harmonics as labeled. The first harmonic (fundamental) is plotted with a heavy line between halves of a "principal maximum." For every other harmonic the amplitude curve between successive "principal maxima" is identical to the curve for the first.

For light normally incident on the grating, positions on the screen are defined by angle θ, the angle with respect to the direction of the incident light beam. The intensity on the screen is calculated, as above, by summing a geometric series. For light with frequency f the intensity is given by

$$I = A^2(f)(\sin^2 N\gamma/\sin^2\gamma), \quad (14)$$

where $A(f)$ is the amplitude of the incident light of frequency f and

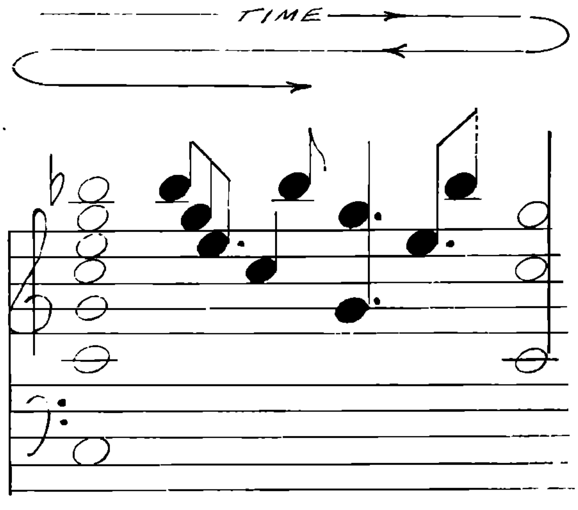

FIG. 2. Peaks in Fig. 1 replotted in musical notation showing the ordered sequence of seven harmonics analyzed by summing seven complex tones. The fundamental is plotted as C3 for convenience only.

$$\gamma = (\pi f/c)\, d \sin\theta, \quad (15)$$

where c is the speed of light.

Equation (14) is identical to Eq. (10). The acoustical and optical effects are analogous, as given in Table I. The table of analogous quantities (Table I) leads to a one-to-one correspondence between the rules for the acoustical effect and standard rules for the diffraction grating, e.g.:

(1) There are $N-1$ intensity zeros between successive principal maxima.

(2) There are $N-2$ secondary maxima between successive principal maxima.

(3) The half-width of principal maxima $\Delta\theta$ is given by

$$\Delta \sin(\theta) \sim (\Delta\theta)\cos\theta = c/(Nfd). \quad (16)$$

Analogous to the durations of harmonic peaks,

$$\Delta t = 2\pi/Nh\,\delta\omega. \quad (17)$$

(4) The spacing of principal maxima is

$$\Delta \sin(\theta) = dc/f. \quad (18)$$

Analogous to the spacing from Eq. (11),

$$\Delta t = 2\pi/h\,\delta\omega. \quad (19)$$

(5) At any point where there is a principal maximum for light of frequency f there is also a principal maximum (of higher order) for harmonics of f. Equation (11) shows that this is true in the acoustical case, however an exception to the rule will be shown in Sec. IV C.

Because the frequency range of visible light is narrow, spanning less than an octave, there is no coincidence of harmonics involving the first order ($m = 1$). For higher orders, harmonics of a "missing fundamental" lead to the troublesome "overlapping of orders."

IV. ACOUSTICAL VARIATIONS

A. Amplitudes, phases, and spectral composition of the complex tone

In applying the "grating" to the analysis of a complex tone, there are no restrictions on the amplitudes and phases of the harmonics of the tone. So long as all the complex tones to be summed have the same values of a_h and ϕ_h, the grating analysis occurs. The amplitude of each harmonic peak in the analysis is proportional to a_h. The phase ϕ_h similarly appears as a simple constant in Eq. (4). The order of the sequence of harmonics, shown in Figs. 1 and 2, does not depend upon the amplitudes or phases of the harmonics. That sequence was calculated for the simple case, $a_h = 1$ and $\phi_h = 0$, but the order would have been the same for any other choice of the seven values of a_h and the seven values of ϕ_h. The order of the sequence also does not depend upon N, the number of complex tones which are summed.

The component frequencies of the complex tone are also unrestricted. For example, the tone may be inharmonic. It may even have a dense spectrum, as for noise, in which case the grating analysis produces a whooshing sound, as though a narrow filter were moved about within the noise spectrum.

In the above example and its variations, the N complex tones are identical except for the frequency shift. Therefore, the sum over l in Eq. (2) could be done because the sum forms a geometric series. There are two conditions, B and C below, which produce a geometric series even though the complex tones which are summed are *not* identical.

B. Amplitudes of the complex tones

A geometric series results if the amplitudes of the corresponding components of the tones are geometrically related:

$$a(l,h) = a(1,h) r_h^{(l-1)}, \quad (20a)$$

where r_h is the common amplitude ratio for the h th harmonic. A special case of this one in which the tones are identical except for attentuation. Then r_h is independent of h. Successive tones are attenuated by a common factor, for example, compared to the first tone, the second tone might be attentuated by 3 dB, the third by 6 dB, the fourth by 9 dB, etc. The main effect of such attentuation is to reduce the size of the harmonic peaks relative to the background. It does not lead to a change in the temporal order of the peaks.

C. Phases of the complex tones

There is only one form of manipulation for the *relative* phases of the components of the tones which leads to a summable series, and that is

$$\phi(l,h) = \phi(1,h) + (l-1)\Delta\phi_h, \quad (20b)$$

a phase shift linear with the ordinal number of the tone. The constant increment $\Delta\phi_h$ may be different for different harmonics. This form of phase shift is actually a time shift for each harmonic given by $\delta t_h = \Delta\phi_h/h\delta\omega$. The average phase angle [cf. Eq. (4)] is

$$\bar{\phi}_h = \phi_h + [(N-1)/2]\Delta\phi_h, \quad (21)$$

and the amplitude A_h is given by Eq. (8) with γ given by

$$\gamma(h) = (h\delta\omega t + \Delta\phi_h)/2 \quad (22)$$

or

$$\gamma(h) = h\delta\omega(t + \delta t_h)/2. \quad (23)$$

In this case, it is possible to reorder the harmonic peaks in time. For example, it is possible to violate rule 5 from the optical analogy.

V. TWO TONES—THE DOUBLE SLIT

The case of two complex tones (1 and 2) for which $a(1,h) = a(2,h) \equiv a_h$ is a unique case in that any choice of relative phase angles for any harmonic satisfies Eq. (20b), because Eq. (20b) includes two free parameters. The amplitude of the h th harmonic is given by

TABLE I. The table of analogous quantities.

Acoustical	Optical
number of tones N	number of slits N
time t	dispersion $\sin\theta$
frequency shift $h\,\delta\omega$	$2\pi f d/c$
harmonic peaks	principal maxima
$t=0$	central image
harmonic occurrence counter $= m$	order number

$$A_h = 2a_h \cos(h\,\delta\omega\, t + \Delta\phi_h)/2. \quad (24)$$

The situation is analogous to the pedagogical optical example of two slits.

Warren and his colleagues[3] have listened to the sum of two complex tones. They reported two experiments, one in which the two tones were identical ($\Delta\phi_h = 0$ for all h) and another in which the two tones were different ($\Delta\phi_h \neq 0$).

For the first experiment a glissando was heard when the complex tone contained harmonics above the seventh or eighth harmonic. When the higher harmonics were removed the glassando vanished, leaving multiple beats. This result may be understood qualitatively by examining the sequence of amplitude peaks, calculated from Eq. (24) with $\Delta\phi_h = 0$. The sequence begins with a peak at the highest harmonic, M. The next peak is harmonic $M - 1$, and so on until the orderly progression of decreasing harmonic numbers is interrupted, just after the peak for harmonic $M/2 + 1$ (or $M/2 + \frac{1}{2}$ for M odd), by the reappearance of a peak for harmonic M. The sequence ends with a progression in the reverse order (rising glissando). The orderly progression includes $M/2 - 1$ harmonics ($M/2 - \frac{1}{2}$ for M odd). For M less than 7 there are less than three peaks in the progression and it is not surprising that one does not hear a glissando experimentally.

For the second experiment, the sum of *different* tones, Warren and his colleagues reported a "complex periodic pattern" when harmonics above the eighth harmonic were present. The change from a glissando to a pattern could be caused by two effects. First, the orderly sequence of harmonic peaks is broken up by the phase differences $\Delta\phi_h$, which may be different for each harmonic. Second, low-order harmonic peaks, which always coincide with peaks for higher harmonics when $\Delta\phi_h = 0$, may now appear singly because of the phase differences $\Delta\phi_h$.

For both experiments, one expects the effects to be weak because the peaks are very broad when only two tones are added. From Eq. (17) for the durations of the harmonic peaks, for $N = 2$,

$$\Delta t = \pi / h\delta\omega. \quad (25)$$

The peaks are least broad for higher values of h so that the higher harmonics should be responsible for most of the effects observed.

VI. SENSITIVITY OF THE EFFECT TO IMPERFECT CONDITIONS

The analysis of sound by the frequency-domain grating occurs because the complex tones are slightly mistuned. Such mistuning of complex sounds always occurs in vocal or instrumental choruses, and yet the grating analysis is not normally heard. The conditions required for the grating analysis are not precisely fulfilled in choruses because of the following deviations: (1) The amplitudes of the harmonics are not exactly the same in each of the tones of the chorus; Eq. (5) does not hold. (2) The phases of the harmonics are not exactly the same in each of the tones of the chorus; Eq. (6) does not hold. (3) The mistuning of the fundamental frequency does not precisely obey Eq. (3). We consider these effects in turn.

A. Amplitude variation among the tones

Because amplitude is a non-negative quantity, a sum over all the tones in the chorus must lead to a finite average value. For any given harmonic h, the power is then

$$P_h = \bar{a}_h^2\,[\sin^2 N\gamma(h)/\sin^2\gamma(h)] + N(\overline{a_h^2} - \bar{a}_h^2), \quad (26)$$

cf. Eq. (10), for comparison. The first term includes the grating analysis; the second is a constant level. If, for example, the amplitudes of the tone to be summed are drawn from a rectangular distribution, $0 < a_h < 1$, the average gives $\bar{a}_h^2 = \frac{1}{4}$ and the variance is $\overline{a_h^2} - \bar{a}_h^2 = \frac{1}{12}$. The ratio of peaks to constant background level is $3N$. Therefore, one expects that the grating analysis can still be heard for sufficiently large N, e.g., $N = 7$. A random choice of amplitudes leads to a constant drone, but the melody of the analyzed harmonics, with its temporal order unchanged, should still be prominent on top of the drone. This prediction has been verified by a listening experiment using one loudspeaker as a source. We conclude that the fact that the amplitudes of the harmonics are different in the different tones in the chorus is probably not responsible for the fact that the grating analysis is not heard in a chorus.

B. Phase variations among the tones

The frequency-domain grating analyzes complex tones because the sum of N complex tones produces a diffraction pattern with a peak, a principal maximum, which is considerably larger than the secondary maxima. In this section we study the change in the diffraction pattern when the phase angle is a random variable among the tones which are summed, as would occur in a chorus. Our study is a numerical one in which the diffraction pattern is calculated for different random configurations of phase angles and compared with the pattern which produces the grating effect, in which there is no variation in phase angles among the tones. It is only necessary to make the comparison for a single harmonic, for the following reason.

If there is no phase variation among the tones, then, for a given number of tones in the sum, there is a particular diffraction pattern in time for the first harmonic. The heavy line in Fig. 1 shows the particular pattern for $N = 7$ tones. The diffraction pattern for the second harmonic is identical to the pattern for the first, but it happens twice as fast. In general, within the overall period of the analysis pattern for the hth harmonic, the particular pattern occurs h times. Therefore, in order to study the change in the overall pattern caused by random variations in the phases of the tones, it is sufficient to study only a single harmonic, the first. The nature of the change will be stochastically identical for all other harmonics.

Therefore, we studied the squared envelope of the function

$$y = \sum_{l=1}^{N} \cos(\omega_l t + \phi_l),$$

for ω_l chosen according to Eq. (3) and random ϕ_l. This corresponds to the pattern for any harmonic when the terms to be summed have identical amplitudes but random phases for that harmonic. For each random configuration we found the

TABLE II. Statistics for the ratio of the power of peak number N_p to the power of the highest peak for seven tones with random phases. Columns 3–5 show the mean over 512 configurations and the mean $\mp$ a standard deviation. The last column shows the percentage of configurations which have at least N_p peaks. For comparison, column R_G shows the ratio of secondary maxima to the principal maximum for no phase difference among the tones.

N_p	R_G(dB)	Random			
		$\overline{R}$(dB)	$\overline{R}-\sigma$(dB)	$\overline{R}+\sigma$(dB)	% configurations
1	0	0	0	0	100
2	− 13	− 2	− 4	0	100
3	− 13	− 3	− 6	− 2	100
4	− 16	− 5	− 9	− 3	99
5	− 16	− 8	− 14	− 5	75
6	− 17	− 9	− 18	− 6	25

peaks of the pattern and put them in order of decreasing height, labeled by integer N_p. We compared the heights of all the peaks with the height of the largest peak because we reasoned that if all the smaller peaks were considerably smaller than the largest peak, then the grating effect would be preserved in that configuration.

Quantity R is defined as the ratio of a smaller peak height to the largest peak height for a particular configuration. Table II shows the results of the numerical study for $N = 7$ tones. Column R_G compares the levels of the secondary maxima to the principal maximum for the case of the perfect grating, where there is no phase variation. The other columns give information about the statistical distribution of R in the random case, its mean value, and its mean minus and plus a standard deviation. The last column shows the percentage of all the configurations for which N_p peaks appeared in the squared envelope for the random case.

The table shows that the highest peak is rarely appreciably taller than several other peaks in the configuration, which suggests that random phases will considerably disrupt the grating effect. In another numerical study we examined 5300 configurations and did not find one for which the ratio R for the second peak was as small as the ratio for the largest secondary maximum in the case of a perfect grating.

As the number of tones in the sum N increases, the average $\overline{R}$ for a particular peak increases slightly. By contrast there is a slight decrease in the ratio of a particular secondary maximum to the principal maximum for the perfect grating. [As N increases, from the smallest value for which a particular secondary maximum first appears, to infinity, the ratio decreases by $20 \log (2/\pi) = -3.9$ dB.] Further, as N increases, the variance of the distribution of R decreases for the random case, so that configurations with small values of R become more and more improbable.

We conclude that for $N = 7$ the grating effect is badly disrupted by random phase angles among the tones. For larger N the grating effect is even more seriously disrupted by randomization. For small values of N, e.g., $N = 3$, there is increased likelihood that a phase configuration chosen at random will produce a diffraction pattern with smaller peaks which are about as small as the secondary maxima of the perfect grating. But for small N, the large width of the peaks tends to destroy the grating effect. It should be noted that even if the sources are phase locked, the fact that they are at different locations in a room will randomize their phases and tend to destroy the grating effect.

C. Nonlinear variation in fundamental frequency among the tones

In choruses, the linear variation in fundamental frequency [Eq. (3)] will not generally occur. The actual variation can be described as a best-fit linear variation plus deviations from linearity. The deviations then appear as random phase angles among the tones, as discussed above, but with a slow temporal variation. Thus the grating effect, already disrupted by random phases, is further disrupted by the variation of the pattern in time.

ACKNOWLEDGMENTS

I am grateful to Professor Jon Pumplin for useful comments on an earlier version of this paper.

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