Digital waveform generation by fractional addressing

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This article concerns the generation of waveforms by a digital oscillator in which sampled data in a memory buffer are recycled. The buffer contains a fixed waveform and the output sample rate is also fixed. Despite these constraints, the oscillator is capable of arbitrarily high frequency resolution if the technique of fractional addressing is used. However, fractional addressing introduces distortion. This article gives a theory of fractional addressing, resembling the theory of diffraction in crystal lattices with a basis. The theory shows how the spectrum of the distortion components can be calculated and how the distortion can be minimized. Attention is called to numerous symmetries in the distortion spectrum. These symmetries are especially interesting if the purpose of the system is to make use of the distortion components to create inharmonic signals. Of particular importance is the γp symmetry theorem, which makes it possible to derive simple formulas for the level of the largest distortion component and for the total distortion power.

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INTRODUCTION

This article is concerned with the generation of waveforms by the continuous recycling of sampled data. To introduce the terminology, the article begins with several examples.

Suppose that a sine signal with frequency $f = 1024$ Hz is recorded into a digital buffer, having $L = 32\,768$ memory locations, at a sample rate of $R = 32\,768$ samples per second. Then, to produce an output sine tone with frequency $f' = 1024$ Hz, one can convert the digital representation of the signal to analog form by a digital-to-analog converter, which converts successive samples to an analog voltage at a rate $R' = 32\,768$ samples per second. In order to convert successive samples, the system has an address register that points to the particular memory location that is converted to analog form at a given instant. The address register will be able to access any of the memory locations (numbered 0–32 767) if it is 15 bits long. After each sample conversion, the address register is incremented by $\Delta = 1$ to point to the next sample in the buffer. After sample number $L - 1 = 32\,767$ has been converted, the address register overflows and points again to sample 0. In this way, the buffer is recycled endlessly. The system is thus a digital oscillator.

Suppose, now, that after each conversion the address register is incremented by $\Delta = 2$. Because the output sampling rate R' is the same as the input sampling rate R, the entire buffer will be reproduced in half the time taken to record it. The output sine tone will have a frequency $f' = 2048$, twice that of the input frequency of f. Thus, by changing the address increment, one can control the frequency of the output signal. In general, the relation between the output frequency and the input frequency is given by

$$f' = f(R'/R)\Delta. \qquad (1)$$

This technique holds good so long as the sampling theorem is satisfied on output, namely, $f' < R'/2$. Because the number of cycles of the sine signal in the buffer is equal to $C = fL/R$, the output frequency can be written as

$$f' = (C/L)R'\Delta. \qquad (2)$$

In order to satisfy the sampling theorem on input, there must be at least two samples per input cycle, i.e., $C \leqslant L/2$. For the example at hand, $C = 1024$ cycles and the system can create output frequencies of 1024,2048,...,16 384 Hz, with increment $\Delta = 1,2,...,16$. In other words, the frequency resolution of such an oscillator is $\Delta f = (C/L)R'$, namely, 1024 Hz.

The frequency resolution can be greatly improved by recording only 1 cycle ($C = 1$) in the buffer. Then, to generate a 1024-Hz sine tone, one sets the address increment Δ equal to 1024. The frequency resolution is, then, 1 Hz.

To make further improvements in the frequency resolution is more difficult. The value of C cannot be reduced below 1; there must be at least 1 cycle of the input waveform in the buffer or else the output will be distorted by a waveform discontinuity between the end of the buffer at $l = L - 1$ and the start of the buffer at $l = 0$.

Improved resolution can be obtained by decreasing the output sampling rate R', but then the maximum frequency of the oscillator is correspondingly reduced. Alternatively, one could increase the length of the buffer by adding more hardware memory. This increases the cost of the oscillator and, perhaps more important, increases the time required to load the buffer.

Fortunately, there is a better way to improve the frequency resolution, namely, by fractional addressing. The technique was introduced by Mathews (1969) in the construction of unit oscillators for a music synthesis program. In the fractional addressing technique, the increment Δ is not an integer; instead, it includes an integer part I and a fractional part γ/M, where γ and M are integers, i.e.,

$$\Delta = I + \gamma/M. \qquad (3)$$

It is always assumed that γ/M is a reduced fraction, i.e., that $\gamma < M$ and γ and M have no factors in common.

An efficient way to implement fractional addressing is shown in Fig. 1. There is a 24-bit address register and a 24-bit increment register. After each successive sample, the ad-

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```
Buffer Address ( 0 to L-1 = 32767)
|15  14  13  . . .  2   1 |
Address Register            ↑
|24  23  22  . . .  11  10   9   8   7  . . .  2   1|
                             +
Address Increment  (Δ)
|24  23  22  . . .  11  10   9   8   7  . . .  2   1|
16384 8192 4096 ...  2   1  1/2 1/4 1/8 ... 1/256 1/512
Bit pattern
  0   0   0  . . .  0   1   1   1   0  . . .  0   0
 msb                                             lsb
```

FIG. 1. Registers for addressing a 32K buffer with a fractional part that may be as small as 1/512. The bit pattern in the increment register (bottom line) represents the increment $\Delta = 1.75$.

dress register is incremented by the increment register. Because only the most significant 15 bits $[\log_2(L)]$ of the address register are used to address the buffer locations, the least significant bits of the address register are fractional addresses. There are $N_F = 24 - 15 = 9$ such fractional bits. The smallest nonzero fractional part is $2^{-9} = 1/512$; the largest possible fractional part is 511/512.

The frequency resolution is determined by the number of bits in the fraction N_F. It is given by

$$\Delta f = (C/L)R'2^{-N_F}. \tag{4}$$

If, for example, there is 1 cycle in a 32 768-sample buffer and N_F equals 9, then the frequency resolution, in terms of the output sample rate R', is 60 parts per billion. If the output sample rate is 50 000 samples/s, then $\Delta f = 0.003$ Hz. The precision of 60 parts per billion can be compared to the inherent stability of crystal oscillators used to generate the sampling clock signal, about 100 parts per billion for a high-stability oven-controlled oscillator.

In sum, the fractional addressing technique leads to excellent frequency resolution while maintaining a maximum allowed frequency equal to half the output sampling rate. It does not require a very long buffer; in fact, for many purposes, the length of the buffer can be considerably shorter than the 32K used in our example.

The remainder of this article is a mathematical analysis of fractional addressing. Such an analysis is necessary because it is not evident *a priori* that the technique should work. It will be shown below that fractional addressing, in which the denominator of the fractional part is M, actually has the effect of reducing the sample rate by a factor of M. Therefore, one might expect that fractional addressing should lead to gross distortion and, in some cases, it does. The theory derives formulas that can be used to calculate the distortion and shows how it is possible to construct digital oscillators, using fractional addressing, that reduce the distortion to acceptable levels.

The theory is quite similar to the theory of crystallography, the study of crystal structure by means of x ray or neutron diffraction (Kittel, 1966). The correspondence is so close that it is possible to refer to elements of the digital theory using the terms of crystallography. This we have done below without further apology.

I. THE SPECTRUM OF A SAMPLED SIGNAL

Our analysis is entirely concerned with the spectrum of the output of a digital oscillator because the spectrum most clearly reveals the nature of the distortion. The analysis is primarily directed to the creation of sine tones. The distortion products that occur in the generation of complex waveforms can be computed by using the sine tone results and the law of superposition.

A temporal view of the sampling situation is given in Fig. 2, which serves to define some of the terms. The figure shows the particular case when the increment is $\Delta = 1.75$, but the definitions of the terms are completely general.

A. The general formalism

As shown in Fig. 2, the buffer contains input samples v_l. The output of the oscillator, as a function of time, is given by the convolution of a sequence of input samples with a sample-hold function h. Depending upon the width of the sample-hold function, the output $v'(t)$ may be a series of delta functions, a bar graph, or a series of steps. The output is

$$v'(t) = \int_{-\infty}^{\infty} dt_1 h(t-t_1) \sum_{m=0}^{\infty} \delta(t_1 - m\tau')v_{l(m)}. \tag{5}$$

Here, τ' is the time interval between output samples, labeled m. It is the reciprocal of the output sample rate R'. The symbol $v_{l(m)}$ stands for the particular input sample l, which is converted on output sample m. Because $v'(t)$ is a convolution in time, its Fourier transform is a product, where the Fourier transform of $h(t)$ is a factor; i.e.,

$$V'(\omega') = H(\omega') \sum_{m=0}^{\infty} e^{-i\omega'\tau' m} v_{l(m)}. \tag{6}$$

Function $H(\omega')$ is called the "form factor." Because $h(t)$

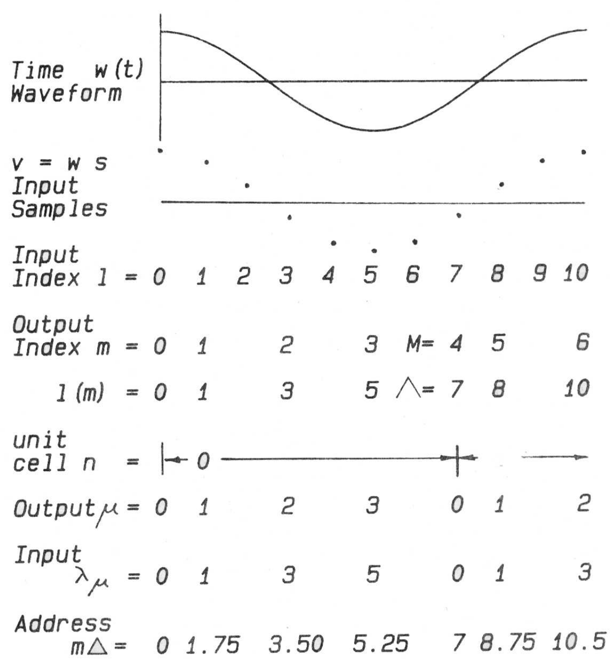

FIG. 2. Cartoon showing the input of a sampled cosine and the output with an address increment of $\Delta = 1.75$. The symbols w, v, s, l, m, $l(m)$, n, μ, Δ, λ_μ, M, and Λ correspond to those in the equations of the text.

describes time-dependent details on a scale that is as short or shorter than the sample time, function $H(\omega')$ depends slowly upon the spectral angular frequency $\omega' (= 2\pi f')$. For the rectangular sample-hold function of Fig. 3, the form factor is

$$H(\omega') = \{[\sin(\omega' g\tau'/2)]/[\omega' g\tau'/2]\} e^{-i\omega' g\tau'/2}. \tag{7}$$

The form factor is a rather benign low-pass filter. The worst case occurs when the duty factor g has its maximum value ($g = 1$) and ω' has its maximum value, $2\pi R'/2$. Then, compared to the low-frequency limit of H, the attenuation is

$$20 \log|H| = 20 \log \frac{\sin(\pi/2)}{(\pi/2)} = -3.9 \text{ dB}, \tag{8}$$

and the worst case phase shift is 90 deg.

The input samples v_l are obtained by sampling an input waveform $w(t)$. It is convenient to represent v_l by means of an inverse Fourier transform,

$$v_l = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega t_l} W(\omega). \tag{9}$$

Then, substituting Eq. (9) into Eq. (6), we find the output spectrum V' in terms of the input spectrum W; i.e.,

$$V'(\omega') = H(\omega') \int_{-\infty}^{\infty} d\omega\, S(\omega',\omega) W(\omega), \tag{10}$$

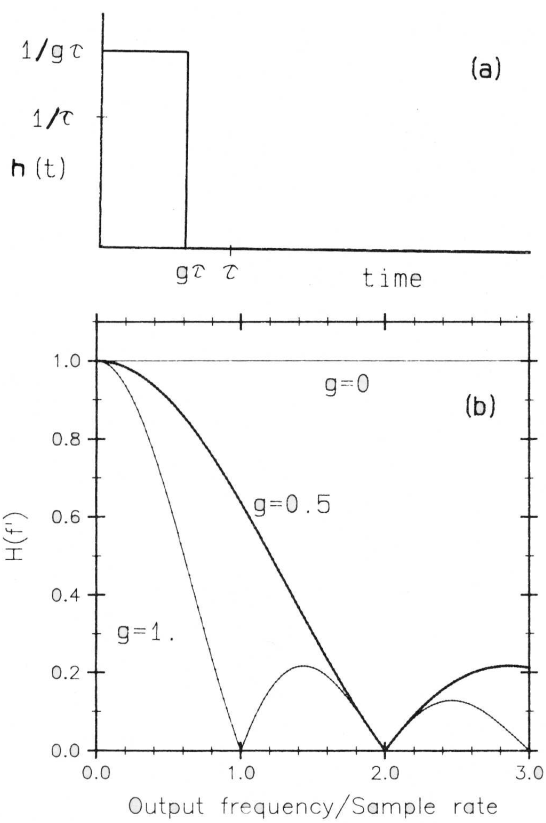

FIG. 3. (a) The sample-hold function h has unit area. It is nonzero for a fraction g of the interval τ' between output samples. (b) The Fourier transform of h is the form factor H. Its absolute value is shown for three different values of g: 0, $\frac{1}{2}$, and 1.

where

$$S(\omega',\omega) = \frac{1}{2\pi} \sum_{m=0}^{\infty} e^{i[\omega\tau l(m) - \omega'\tau' m]}. \tag{11}$$

Function S comprehends the case of ordinary digital recording and reproduction, where input and output sample rates are the same ($\tau = \tau'$), and successive input samples are converted on output, $l(m) = m$. In that case, function S becomes

$$S(\omega',\omega) = S(\omega' - \omega) = \sum_{p=-\infty}^{\infty} \delta(\omega' - \omega - 2\pi pR). \tag{12}$$

The output spectrum V' is the convolution of the input spectrum with the sampling function S, a series of delta functions at multiples of the sample rate. Peaks in the input spectrum appear as satellites to the left and right of the peaks of S. When the output is low-pass filtered with a cutoff at half the sample rate, only the satellites of the $p = 0$ term survive the filtering, and the output spectrum is equal to the input spectrum.

Equation (12) shows that, in ordinary recording and reproduction, function $S(\omega',\omega)$ is a function of the difference $\omega' - \omega$. This is characteristic of a "local" sampling operator. If input and output sample rates are different, then $S(\omega',\omega)$ is a function of $(\omega' - \tau\omega/\tau')$, a simple scaling, and the sampling operator can again be considered to be "local." A similar scaling applies if the increment Δ is an integer. Then $l(m) = m\Delta$ and S is simply a function of $(\omega' - \Delta\tau\omega/\tau')$. But, in the general case of fractional addressing, function $S(\omega',\omega)$ is not only a function of $\omega' - \omega$; the sampling operator is "nonlocal." A scheme for dealing with this nonlocal operator is described next.

B. The unit cell

The address of input sample l, which is converted on output sample m, is given by the function $l(m)$. This is not a trivial function, as is shown in Fig. 2 for the special case of $\Delta = 1.75$. Function $l(m)$ is the integer part of the number $m\Delta$. For successive integer values of m, starting at zero, the values of l are equal to 0,1,3,5,7,8,10,... .

But, although the pattern of l values may appear to be irregular, the pattern must repeat itself after precisely M output samples (recall that γ/M is a reduced fraction). The spacing between successive l values for a fractional part of $\frac{3}{4}$ repeats after four output samples. The basic repetitive pattern is called a "unit cell." Each output sample number m can be represented by a cell number n and a within-cell number μ, which is equal to m modulo M; i.e.,

$$m = nM + \mu \qquad \begin{array}{l} 0 \leqslant n < \infty, \\ 0 \leqslant \mu \leqslant M-1. \end{array} \tag{13}$$

Each input sample number can be represented similarly,

$$l(m) = n\Lambda + \lambda_\mu, \tag{14}$$

where Λ is the number of input samples in the unit cell,

$$\Lambda = MI + \gamma = M\Delta. \tag{15}$$

Thus λ_μ is given by the set of allowed l values modulo Λ.

The representation of Eqs. (13) and (14) allows one to write the sum in Eq. (11) as a product of a periodic part and a sum over the unit cell, containing only M terms; i.e.,

$$S(\omega',\omega) = \frac{1}{2\pi} \sum_{n=0}^{\infty} e^{i(\omega\tau\Lambda - \omega'\tau' M)n} \times \frac{1}{M} \sum_{\mu=0}^{M-1} e^{i(\omega\tau\lambda_\mu - \omega'\tau'\mu)}. \quad (16)$$

The first sum is a series of delta functions; the second sum is the "structure factor" σ; i.e.,

$$S(\omega',\omega) = \sum_{p=-\infty}^{\infty} \delta(\omega'\tau' M - \omega\tau\Lambda - 2\pi p)\sigma(\omega',\omega), \quad (17)$$

where

$$\sigma(\omega',\omega) = \frac{1}{M} \sum_{\mu=0}^{M-1} e^{i(\omega\tau\lambda_\mu - \omega'\tau'\mu)}. \quad (18)$$

The factor $1/M$ in Eq. (16) renormalizes the sum on n to correspond to the number of time points in Eq. (11).

Because the basic periodic unit in time has been expanded by a factor of M, the sample rate is effectively divided by M. The series of delta functions in Eq. (17), and the output spectrum, has peaks that are satellites of integer multiples of R'/M. This is particularly evident if input and output sample rates are the same, i.e., $\tau = \tau' = 1/R'$.

Then,

$$S(f',f) = \sum_{p=-\infty}^{\infty} \delta\left(f' - f\Delta - p\frac{R'}{M}\right)\sigma(f',f). \quad (19)$$

Equation (19) has converted angular frequency to frequency in hertz, and has used the fact that $\Delta = \Lambda/M$. The output spectrum consists of satellites, formed by peaks of the input spectrum, around low-frequency values pR'/M. This effect is shown in Fig. 4, which shows the delta functions in the first zone of the spectrum, from 0 to R'. In Fig. 4(a), the address increment Δ has no fractional part ($M = 1$) and only the input sine component, at frequency f, passes the low-pass filter with cutoff at $R'/2$. In Fig. 4(b), the address increment Δ equals 1.75 ($M = 4$) and the output spectrum contains satellites of $R'/4$. Three distortion components are passed by the low-pass filter. It is clear that if the fractionally addressed digital oscillator is to work without gross distortion, the low-frequency satellites must be wiped out by the structure factor σ.

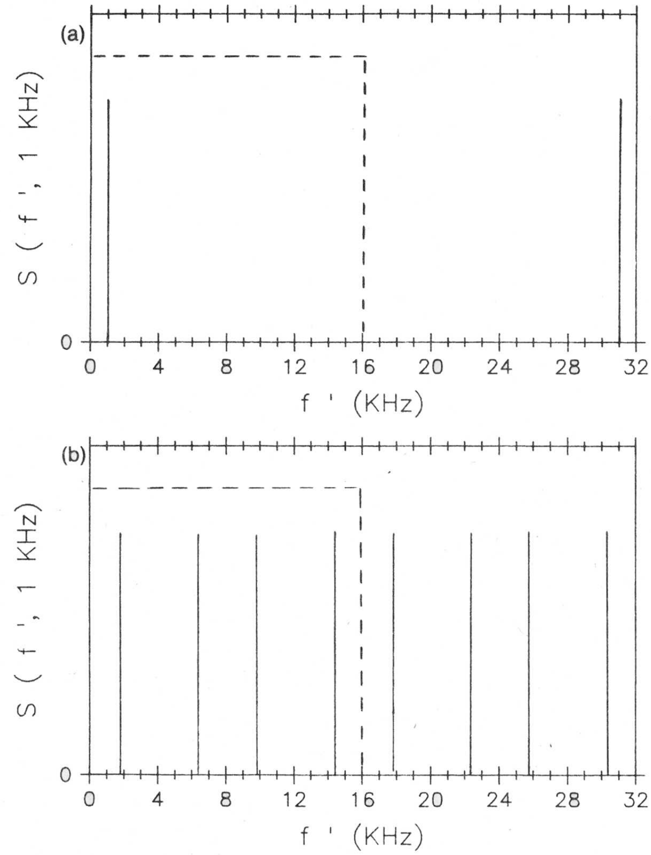

FIG. 4. The series of delta functions from Eq. (19) is the output spectrum if the structure factor $\sigma(f',f)$ equals 1. For both (a) and (b), the input and output sample rates are $R = R' = 32$K, and there are $C = 1$K cycles recorded in a buffer of length $L = 32$K. The figures show the first zone from 0 to 32K. All other zones are identical to the first. Dashed lines show the passband of a low-pass filter set at $R'/2 = 16$K. In Fig. 4(a), the address increment is $\Delta = 1.0$. In Fig. 4(b), the address increment is $\Delta = 1.75$. (Frequency is expressed in units of 1K = 1024 Hz.)

C. The structure factor

The structure factor is given by Eq. (18). It is generally a function of ω' and ω, but, because of the delta function in Eq. (17), $\omega'\tau'$ can be replaced by

$$\omega'\tau' = \omega\tau\Delta + 2\pi p/M. \quad (20)$$

Further, if the input waveform $w(t)$ is a cosine with angular frequency ω_s ($= 2\pi f_s$), then

$$W(\omega) = \pi[\delta(\omega - \omega_s) + \delta(\omega + \omega_s)] \quad (21)$$

and, by Eq. (10), values of ω can be limited to $\pm\omega_s$. It is also convenient to write $\omega_s\tau$ in terms of the number of cycles per buffer, $\omega_s\tau = 2\pi C/L$. We then have

$$\sigma = \sigma(p,\pm) = \frac{1}{M}\sum_{\mu} \exp\left[2\pi i\left(\pm\frac{C}{L}(\lambda_\mu - \Delta\mu) - \frac{p\mu}{M}\right)\right]. \quad (22)$$

Then, for an input cosine, the output spectrum becomes

$$V(f') = H(f') \times \sum_{p=-\infty}^{\infty} \delta\left[f' - \frac{R'}{M}\left(p \pm \frac{C\Lambda}{L}\right)\right]\sigma(p,\pm), \quad (23)$$

where both (+) and (−) signs must be used.

Equation (22) can be simplified by noting that λ_μ is the integer part of $\Delta\mu$. Then the difference $\Delta\mu - \lambda_\mu$ is the fractional part of $\Delta\mu$ or

$$\Delta\mu - \lambda_\mu = (1/M)\langle\gamma\mu\rangle_M, \quad (24)$$

where we use the notation $\langle x\rangle_M$ to indicate the value of x modulo M. The final expression for σ then becomes

$$\sigma = \sigma(\gamma,p,\pm) = \frac{1}{M}\sum_{\mu=0}^{M-1} \exp\left[\frac{-2\pi i}{M}\left(p\mu \pm \frac{C}{L}\langle\gamma\mu\rangle_M\right)\right]. \quad (25)$$

Explicit dependencies are: γ, the numerator of the fractional part; p, the satellite index; and (+) or (−) for satellites to the right or left of frequency $(R'/M)p$.

D. Special case $\gamma=1$

Because of the term in $\langle\gamma\mu\rangle_M$, it is not easy to evaluate the sum in Eq. (25) in closed form. However, a closed-form solution can be obtained in the special case that $\gamma=1$, i.e., when the numerator of the fractional part of Δ equals 1. Then $\langle\gamma\mu\rangle_M$ can simply be replaced by μ. The sum becomes a geometric series, with the result that

$$|\sigma(1,p,\pm)| = \frac{1}{M}\frac{\sin(\pi C/L)}{|\sin[(\pi/M)(C/L\pm p)]|}. \tag{26}$$

E. Symmetries of the structure factor

The general expression for the structure factor in Eq. (25) exhibits symmetries as follows.

(1) Periodic symmetry:

$$\sigma(\gamma,p+M,\pm) = \sigma(\gamma,p,\pm). \tag{27}$$

This symmetry occurs because it is possible to add any multiple of M to p and leave the sum in Eq. (25) invariant. This periodicity is the same as the periodicity that occurs when there is no fractional part in the address increment.

(2) Inversion symmetry:

$$|\sigma(\gamma,-p,\pm)| = |\sigma(\gamma,p,\mp)|. \tag{28}$$

This holds because changing the sign of all the exponents in Eq. (25) leaves the absolute value of the sum unchanged.

(3) Reflection symmetry:

$$|\sigma(\gamma,M-p,\pm)| = |\sigma(\gamma,p,\mp)|. \tag{29}$$

This symmetry is easily proved from symmetries (1) and (2) above. It means that the spectrum in any zone $(0\leqslant p<M)$ is symmetrical about the center of the zone at $p=M/2$.

(4) Zone-center invariance A:

If M is even, then

$$\sigma[\gamma,(M/2),\pm] = \sigma[\gamma',(M/2),\pm] \quad \text{for all } \gamma,\gamma'. \tag{30}$$

This invariance says that the value of the structure factor, when p is at the center of the zone, is independent of the numerator γ. To prove this symmetry, we first note that

$$\sigma\left(\gamma,\frac{M}{2},\pm\right) = \frac{1}{M}\sum_{\mu}(-1)^{\mu}e^{\pm(C/L)\langle\gamma\mu\rangle_M}. \tag{31}$$

The next step requires theorem A from Appendix A. Theorem A says that for γ and M relatively prime, the set of integers $\{\langle\gamma\mu\rangle_M\}$ is independent of γ. Therefore, Eq. (31) is of the form

$$\sigma\left(\gamma,\frac{M}{2},\pm\right) = \frac{1}{M}\sum_{\mu_1}(-1)^{\mu_2}e^{\pm(C/L)\mu_1}, \tag{32}$$

where, for each value of μ_1, the value of μ_2 is chosen such that

$$\langle\gamma\mu_2\rangle_M = \mu_1. \tag{33}$$

If M is even, as usual in fractional addressing, then γ must be odd. Then Eq. (33) shows that, for any γ, μ_2 is even (odd) if μ_1 is even (odd). But the terms in Eq. (32) depend upon μ_2 only through its evenness or oddness. Therefore, $\sigma(\gamma,M/2,\pm)$ is independent of γ.

(5) Zone-center invariance B:

Combining symmetry (3) with symmetry (4) leads to an extension of (4), namely,

$$|\sigma[\gamma,(M/2),\pm]| = |\sigma[\gamma',(M/2),\mp]| \tag{34}$$

for all γ,γ'. The above symmetries reduce the number of possible structure factors. For every value of γ there are only M possible values, and there are $M/2$ possible values of γ. The total number of possible structure factors is further limited by the following remarkable theorem.

F. The γp symmetry theorem

The γp symmetry theorem says that

$$\{\sigma(\gamma,p,\pm)\} = \{\sigma(\gamma',p,\pm)\} \tag{35}$$

for all γ and γ'; i.e., the set of structure factors, for given M, is independent of γ.

In particular, the theorem says that once one has calculated the structure factors for the special case $\gamma=1$, using the simple closed-form expression of Eq. (26), one knows what the structure factors will be for any value of γ. One does not immediately know, however, in what order they will appear.

The theorem is proved in Appendix B. The appendix shows that, for any γ,

$$\sigma(\gamma,\langle\gamma p'\rangle_M,\pm) = \sigma(1,p',\pm). \tag{36}$$

Theorem A guarantees that, for any value p, there is a value of p' such that $p=\langle\gamma p'\rangle_M$. Therefore, Eq. (36) covers all possibilities. By working out the values of $\langle\gamma p\rangle_M$, one may learn the order of appearance of the values of $\sigma(1,p,\pm)$ in the sequence of $\sigma(\gamma,p,\pm)$.

II. APPLICATIONS

There are two applications of the fractional addressing technique. The first is a digital oscillator that produces a sine tone or other periodic waveform. In this case, the goal is to minimize the distortion.

The second application is a complex tone generator that makes use of the distortion components introduced by fractional addressing. This application is similar to the FM synthesis technique described by Chowning (1973). Starting with a sine wave in the buffer, the technique provides an efficient way to create complex inharmonic spectra, spectra of bell tones, for example. In this application, the symmetries described above are of particular interest.

Fractional addressing and FM synthesis also share a common liability. Although it is straightforward to calculate a spectrum as a function of the input parameters, it is not at all clear how to invert this relationship. The present article will have little more to say about complex tone synthesis.

A. The digital oscillator

Equation (26) is a simple closed-form formula by which the structure factor can be calculated for the case $\gamma=1$. If the input waveform is a sine or cosine wave, then the structure factor is equivalent to the output spectrum, apart from the form factor H. Because of the γp symmetry theorem, one knows that the spectral levels of distortion products calculated for $\gamma = 1$ actually apply for any value of γ.

In a successful digital oscillator, the level of each distortion component must be small. However, the denominator of Eq. (26) is never greater than 1. Therefore, the numerator must be made small by choosing a small C/L. In that case, to an excellent approximation, the structure factor is given by

$$|\sigma(\gamma,p,\pm)| = 1, \tag{37a}$$

for p an integer multiple of M, and

$$|\sigma(\gamma,p,\pm)| = \frac{\pi C/LM}{|\sin(\pi p'/M)|}, \tag{37b}$$

otherwise, where $\langle \gamma p \rangle_M = p'$.

Therefore, spectral satellites to the left ($-$) and to the right ($+$) of multiples of R'/M are equal. There are only $M/2+1$ different values of σ. Unlike the exact symmetries presented in Sec. I, the ($\pm$) symmetry depends upon C/L being small. This symmetry is nearly exact, however, for any practical digital oscillator.

Equation (37) shows the advantage of long buffers (large L/C). If the buffer is lengthened by a factor of 2, the level of each distortion component is decreased by 6 dB.

The equations in this article depend upon C and L only through their ratio. For the digital oscillator, there is never any advantage to choosing C to be greater than 1. A value of C greater than 1 simply decreases the effective length of the buffer, which decreases the frequency resolution [Eq. (4)] and increases the distortion [Eq. (37)].

Equation (37) shows that the largest distortion component always corresponds to $p' = 1$. Values of this component are given in Table I for several values of L, and for essentially all values of M.

Figure 5 shows the spectrum of a digital oscillator for a simple case: The output sample rate is 16 384 Hz, and there is a single cycle in a buffer of length 2048. The increment is $\Delta = 100 + \frac{1}{8}$ so that the oscillator frequency is 801 Hz. The frequency range in Fig. 4 extends to half the sample rate. The other half of the first zone is the mirror image of Fig. 4 (symmetry 3), and all other zones are identical to the first (symmetry 1). Because the oscillator frequency is less than half the effective sample rate of 16 384/8, all distortion components appear above the oscillator frequency. Because $\gamma = 1$, the spectrum decreases monotonically.

Figure 6 shows the result of increasing the increment to $\Delta = 600 + \frac{3}{8}$, so that the oscillator frequency is 4803 Hz.

TABLE I. Worst case distortion component ($p = 1$) for the case of $C = 1$ cycle in a buffer of length L samples. Levels are given with respect to the level of the signal at the main oscillator frequency, corresponding to $p = 0$.

M	Distortion (dB) $L = 2048$	Distortion (dB) $L = 32\,768$
2	$-$ 62.3	$-$ 86.3
4	$-$ 65.3	$-$ 89.3
8	$-$ 66.0	$-$ 90.0
16	$-$ 66.2	$-$ 90.2
∞	$-$ 66.2	$-$ 90.2

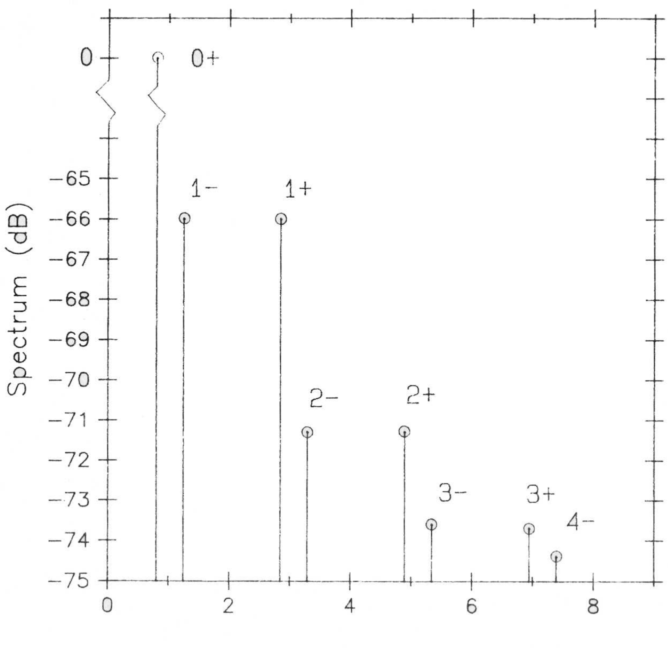

FIG. 5. The spectrum of the lower half of the first zone includes frequencies up to half the sample rate. There is 1 cycle in a 2K buffer. The sample rate is 16 384 Hz, and the increment is $100 + \frac{1}{8}$. The desired sine is thus at 801 Hz. Spectral components are labeled with $p \pm$.

More than half of the distortion components now appear below the oscillator frequency, and, because $\gamma = 3$, the component levels appear to be scrambled. However, the symmetries of Sec. I are sufficient to show that the component levels in Fig. 6 are actually identical to those in Fig. 5.

Figures 5 and 6 and Table I do not include the form factor. Because the form factor decreases monotonically over the frequency range shown, it breaks all the symmetries in the figures, but not by much. Including the form factor

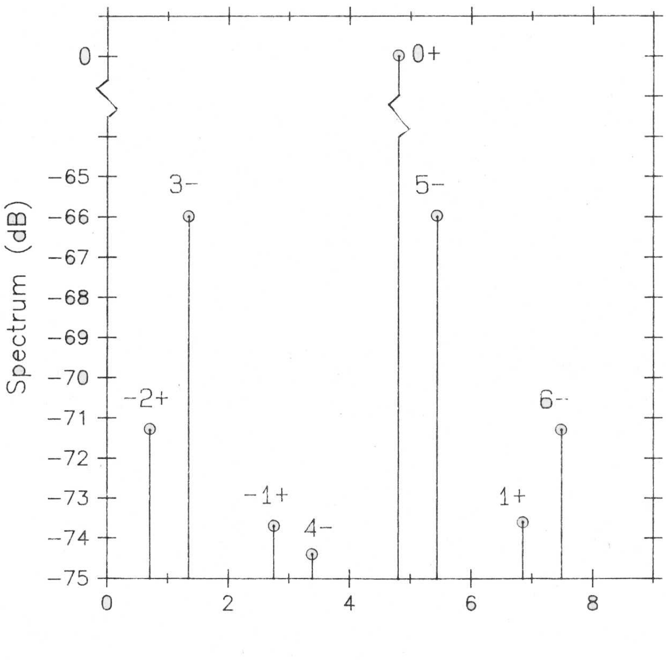

FIG. 6. The same as Fig. 4, except that the increment is $600 + \frac{3}{8}$. Thus the desired sine is at 4803 Hz.

improves the relative distortion figures for Fig. 5, where the oscillator frequency is low. It has the reverse effect if the oscillator frequency is higher than most of the distortion component frequencies.

B. Total distortion

Table I shows that the largest single distortion component occurs for the simplest fractional part, namely, $\gamma/M = \frac{1}{2}$. However, for $M = 2$, there are only two distortion components in the zone, and one of these lies above half the sample rate. For larger values of M, there are more distortion components. One might expect, therefore, that the total distortion increases for increasing M.

The total distortion, relative to the level of the sine at $p = 0$, is given by the sum of the squares of the individual distortion components in the lower half of the first zone, from zero up to half the sample rate. Distortion components at higher frequencies are presumed to be removed by lowpass filtering.

Initially it would seem to be impossible to find a simple form for this sum. If one ignores the slowly varying form factor, however, the symmetries discussed above lead to a very simple result. By $\pm$ symmetry, one knows that there will be two equal distortion satellites for every value of p, except for $p = M/2$, where there is only one. But special treatment for the component at $p = M/2$ is possible, in general, because the zone-center invariance theorem says that the value of this component is independent of γ. Further, γp symmetry ensures that all the other components in the sum are independent of γ. Therefore, the total distortion power (TDP) can be obtained simply by summing the components given by Eq. (26); i.e., for any γ, and C/L small,

$$\mathrm{TDP} = \left(\frac{C}{L}\right)^2 \left(\frac{\pi}{M}\right)^2 \left\{1 + 2 \sum_{p=1}^{M/2-1} \left[\sin\left(\frac{p\pi}{M}\right)\right]^{-2}\right\}. \tag{38}$$

It is convenient to think of TDP as the product of $(C/L)^2$ and a factor depending only on M. Then, for $C = 1$, the TDP in decibels is given by

$$\mathrm{TDP(dB)} = Q(M) - 20\log(L). \tag{39}$$

Function $Q(M)$ is given in Table II. For example, for the cases of Figs. 5 and 6, where $M = 8$ and $L = 2048$,

$$\mathrm{TDP(dB)} = 5.1 - 66.2 = -61.1\ \mathrm{dB}. \tag{40}$$

As expected, Table II shows that the total distortion does increase slightly for increasing M.

C. Complex waveforms

Calculations of the levels of the individual distortion components and of the total distortion for a complex waveform with harmonic partials are rather straightforward because of the linearity of the general formalism. If there is 1 cycle of the waveform (i.e., of the fundamental) in the buffer, then there are 2 cycles of the second harmonic, etc. Generally, then, the levels of the distortion components attributable to the nth harmonic are given by Eq. (26) with C replaced by nC. The distortion spectrum can be computed in this simple way, except in the unlikely case of accidental frequency degeneracies.

TABLE II. Total distortion power term Q as a function of the denominator of the fractional part of the address increment.

M	Q(dB)
2	3.92
4	4.89
8	5.10
16	5.15
32	5.17
∞	5.17

A distortion component must also be weighted by the amplitude of the harmonic n that produces the component. In the case that the amplitudes of the harmonics are all the same, a component, labeled by $p, \pm$, increases with increasing n at a rate of 6 dB per octave. In this case, the total distortion, relative to the total waveform power, is equal to that for the fundamental alone multiplied by $(N+1)(2N+1)/6$, where N is the number of harmonics. The above expression is valid so long as NC/L is small. For most practical waveforms, however, the harmonic amplitudes eventually decrease with increasing harmonic number and the actual total distortion is less than given above.

D. An alternative calculation of total distortion

In this article, the total distortion due to fractional addressing was computed by summing the individual components in the distortion power spectrum. An alternative calculation of the total distortion can be done in terms of the waveform. Here, one sums the squared differences between the desired waveform at successive instants in time and the actual values that come from the fractional addressing scheme. Moore (1977) has performed such calculations for digital oscillators in which the fractional address is either truncated, rounded to the nearest integer, or used to interpolate between successive input samples. Moore's calculations show that the total distortion is least for the interpolating oscillator and greatest for the truncating oscillator. However, Rossum (1985) has made the (somewhat counterintuitive) observation that both truncating and rounding actually lead to the same total distortion. This occurs because rounding is equivalent to adding a constant of 0.5 to each fractional address and then truncating. Therefore, the two procedures should differ only by a phase shift in the output, which does not affect the levels of the distortion components. The present work confirms Rossum's observation; calculations of the total distortion from Eq. (25) are unchanged if a constant is added to $\gamma\mu$.

Therefore, Moore's calculation for rounding oscillator is the correct approach for the distortion in both rounding and truncating oscillators, and this calculation should agree with Eq. (39) above. A comparison shows that the two do agree to within 1 dB, except for a buffer length $L = 32$ where Moore's calculated distortion is 1.4 dB too large.

III. CONCLUSIONS

The technique of fractional addressing makes it possible to build an efficient digital oscillator. With a single cycle of a waveform fixed in a buffer memory and with a fixed sample rate one can obtain arbitrarily good frequency resolution over the entire frequency range up to half the sample rate. The resolution is given by the sample rate divided by $2N_A$, where N_A is the number of bits in the address register [Eq. (4)]. The cost of adding more bits to the register is negligible. For most purposes, there is probably little point in going beyond 24 bits, because the corresponding frequency resolution is comparable to the stability of the crystal oscillator.

The fractional addressing technique introduces distortion, because an address increment with a fractional part equal to γ/M effectively reduces the sample rate by a factor of M. Sections I and II of this article show how the spectrum of the distortion components can be calculated. The distortion spectrum depends upon the "structure factor," which relates the input spectrum to the output spectrum. Calculations are particularly simple if the input spectrum is a sine wave, but the relation is a linear one, and the formalism may be used to calculate the output spectrum for a complex input waveform by superposition. With the right structure factor, the distortion components can all be reduced to acceptable levels.

The structure factor is nonlocal, but it exhibits a number of symmetries that simplify calculations using it. The most remarkable of these is the γp symmetry theorem, which says that the set of levels of the distortion components is independent of the numerator of the fractional part of the increment, and depends only upon the denominator. The symmetries make it possible to calculate both the level of the largest distortion components and the level of the total distortion power. The result for the total distortion power is a simple one. Indeed, for some purposes, the entire contents of this article can be summarized by the statement that the total distortion power, in decibels relative to the power of the desired signal, is equal to

$$5 - 20\log(L), \tag{41}$$

where L is the length of the buffer. As shown in Table II, this expression is good to within 1 dB, and it applies *for any fractional address*.

The design of a digital oscillator is thus mainly a choice of buffer length. For signal-to-noise ratios typical of analog systems, about 60 dB, a buffer length of 2 or 4K words is adequate. To maintain consistency with 16-bit digital audio systems (theoretically 96 dB), a buffer length of 32 or 64K is required.

It seems evident that, whenever a theory exhibits symmetries that lead to a result of such dazzling simplicity as Eq. (41), there must then be a higher symmetry operating. In the present case, that higher symmetry is almost surely the law of conservation of energy, but we have been unable to make the theoretical connections necessary to use it.

A few caveats are also in order. The presence of the form factor complicates matters because it breaks all the symmetries. The main practical result of this is that the distortion now depends somewhat (± 4 dB) on the frequency of the oscillator output and, in particular, on the integer part of the address increment. By contrast, the structure factor is entirely independent of the integer part. The form factor is actually not present if the DAC output is a spike train instead of a staircase. The form factor is similarly irrelevant if the output filter includes $\sin(x)/x$ correction.

This article has, of course, considered only an "unintelligent" signal-generating system. Samples were recorded into the buffer and subsequently simply sent to the DAC, without interpolation. Commercially available arithmetic processors can perform interpolation as well as other signal control functions, such as amplitude control. Distortion figures in intelligent systems are not limited by the calculations in this article. However, if local intelligence is not needed for other functions, it may be more cost effective to build an unintelligent digital oscillator and pay the price of a longer buffer, as described, for example, by Eq. (41).

Note added in proof: After the present article was accepted, Robert Maher discovered the article "Noise spectra of digital sine generators using a table lookup method," by S. Mehrgardt [IEEE Trans. Acoustics, Speech Signal Process. **ASSP-31**, 1037–1039 (1983)]. Although the final equations are somewhat difficult to interpret, the article by Mehrgardt appears to include some of the results of the present work.

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APPENDIX A: PROOF OF THEOREM A

Theorem A is as follows. "If γ and M are relatively prime integers, then the set of integers given by

$$\text{set } \gamma = \{\langle\gamma\mu\rangle_M\}, \quad \mu = 0,1,2,...,M-1$$

is identical to

$$\text{set } 1 = \{0,1,2,..., M-1\}."$$

Proof: Because the notation $\langle\gamma\mu\rangle_M$ means $\gamma\mu$ modulo M, the only possible members of set γ are the integers zero through $M-1$.

To show that set γ is equal to set 1, it is sufficient to show that no two members of set γ can be the same. To do this, we suppose that the contrary is true, namely, that there are two values of μ, μ_1, and μ_2, ($\mu_1 \neq \mu_2$), such that

$$\langle\gamma\mu_1\rangle_M = \langle\gamma\mu_2\rangle_M. \tag{A1}$$

Then there must be some integer K, such that

$$\gamma(\mu_1 - \mu_2) = MK. \tag{A2}$$

A central theorem of number theory (Schroeder 1986), however, states that if γ and M are relatively prime, then Eq.

(A2) can hold only if $(\mu_1 - \mu_2)$ is an integer multiple of M. But μ_1 and μ_2 are both in the range zero to $M-1$; hence, their difference can never be an integer multiple of M. Therefore, Eq. (A1) cannot hold, and all the members of set γ must be distinct. Therefore, set γ must be equal to the set of successive integers from zero to $M-1$.

Q.E.D.

APPENDIX B: PROOF OF THE γ-p SYMMETRY THEOREM

The theorem applies to Eq. (25),

$$\sigma(\gamma,p,\pm) = \frac{1}{M}\sum_{\mu=0}^{M-1} \exp\left[\frac{-2\pi i}{M}\left(p\mu \pm \frac{C}{L}\langle\gamma\mu\rangle_M\right)\right]. \tag{B1}$$

For convenience, we replace notation $\langle\ \rangle_M$ by $\langle\ \rangle$ below. The theorem is as follows.

"For any values of γ and p_2 there exists a p_1 such that

$$\sigma(\gamma,p_2,\pm) = \sigma(1,p_1,\pm)." \tag{B2}$$

To prove this theorem, we consider a particular p_2 given by $\langle\gamma p_1\rangle$. (Theorem A shows that, for any value of p_2, a value of p_1 exists.) Then

$$\sigma(\gamma,p_2,\pm) = \frac{1}{M}\sum_{\mu_1=0}^{M-1} \exp\left[\frac{-2\pi i}{M} \times\left(\langle p_1\gamma\rangle\mu_1 \pm \frac{C}{L}\langle\gamma\mu_1\rangle\right)\right]. \tag{B3}$$

By theorem A, the sum over μ_1 is equivalent to a sum over μ_2, where

$$\mu_2 = \langle\gamma\mu_1\rangle.$$

We may also replace $\langle p_1\gamma\rangle\mu_1$ by $\langle p_1\gamma\mu_1\rangle$ because of the factor $2\pi i/M$. Then

$$\sigma(\gamma,p_2,\pm) = \frac{1}{M}\sum_{\mu_2=0}^{M-1} \exp\left[\frac{-2\pi i}{M}\left(\langle p_1\gamma\mu_1\rangle \pm \frac{C}{L}\mu_2\right)\right]. \tag{B4}$$

Now $\langle p_1\gamma\mu_1\rangle = \langle p_1\langle\gamma\mu_1\rangle\rangle = \langle p_1\mu_2\rangle$.

Hence, we have

$$\sigma(\gamma,p_2,\pm) = \frac{1}{M}\sum_{\mu_2=0}^{M-1} \exp\left[\frac{-2\pi i}{M}\left(\langle p_1\mu_2\rangle \pm \frac{C}{L}\mu_2\right)\right]. \tag{B5}$$

But the right-hand side of Eq. (B5) is simply equal to $\sigma(1,p_1,\pm)$, where p_1 is chosen to be such that

$$\langle\gamma p_1\rangle = p_2. \tag{B6}$$

Q.E.D.

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