Noise power fluctuations and the masking of sine signals

W. M. Hartmann and Jon Pumplin
Physics Department, Michigan State University, East Lansing, Michigan 48824-1116

(Received 12 January 1987; accepted for publication 22 December 1987)

This article is concerned with fluctuations in noise power and with the role that such fluctuations play in the masking of sine signals by noise. Several measures of noise fluctuations are discussed: the fourth moment of the waveform, the fourth moment of the envelope, and the crest factor. Relationships among these quantities are found for cases of equal-amplitude random-phase noise and Rayleigh-distributed-amplitude noise. Of particular interest is a special non-Gaussian noise called low-noise noise in which the fluctuations are small by any of our measures. The results of frozen-noise masking experiments are reported, where the noise waveform was fixed for all stimulus presentations. In separate experiments, equal-amplitude random-phase Gaussian noise, with typical fluctuations, and low-noise noise, with almost no fluctuations were used. The data show that for a noise bandwidth less than the critical bandwidth, the masked threshold is about 5 dB lower for low-noise noise than for Gaussian noise. When the noise bandwidth is larger than the critical bandwidth, the masked threshold is the same for both kinds of noise. It is concluded that noise power fluctuations increase masked threshold by about 5 dB and that filtering by the auditory system reintroduces fluctuations into broadband low-noise noise.

PACS numbers: 43.66.Dc, 43.66.Nm, 43.60.Cg [WAY]

INTRODUCTION

The problem of the detection of a sine signal in masking noise can be approached in terms of signal detection theory (cf. Green and Swets, 1966). According to the theory listeners make their decisions on the basis of a likelihood ratio, the ratio of the probability that the stimulus comes from a population of signal-plus-noise stimuli to the probability that the stimulus comes from a population of noise-alone stimuli. In principle, the probabilities might depend upon a number of dimensions of the stimulus (Ahumada and Lovell, 1971; Ahumada *et al.*, 1975). If the phase of the signal is unknown by the listener, and if the signal cannot be spectrally resolved from the noise, then total stimulus energy is a basis for an optimal detection strategy (Bos and deBoer, 1966). In this case, signal detection theory becomes an energy detection theory, whereby the listener's performance is limited by the variance of a probability distribution for an internal representation of energy or for some quantity that increases monotonically with increasing energy.

In a typical masking experiment the noise is created by a thermal noise generator. Because the output power of the generator fluctuates, the energy in the stimulus varies from one experimental presentation to the next. If the noise has a Gaussian distribution, then energy detection theory makes a strong prediction for detectability: For a d' of unity the detectable signal energy is approximately equal to the noise power density multiplied by the square root of the product of the noise bandwidth and the stimulus duration. The latter quantities are normally well defined physically, but the values corresponding to the internal representation may be influenced by the critical bandwidth and the auditory integration time. Incorporating these elements of the auditory system into the energy detection model introduces some uncertainty concerning the predictions of the model. However, the sum of the evidence is that the model predicts performance that exceeds the performance of human listeners (Green, 1960; Raab and Goldberg, 1975). The model can be brought into agreement with experiment by introducing the concept of an internal noise, statistically independent of the external noise, which adds a further source of variance to the internal distribution (de Boer, 1966; Green and Swets, 1966). To fit the data usually requires that the internal noise be as large as or several times larger than the external noise (Green, 1964; Raab and Goldberg, 1975; Gilkey, 1981).

An alternative kind of masking experiment replaces the thermal noise generator with a source of frozen noise. With frozen noise the masker waveform is the same on each experimental interval, there is no stimulus variance, and detection performance should be limited by internal noise alone. If internal noise does in fact exceed the external noise, one expects that this replacement should have little effect on detection thresholds. Experimentally it is found that thresholds are actually reduced when frozen noise is used (Pfafflin, 1968), though the amount of the reduction varies with different noise samples and with the relative phases of the signal (Hanna and Robinson, 1985).

The distinction between external noise and internal noise becomes blurred when stimulus intervals are longer than the auditory integration time. In that case the listener must sample the stimulus. Variations in sampling strategy, which is a form of internal noise, lead to variations in the internal representation of the noise energy, and these depend upon the momentary fluctuations in the stimulus power. In the general case it may be possible to reduce thresholds further by using non-Gaussian noise in which the fluctu-

 0001-4966/88/062277-13$00.80

ations have been reduced. This is not difficult to do. A random telegraph wave, namely, a random sequence of $+1$ and -1, is an example of a waveform with zero power fluctuation (Feth, 1970). Of course, such a noise has a broad bandwidth. If one tries to control the bandwidth by filtering, then power fluctuations are reintroduced. A noise generated by frequency modulating a tone also has no fluctuation in envelope power. Margolis and Small (1974) attempted to control both power fluctuation and bandwidth by frequency modulating a tone with thermal noise that had been peak clipped. There are two sources of band broadening in such a waveform, the harmonic distortion introduced by the clipping and the higher-order FM sidebands. It was apparently possible though to reach some compromise in this case between power fluctuation and bandwidth control. Margolis and Small found that masked thresholds within the noise passband were 5 to 10 dB smaller for the FM noise than for thermal noise with the same power.

The present article is also concerned with the role that noise fluctuations play in masking. Section I compares several different measures of noise fluctuation and briefly describes the low-noise noise algorithm developed by one of us (Pumplin, 1985). The algorithm generates noise with very small fluctuations according to any of our measures, approaching those for FM, while maintaining strict control over the bandwidth. Sections II and III describe masking experiments in which masking by low-noise noise is compared with masking by noises having statistically probable fluctuations.

I. NOISE FLUCTUATIONS

There are a variety of different measures of noise fluctuations. The purpose of this section is to discuss and compare some of them. The comparisons are based upon studies of bandlimited periodic waveforms with period T. Such a noise waveform can be written as the sum of cosine components, separated in frequency by the reciprocal of T:

$$x(t)=\frac{1}{\sqrt{N}}\sum_{n=N_1}^{N_2}X_n\cos\left(\frac{2\pi nt}{T}+\phi_n\right),\tag{1}$$

where ϕ_n is the phase angle of the nth component, and the number of components is $N=N_2-N_1+1$.

Below we will consider moments of the waveform; the mth moment is defined as

$$\overline{x^m}=\frac{1}{T}\int_0^T x^m(t)dt.\tag{2}$$

The second moment ($m=2$) is the average power, which is independent of phases. If all component amplitudes X_n are equal to unity, the second moment is 1/2. The odd moments of x are zero, and the higher even moments describe fluctuations.

A. The waveform fourth moment

The fourth moment of the waveform is a measure of fluctuation that is closely related to the variance in instantaneous power. It is convenient to work with a fourth moment that is normalized by the square of the average power, namely,

$$W=\overline{x^4}/(\overline{x^2})^2.\tag{3}$$

The variance of the power, as a fraction of the square of the average power, is equal to $W-1$. The square root of $W-1$ is the standard deviation of the instantaneous power, in units of the average power.

The ensemble-averaged value of W, for a waveform with N components of equal amplitude, can be calculated exactly. It is given by

$$\langle W\rangle=3-3/(2N).\tag{4a}$$

If the component amplitudes are Rayleigh distributed, then the ensemble-averaged W is equal to

$$\langle W\rangle=3-3/(N+1).\tag{4b}$$

Equations (4a) and (4b) are proved in Appendix A. In the limit of large N they are the same, as expected, because in this limit, equal-amplitude noise and Rayleigh-amplitude noise have the same statistical properties (Rice, 1954).

We have studied the distribution of W for a variety of random-phase noises of the type given by Eq. (1) by performing numerical experiments, summarized in Table I. Each entry in this table was obtained by generating an ensemble of 2000 waveforms, using equal amplitudes and random phases which were computed anew for each waveform. The computed mean agrees with Eq. (4a). Table I also gives the standard deviation about the mean and the median value of W. The median is less than the mean because the distribution of W has a tail to the right; plots of this distribution were given by Hartmann (1987). The quantity W is the basis for the low-noise noise algorithm (Pumplin, 1985) used to calculate noise maskers for the experiments described in Sec. II.

B. The envelope fourth moment

The fourth moment of the envelope provides an alternative measure of noise power fluctuations. For the waveform of Eq. (1) the envelope is

$$E(t)=\left|\frac{1}{\sqrt{N}}\sum_{n=N_1}^{N_2}X_n e^{i(2\pi nt/T+\phi_n)}\right|.\tag{5}$$

The envelope power is given by the square

$$E^2(t)=\frac{1}{N}\sum_{n=N_1}^{N_2}X_n^2+\frac{2}{N}\sum_{n=N_1}^{N_2}\sum_{n'=N_1}^{n-1}X_nX_{n'}\times\cos[2\pi(n-n')t/T+\phi_n-\phi_{n'}].\tag{6}$$

Its time average is equal to twice the average waveform power, while the sum over frequency differences describes instantaneous fluctuations.

It is useful to compare the fluctuation in the envelope power with the fluctuation in the waveform power discussed in Sec. I A. As shown in Eq. (6), the envelope fluctuation involves only differences between the component frequencies. For example, for $N=2$ the envelope fluctuation corresponds to the familiar beats at the difference frequency. When there are many components, the envelope fluctuation derives from beats among all possible pairs of components. By contrast, the fluctuation in waveform power involves both frequency differences and frequency sums.

In the standard model of energy detection, a temporal integrator follows the square-law device (Green and Swets,

TABLE I. Properties of the distributions for the normalized fourth moment (W) and crest factor (C) for equal-amplitude random-phase noise, as described by Eq. (1). The lowest frequency spectral component is N_1; the highest is N_2. The distributions were determined by numerical experiments with 2000 noise waveforms. Entries labeled "Equations" were calculated from Eqs. (4a), (C6), (C4), and (C5) of the text, using $K = 8N/3$.

N_1	N_2	$N = N_2 - N_1 + 1$	$M = (N_1 + N_2)/2$	$W = \overline{x^4}/(\overline{x^2})^2$			$C = \max\lvert x\rvert/(\overline{x^2})^{1/2}$		
				Median	Mean	s.d.	Median	Mean	s.d.
1	11	11	6	2.73	2.86	0.71	2.52	2.55	0.35
15	25	11	20	2.73	2.86	0.62	2.67	2.72	0.35
495	505	11	500	2.70	2.86	0.67	2.72	2.78	0.37
995	1005	11	1000	2.70	2.86	0.67	2.72	2.78	0.37
Equations		11			2.86		2.73	2.79	0.42
1	51	51	26	2.90	2.97	0.38	3.02	3.06	0.36
75	125	51	100	2.91	2.97	0.30	3.21	3.25	0.34
475	525	51	500	2.92	2.97	0.33	3.24	3.29	0.36
975	1025	51	1000	2.92	2.97	0.33	3.24	3.29	0.36
Equations		51			2.97		3.25	3.30	0.37
1	151	151	76	2.97	2.99	0.23	3.35	3.39	0.35
125	275	151	200	2.97	2.99	0.20	3.49	3.55	0.35
425	575	151	500	2.97	2.99	0.20	3.54	3.60	0.35
925	1075	151	1000	2.97	2.99	0.20	3.54	3.61	0.35
Equations		151			2.99		3.57	3.61	0.34

1966, Chap. 8). Fluctuations in the output of the integrator come only from the frequency difference terms; frequency sum terms vary too rapidly to appear as fluctuations in the output. Therefore, the envelope fourth moment is a description of fluctuation which is more consistent with the standard model than the waveform fourth moment. The calculation of noise fluctuations by Hartmann *et al.* (1986) was based on the fourth moment of the fluctuation in the envelope, as passed through an integrator, a model similar to Jeffress (1968).

The fourth moment of the envelope, normalized by the square of the envelope power, is defined as

$$Y = \overline{E^4}/(\overline{E^2})^2. \tag{7}$$

The variance of the envelope power, as a fraction of the average envelope power, is given by $Y - 1$. The standard deviation is $(Y-1)^{1/2}$.

The ensemble-averaged value of Y, for a waveform with N components of equal amplitude, can be calculated analytically:

$$\langle Y \rangle = 2 - 1/N. \tag{8}$$

For a sine waveform ($N = 1$) the ensemble-average variance is zero, as expected. The ensemble-averaged standard deviation of the envelope power E^2 is always less than the envelope power itself, though it approaches the envelope power for large N.

There is a useful theorem which applies in the case that the noise is narrow band. In that case, the envelope fourth moment and the waveform fourth moment are related by a simple factor, for any waveform of the form of Eq. (1). The result, proved in Appendix B, is that

$$Y = \tfrac{2}{3} W. \tag{9}$$

Equation (9) holds true as long as the total bandwidth is less than the center frequency of the band. Unlike other conclusions in this section, which apply to ensemble averages, Eq. (9) applies to any individual waveform. Because the envelope, and in particular its fourth moment, is independent of the band center frequency, the fourth moment of the waveform is also independent of the center frequency, as long as the narrow-band condition is met.

C. The crest factor

The crest factor, or peak factor, is the ratio of the absolute maximum (or minimum, whichever has the larger absolute value) of a waveform to its root-mean-square value. It is a common way to describe fluctuations in engineering applications. In electronic circuits employing junction transistors, hard clipping of the waveform begins at a fixed voltage level. Therefore, the crest factor gives a useful insight to the power which can be transferred without gross distortion. For the human auditory system, where the onset of distortion is more gradual, it may be a less useful measure. Schroeder's algorithm (1970) for synthesizing waveforms with small fluctuations was based upon considerations of the crest factor.

There are few things which can be said analytically about the crest factor: For a sine wave it is equal to $\sqrt{2}$. For a waveform with N sine components of equal amplitude, the maximum possible crest factor is $(2N)^{1/2}$, which is realized when all the values of ϕ_n in Eq. (1) are zero. Crest factors for random-phase noise are considerably less than this maximum value.

Numerical experiments on the distribution of crest factors for equal-amplitude random-phase noise are summarized in Table I. The crest factor for each waveform was calculated by using a fast Fourier transform (FFT) to determine the approximate location of each local maximum in $x(t)$, followed by a gradient search to obtain its precise value. The largest of the local maxima gives the crest factor.

Table I shows that, for narrow bands, the crest factor depends only upon the number of components N and not upon the center frequency. It also shows that the crest factor increases slowly with increasing N. These results can be understood from the discussion of the distribution of crest factors in Appendix C. There it is shown that statistical properties of the crest factor depend upon the square root of the log of N, a slow dependence indeed.

D. Fourth moment and crest factor compared

To compare the fourth moment with the crest factor, we performed numerical experiments for equal-amplitude random-phase waveforms with $N = 51$ and $N = 151$ components. The results are shown in Figs. 1 and 2, respectively. Each graph shows 2000 dots; each dot corresponds to one waveform, i.e., one set of the random phases. The y coordinate of the dot is the crest factor and the x coordinate is the fourth moment. The two-dimensional distribution has an elliptical shape, with an orientation which shows that there is a tendency for a waveform which has a high crest factor also to have a high value of the fourth moment. The correlation coefficient is 0.84 for $N = 51$ and 0.75 for $N = 151$.

In a moment expansion of the fluctuation, the fourth moment of the waveform is the moment of lowest order, while the crest factor is determined by the infinite-order moment. [This can be seen from Eq. (2) for the mth moment. As m increases, the integral is increasingly dominated by extreme values of $x(t)$. As m becomes infinite, only the largest value, essentially the crest factor, contributes to the integral.] The fact that the crest factor and the fourth moment correlate rather well in the numerical experiments leads us to conjecture that moments of all orders between the lowest and the highest should also correlate well. This means that a noise waveform that appears to have a large (small) fluctuation according to one measure of noise power fluctuations will tend to have a large (small) fluctuation according to any measure. This conclusion applies to noise made by adding sine waves closely spaced in frequency, where the waveform distribution is approximately Gaussian. It does not apply to non-Gaussian noises such as the inverted triangular distribution (small crest factor but large fourth moment) used by Sorkin *et al.* (1979).

E. Low-noise noise

The generation of signals with small fluctuations is a problem of long-standing interest to communications engineers. The increasing use of digital representations for signals has increased the significance of this problem. In 1970, Schroeder presented an algorithm for calculating waveforms with low crest factors. For the case of equal-amplitude noise the algorithm is extremely simple to describe. One chooses the phase angles according to the formula

$$\phi_n = \phi_0 - \pi n^2/N, \tag{10}$$

where ϕ_0 is an arbitrary constant.

However, the waveform created by this algorithm is peculiar in that it does not sound like noise. As noted by Schroeder, it is similar to a swept sine wave, beginning at the lowest frequency of the band and arriving, near the end of the waveform period, at the highest frequency of the band. If the components of the band are separated by 1 Hz then the duration of the sweep is 1 s. The response of a tuned system to this waveform shows one peak per waveform period, occurring at a time when the instantaneous frequency of the waveform agrees with the resonant frequency of the system. If this waveform is used to mask a sine signal in the band, then detection of the signal is aided by beats when the instantaneous masker frequency is close to the signal frequency.[1]

An alternative choice is the low-noise noise algorithm (Pumplin, 1985), which minimizes the fourth moment of the waveform (equivalent to minimizing the standard deviation in power) with respect to the phases ϕ_n. The algorithm is tractable computationally because it is possible to calculate all necessary derivatives of the fourth moment for use in a gradient search, by using a single FFT. The algorithm results in a waveform with a fluctuation which is almost zero, smaller in fact than the fluctuation resulting from Schroeder's algorithm; and the waveform is free of the notable regularity found in Schroeder's waveform. Symbol NL in Fig. 1 and symbol WL in Fig. 2 show the crest factors and fourth moments for low-noise noises used in the following experiments.

II. MASKING EXPERIMENT METHOD

The purpose of the experiment was to determine the contribution which noise power fluctuations make to masking. We asked listeners to detect a 1000-Hz sine tone in the presence of frozen masking noise. We compared detection performance for maskers made with random-phase noise, in which the power fluctuations are often large, with performance for low-noise noise, in which the power fluctuations are nearly zero.

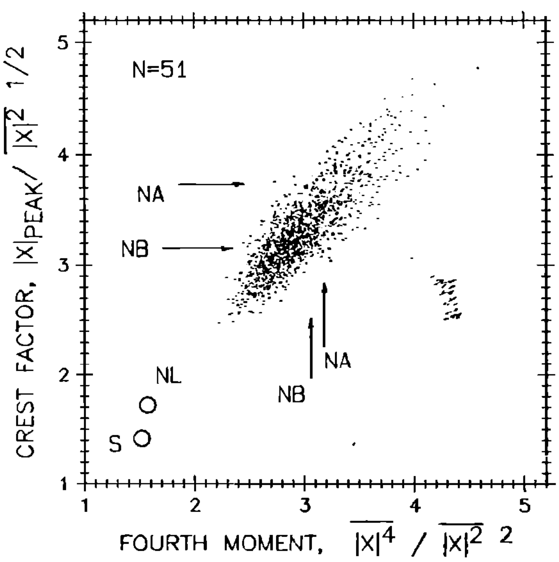

FIG. 1. Scatter plot showing the crest factor and the waveform fourth moment for 2000 narrow-band noise waveforms with 51 components with frequencies from 950 to 1050 Hz. Noises NA and NB from the masking study appear at the intersection of the corresponding vertical and horizontal arrows. Low-noise noise (NL) and a sine wave (S) are shown by circles.

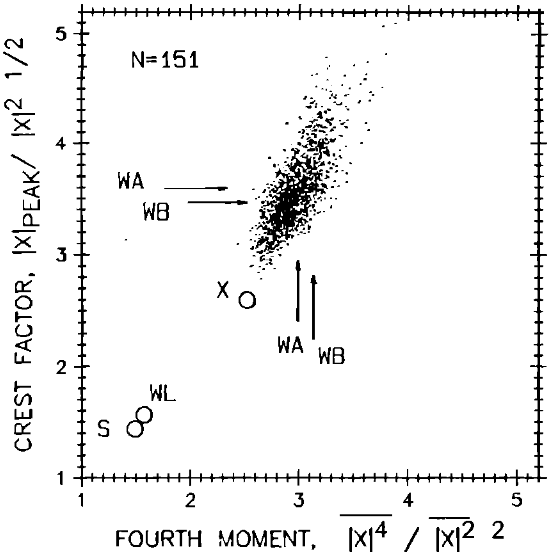

FIG. 2. Same as Fig. 1 but for wideband noise waveforms with 151 components from 850 to 1150 Hz. Noises WA and WB are shown by arrows. Low-noise noise (WL) and a sine wave (S) are shown by circles. Low-noise noise passed through a rectangular critical-band filter is labeled by an X.

A. Stimuli

Both the signal and the noise were generated digitally from sound files, 2048 samples long, with a sample rate of 4096 Hz. The sounds were converted by 12-bit DACs and low-pass filtered from 1.8 kHz at -115 dB per octave. Subjects were seated in a sound-attenuating room and listened to the stimuli diotically through Yamaha YH-1000 headphones. The noise level was always 60-dB SPL.

There were six different noises, all centered on 1000 Hz. Three of them had a bandwidth of 100 Hz (spectrum level of 40 dB). These noises were called "narrow" and were given the names NA, NB, and NL. The other three had a bandwidth of 300 Hz (spectrum level of 35 dB); they were called "wide" and were given the names WA, WB, and WL. The narrow-band noises had 51 sine components separated by 2 Hz, i.e., 950,952,...,1050 Hz. The wide noises had 151 components, also separated by 2 Hz, 850,852,...,1150 Hz. The repetition time for the noises was, therefore, 500 ms, but the listeners were not aware of the periodicity because the experimental stimulus intervals were slightly shorter than 500 ms.

Noises NA, NB, WA, and WB were made by adding components with equal amplitudes and with phases chosen by a random number generator (0 to 360 degrees). Noises NL and WL were made by adding components with equal amplitudes and with phases chosen by the low-noise noise algorithm. Table II shows the crest factors and the fourth moments for these noises. In informal listening tests, the low-noise noises could easily be distinguished from the random-phase noises in an A–B comparison. For the former, the loudness fluctuations were less prominent and momentary pitch fluctuations were more prominent. However, the low-noise waveforms did sound "noisy," and not at all like a swept sine.[2] The values of 1.58 and 1.60 for the normalized fourth moments of NL and WL come close to the absolute lower limit, 1.50, for narrow-band signals. Hence, the fluctuation in envelope power was very small $\approx \pm 1$ dB.

Masking experiments using frozen noise waveforms, such as ours, generally find that masked thresholds depend upon the starting phase of the sine signal (Dolan *et al.*, 1981; Gilkey *et al.*, 1985). To study this effect and to eliminate it in our final results we used six different 1000-Hz sine signals, having starting phases which were integral multiples of 60 degrees. For signal number j, the starting phase in degrees was $\phi_j = 60j$ ($j = 1,2,...,6$).

B. Procedure

The experiment employed a two-interval forced-choice task, with signal and noise on one interval and noise alone on the other. On each interval of the trial the noise waveform was the same. The noise waveform was also the same on each trial of an experimental run. Intervals were indicated by lights on the response box; the listener pressed the corresponding button on the box to record his decision as to which interval included the signal. Feedback was given after each response on training runs; feedback was not given for data runs.

Each experimental trial consisted of an initial delay (700 ms) and two stimulus intervals, each 490 ms, separated by a gap of 250 ms. The response interval was subject controlled. The noise, present on both intervals, was turned on and off with a raised-cosine envelope of 10-ms duration. The signal, present on one interval, was turned on and off together with the noise, but with a 30-ms raised-cosine envelope. The zeros of the 10- and 30-ms envelopes occurred at the same time. Therefore, the signal envelope was entirely enclosed by the noise envelope.

The experimental paradigm was the staircase procedure described by Levitt (1971). After two correct responses the signal level was reduced by 2 dB; after one incorrect response the level was increased by 2 dB. The initial signal level was equal to the noise level, 60 dB SPL. In a given experimental run the subject reversed the direction of the staircase 12

TABLE II. Crest factors and relative fourth moments for the six masking noises used in the masking experiment.

	BW = 100 Hz			BW = 300 Hz		
Name	NA	NB	NL	WA	WB	WL
Crest factor	3.78	3.19	1.75	3.63	3.49	1.58
$\overline{x^4}/(\overline{x^2})^2$	3.19	3.09	1.58	3.01	3.14	1.60

times. The first four reversals were discarded from the data and the remaining eight were averaged to find an estimate of the detection threshold.

Each listener completed four runs for each of the six sine phases for each of the six noises (total of 144 runs). For a given experimental session the noise waveform was always the same. A session began with training runs so that the listener could become familiar with the noise. After training runs seemed to indicate asymptotically good performance, a random list of the six signal phases was made and the listener did two runs for each signal phase on the list. Then the listener did two more runs for each signal phase beginning again at

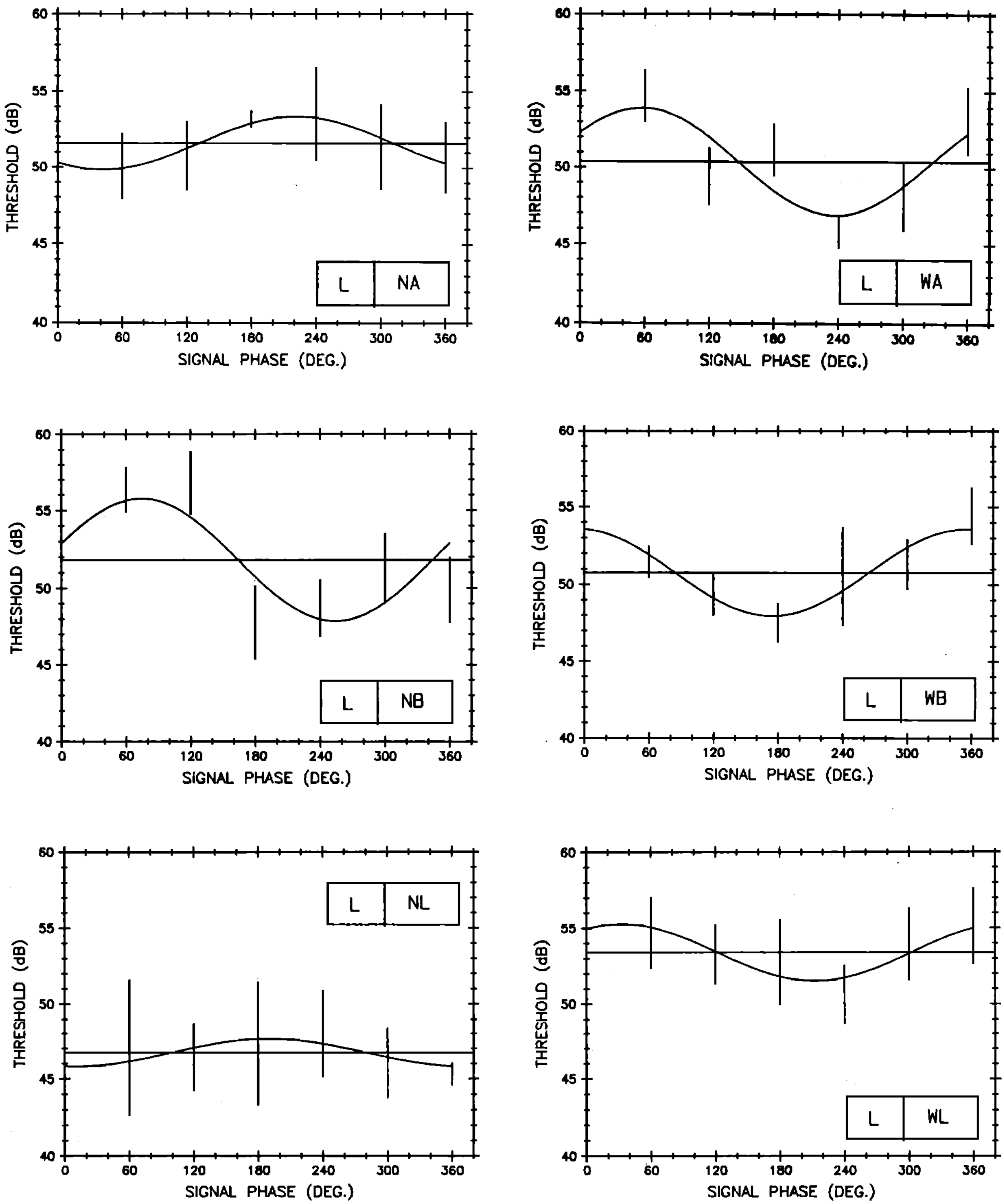

FIG. 3. Masked thresholds as functions of signal phase for subject L for six different noisy maskers. The vertical lines are centered on the average threshold for each signal phase, and their lengths are two standard deviations. The horizontal line is the average threshold over signal phases (parameter c), and the smooth curve is the best-fit sine function.

the top of the list. It was usually possible to do all 24 runs for a given noise (four runs at each of six signal phases) in a single 2-h session. The ordering of the noises among the sessions was different for each listener.

C. Subjects

Six subjects participated in all of the experiments. Subjects B, J, and M were males; I, L, and T were females. Subjects ranged in age from 16 to 34 years, and all of them had normal hearing according to their own reports. B and J had recent experience in a variety of psychoacoustical experiments; the other listeners had never been subjects before. The authors were not subjects.

III. RESULTS

For each signal phase and each noise there were four experimental runs providing four estimates of the threshold. The average of these four was a final threshold, and the standard deviation ($N-1=3$ weight) provided an estimate of the error.[3] Typical data, for listener L, are shown in Fig. 3. The threshold values show the effects of signal phase and of the different noises. To parametrize the average, and the variation due to signal phase for a given noise, we fitted the data with a function that is the sum of a constant plus a sine curve:

$$F(\phi_j) = c + a\sin(\phi_j + \phi_0). \quad (11)$$

Appendix D shows how the three free parameters, c, a, and ϕ_0, were chosen to provide the best fit, in a least-squares sense, to the data. Because the average of the sine function over the six phases is zero, the constant c is a best estimate of the threshold for a given noise. Values of c, a, and ϕ_0 which best fit the masking data are given in Table III.

The choice of a sine function to fit the phase dependence of the data cannot be justified except as a simple way to describe the fact that the signal is easier to detect for some phase angles than for others, by using a function that is both smooth and periodic. The phase dependence of the threshold does, however, presumably reflect some form of constructive and destructive interference between signal and noise (Pfafflin and Mathews, 1966), and one expects that the phase parameter ϕ_0 for a given noise should be the same for different listeners, as was true for the listeners in the experiments of Hanna and Robinson (1985). Table III, however, shows that the phase parameters ϕ_0 differ considerably for different listeners, as in the experiments of Dolan *et al.* (1981). The spread in ϕ_0 is always considerably greater than the experimental spacing of $\Delta\phi = 60°$ except for noise WA where it is 70°.

Of course, one cannot attach much importance to the phase parameter in the sine curve fit if the sine curve amplitude is less than the average error in the data. Figure 3 for noise NL is an example of such a case. If the best-fit phase parameters are considered to be meaningful only if amplitude a exceeds twice the average standard deviation (the lengths of the error bars in Fig. 3), then many phase parameters are eliminated from consideration, but those which remain show a rather smaller spread across listeners: for NB a spread of 35° ($N=5$), for WA a spread of 25° ($N=2$), but for WB a spread of 200° ($N=3$).

An alternative approach is to consider the best-fit phase

TABLE III. Values of c (average masked threshold), a (sine amplitude parameter), and ϕ_0 (phase parameter) in the best fit to the threshold data for six subjects and six noises. Values of c and a are in dB; values of ϕ_0 are in degrees. Some values of ϕ_0 were shifted by − 360° to minimize the overall spread among subjects for a given noise.

Subject	c	a	ϕ_0	c	a	ϕ_0
	Noise NA			Noise WA		
B	50.7	0.7	−4	48.8	1.7	7
I	50.1	2.3	−135	51.1	2.4	−36
J	47.6	1.9	71	46.9	3.7	3
L	51.6	1.8	−132	50.4	3.5	33
M	53.6	2.9	−104	52.5	4.3	11
T	46.9	2.7	−132	49.4	4.1	−16
	Noise NB			Noise WB		
B	46.4	4.8	20	47.2	1.8	209
I	50.4	5.0	−14	48.9	2.5	209
J	44.4	3.1	118	43.8	4.2	297
L	51.8	4.0	16	50.8	2.8	96
M	52.1	4.7	−9	52.1	2.1	150
T	48.1	4.5	−7	46.9	2.1	139
	Noise NL			Noise WL		
B	42.5	1.8	55	45.2	2.6	86
I	44.8	1.6	138	48.4	2.9	113
J	39.5	5.6	267	47.0	1.6	−41
L	46.7	0.9	261	53.4	1.9	58
M	54.2	1.3	128	54.4	1.7	101
T	42.9	2.1	142	46.3	0.9	149

parameters to be meaningful only if the sine curve gives a good fit to the data, falling within the error bars, as for NA and WB in Fig. 3 but not for NB or WA. This procedure results in a spread across listeners for ϕ_0, which is less than 60° for four of the six noises (total $N = 21$). This latter criterion, therefore, appears to be more successful than the first. It eliminates fewer data, and those data which remain usually have similar values of ϕ_0 for different listeners. We conclude that there is probably a reproducible relationship between properties of the masking noise and the functional dependence of signal thresholds on signal phase. In some cases the dependence can be fitted with a simple sine function, but in other cases it cannot.

It is interesting to ask whether the dependence of threshold on signal phase is systematically related to the noise power fluctuations. Figure 3 suggests that the amplitude parameter a may be unusually small for low-noise noise maskers. This behavior, however, was not generally observed. For example, for listener J the *largest* amplitude parameter occurred for the narrow-band low-noise noise.

The value of the amplitude parameter a depends upon the particular noise waveform. For every subject, a is larger for noise NB than for noise NA. The average for NB is 4.4 dB (s.d. = 0.7), about twice as large as the average for NA which is 2.1 dB (s.d. = 0.8). In the terms of Hanna and Robinson (1985), noise NB would be described by a longer vector than noise NA.

Of primary importance in this study are the average thresholds and their dependences on noise type. Figure 4 shows the thresholds averaged over signal phase angles, for six listeners and three narrow-band noises. Figure 5 shows the corresponding thresholds for the wideband noises. The estimates of error shown in the figures were derived by first calculating the variance for each signal phase ($N - 1 = 3$ weight) and then averaging the six variances to estimate the variance for a given subject and noise. Therefore, this error statistic includes variation due to inconsistent listener performance but excludes the variation attributable to the different signal phases. The error bars in Figs. 4 and 5 show ± 1 standard deviation limits.

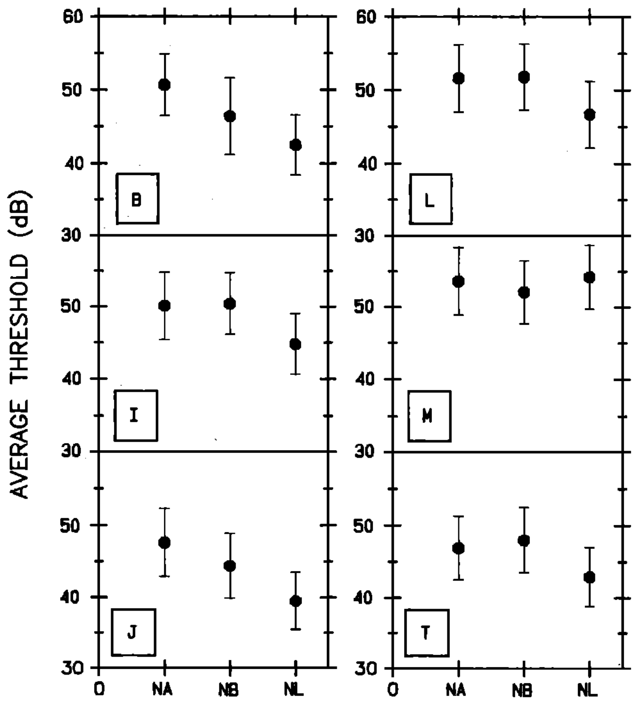

FIG. 4. Masked threshold, averaged over six signal phases, for six listeners for the three narrow-band masking noises.

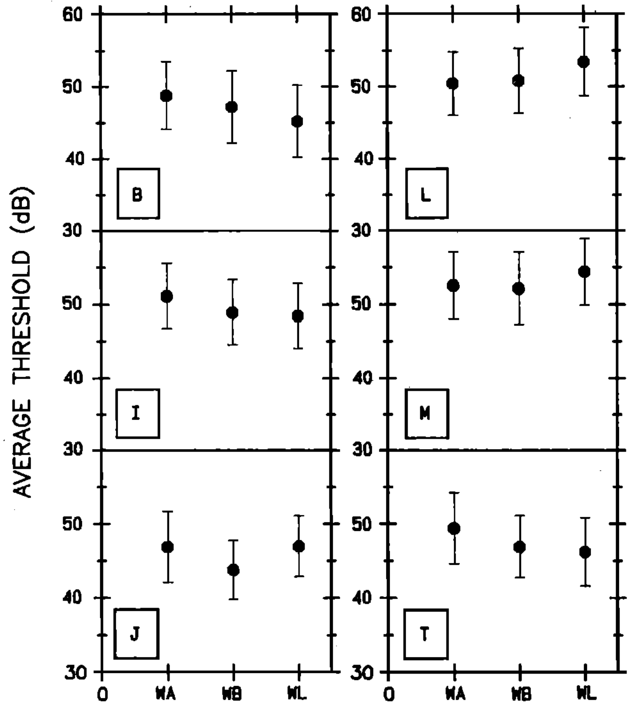

FIG. 5. Same as Fig. 4 but for the three wideband masking noises.

Table IV gives thresholds for each noise averaged over five or six listeners. (Listener M is omitted when the average

TABLE IV. Masked thresholds in dB SPL from column c of Table III averaged over six subjects or over five subjects (subject M omitted). Values in parentheses are standard deviations ($N - 1$ weight) across subjects.

	Noise NA	Noise WA
5s	49.4 (2.0)	49.3 (1.6)
6s	50.0 (2.5)	49.9 (1.9)
	Noise NB	**Noise WB**
5s	48.2 (3.0)	47.5 (2.6)
6s	48.9 (3.1)	48.3 (3.0)
	Noise NL	**Noise WL**
5s	43.3 (2.7)	48.1 (3.2)
6s	45.1 (5.1)	49.1 (3.9)
	Noises NA and NB = N	**Noises WA and WB = W**
5s	48.8 (2.5)	48.4 (2.2)
6s	49.5 (2.8)	49.1 (2.5)

is over five.) The average thresholds for narrow bands ($N =$ NA and NB combined) and for wide bands ($W =$ WA and WB combined) are our best estimates for the effect of bandwidth on masked threshold. The spectral density of N was 5 dB greater than that for W. The noise power in a critical band filter, 160 Hz wide, was 3 dB greater for N than for W. Therefore, one might have expected a larger average threshold for N than for W. Table IV, however, shows that this difference was only 0.4 dB. Thresholds for N were larger than thresholds for W for five of the six listeners, but the largest difference for any listener was only 1.1 dB. We have no explanation for this null result.

IV. DISCUSSION

The purpose of the masking experiment was to study the effect of noise fluctuations on masking, in particular, the effect of a low-noise noise masker. We consider the cases of narrow-band noise and wideband noise separately. Figure 4 shows that for the narrow-band noises, thresholds are smaller for low-noise noise (NL) than for the noises with appreciable power fluctuations (NA and NB). There is one exception to this rule, listener M, for whom the reverse was true. Listener M, however, was the least successful of the subjects in that, for every noise, his average threshold was higher than that for any other subject. We are inclined to regard the performance of listener M as anomalous. The difference between the random-phase noise thresholds (NA and NB) and the low-noise noise thresholds (NL) averaged over the six subjects was 4.4 dB. This difference is close to the average standard deviation of 4.5 dB. Averaged over five listeners, excluding M, the difference between the random-phase noise threshold and the low-noise noise threshold was 5.5 dB.

For the wideband noises there were three listeners for whom the thresholds for low-noise noise (WL) were less than for random-phase noise (WA and WB), and there were three listeners for whom the reverse was true. The threshold difference between random-phase noise and low-noise noise, averaged over six listeners, was -0.1 dB, a completely negligible difference.

V. CONCLUSION

We performed masking experiments in which the task was to detect a 1000-Hz sine tone in a band of noise. There were two kinds of equal-amplitude noise bands, one made with components having random phases, the other made according to the low-noise noise algorithm. We found that the relative masking efficiency of these two kinds of noise depends upon the noise bandwidth.

Narrow-band noise maskers made with low-noise noise are less effective in masking a sine tone than are narrow-band maskers made with random-phase noise. The difference can be attributed to power fluctuations, present in the random-phase noise but not appreciably present in the low-noise noise. This difference amounts to about 5 dB of masking. This result agrees with the difference of about 5 dB found by Margolis and Small (1974) for a signal in the center of their FM band. It can be related to the question of internal versus external noise. If the total variance limiting detection is the sum of the stimulus variance and the variance associated with internal noise (de Boer, 1966), then a difference of 5 dB when noise fluctuations are removed means that the ratio of stimulus variance to internal variance is 9 to 1.

There are several possible explanations for the reduced effectiveness of a low-noise noise masker within the context of energy detection theory. The first begins by noting that the low-noise noise waveform distribution is bimodal, thus resembling the distribution which occurs when a sine pedestal is added to a Gaussian noise masker. It is known that adding such a pedestal improves signal detectability (Green, 1960). This analogy, however, is probably misleading. The effect of the pedestal seems to be well explained by the model of Pfafflin and Mathews (1962), and it occurs because adding the pedestal creates a masker which is correlated with the signal. In our experiment, by contrast, the low-noise noise masker is no more correlated with the signal than is the random-phase noise masker.

A better explanation for our experiment is that for stimulus intervals as long as 500 ms, the listeners do not make their responses on the basis of a single observation. Instead they make multiple observations and reach a final decision based on some combination of these observations. In the case of low-noise noise, the energy within any subinterval of the stimulus is very nearly the same as in any other subinterval. In the case of random-phase noise, the energy in different subintervals is different because of the noise power fluctuations. If the signal can be spectrally resolved from the masker then fluctuations where the noise is a minimum might make the signal more salient and, depending upon the combination rule for subinterval observations, might lead to lower detection thresholds. If the signal cannot be spectrally resolved from the masker, as in our experiment, then energy variations among different subintervals must lead to higher detection thresholds. Such a view is consistent with our observations. If this explanation is correct then one would predict that in frozen-noise masking experiments, the difference between thresholds for random-phase noise and low-noise noise should become smaller as the signal duration is decreased.

For wideband maskers, by contrast, there is no difference in the masking effectiveness of low-noise noise and random-phase noise. The null result for wideband maskers can be understood in terms of the critical-band model of masking. The minimal power fluctuation in low-noise noise results from the synthesis of all the components. If however, some of these components are rejected by an auditory filter then the noise no longer has minimal power fluctuation. To test this idea we resynthesized the low-noise noise masker called WL using only those components within a band 160 Hz wide and centered on 1000 Hz. This band corresponds to estimates of the critical bandwidth at 1000 Hz, based upon masking studies (Zwicker, 1961, Scharf, 1970). The crest factor and fourth moment are shown by the symbol X in Fig. 2. Although the power fluctuations in this filtered noise are not as large as those for typical random-phase noise, the fluctuations are considerably greater than those in the full band of low-noise noise (symbol L). In a qualitative way this increase in fluctuation accounts for the fact that masked thresholds are the same for random-phase noise maskers and for low-noise noise maskers when both are wideband.

Several predictions follow from this critical-band model.

(1) The masking effectiveness of a wideband masker should be reduced if power fluctuations within a critical band, centered on the signal, are reduced.

(2) Although a random-telegraph-noise masker has the lowest possible crest factor and fourth moment (both are equal to one), it should not produce an anomalously low threshold because fluctuations are reintroduced by the critical-band filter.

(3) Masking by peak-clipped noise may result in an anomalously low threshold. The original noise, prior to clipping, must be in a narrow band around the signal frequency. The clipping operation should be hard and symmetrical about the waveform zero. This produces a masker that has little power fluctuation within a critical band, and the harmonic distortion (third order and higher odd order) is at frequencies well removed from the signal frequency.

Finally, we note that the critical bandwidths found in recent masking experiments near 1000 Hz are somewhat less than the value of 160 Hz used in our calculations (Moore and Glasberg, 1983). If so, then point X in Fig. 2 would move closer to the cluster of points for random-phase noise. It is conceivable that the critical bandwidth is even smaller than 100 Hz. In that case, our narrow-band experiment, using noise with a bandwidth of 100 Hz, was not optimum. A greater difference (greater than 5 dB) between masking by random-phase noise and masking by low-noise noise would be obtained if the bandwidth were smaller.

ACKNOWLEDGMENTS

We are grateful to Dr. D. E. Robinson, Dr. R. H. Gilkey, Dr. S. Buus, and Dr. D. M. Green for useful discussions. Dan Lin helped with the data analysis.

APPENDIX A: ENSEMBLE-AVERAGED MOMENTS

This Appendix calculates ensemble-averaged moments of the waveform in Eq. (1) for two cases, equal amplitudes and Rayleigh-distributed amplitudes. The results appear in the text as Eqs. (4a) and (4b).

The average second moment, or power, is given by

$$\langle \overline{x^2} \rangle = \left\langle \frac{1}{N} \sum_{n,n'} X_n X_{n'} \; \overline{c_n c_{n'}} \right\rangle, \tag{A1}$$

where $c_n = \cos(2\pi nt/T + \phi_n)$, and X_n is the amplitude. There are N^2 term in the sum.

The time-averaged value of $c_n c_n$, is $\frac{1}{2}\delta_{n,n'}$ so that (A1) becomes

$$\langle \overline{x^2} \rangle = \frac{1}{2N} \sum_n \langle X_n^2 \rangle = \frac{1}{2} \langle X^2 \rangle, \tag{A2}$$

i.e., the ensemble-averaged power is proportional to the ensemble average of the square of a component amplitude.

Similarly, the averaged fourth moment is given by

$$\langle \overline{x^4} \rangle = \left\langle \frac{1}{N^2} \sum_{n_1 n_2 n_3 n_4} X_{n_1} X_{n_2} X_{n_3} X_{n_4} \; \overline{c_{n_1} c_{n_2} c_{n_3} c_{n_4}} \right\rangle. \tag{A3}$$

The time average is zero unless all the n indices are the same or unless they are equal in pairs. There are three ways for them to be equal in pairs. Then, because $\overline{c_n^4} = \frac{3}{8}$, and $(\overline{c_n^2})^2 = \frac{1}{4}$, we have

$$\begin{aligned}\langle \overline{x^4} \rangle &= \left\langle \frac{3}{N^2} \left(\frac{1}{4} \sum_{\substack{n,n' \\ n \neq n'}} X_n^2 X_{n'}^2 + \frac{1}{8} \sum_n X_n^4 \right) \right\rangle \\ &= \left\langle \frac{3}{N^2} \left(\frac{1}{4} \sum_{n,n'} X_n^2 X_{n'}^2 - \frac{1}{8} \sum_n X_n^4 \right) \right\rangle \\ &= \left\langle \frac{3}{N^2} \left[\frac{1}{4} \left(\sum_n X_n^2 \right)^2 - \frac{1}{8} \sum_n X_n^4 \right] \right\rangle. \end{aligned} \tag{A4}$$

It is not hard to calculate the above ensemble average if the distribution of amplitude X is known. However, it is more pertinent find the average *relative* fourth moment $\langle W \rangle = \langle \overline{x^4}/(\overline{x^2})^2 \rangle$. Because the first term in the last line of Eq. (A4) is proportional to $(\overline{x^2})^2$, the ensemble average has the simple form

$$\langle W \rangle = 3\left(1 - \left\langle \frac{\Sigma_n X_n^4}{2(\Sigma_n X_n^2)^2} \right\rangle\right). \tag{A5}$$

Thus $\langle W \rangle$ involves the ensemble average of the ratio of two sums. For the case of equal-amplitude noise, the ensemble average is trivial and

$$\langle W \rangle = 3[1 - 1/(2N)]. \tag{A6}$$

For the case of Rayleigh-distributed noise, there is a simple approximate solution and an exact solution. The approximate solution applies when there are a large number of terms in each sum in Eq. (A5) (large N). Then each sum approximates its ensemble average because the distribution of amplitudes is independent of index n.

For Rayleigh-distributed noise, the amplitudes have the probability density

$$R(X) = Xe^{-X^2/2}. \tag{A7}$$

The even moments of this distribution are given by

$$\langle X^{2\mu} \rangle = \int_0^\infty dX\, X^{2\mu} R(X) = \frac{\mu!}{2^\mu}. \tag{A8}$$

Therefore, $\langle X^2 \rangle = \frac{1}{2}$ and $\langle X^4 \rangle = \frac{1}{2}$, and in the large-$N$ approximation we find

$$\langle W \rangle \approx 3 - 3/N. \tag{A9}$$

The exact form of $\langle W \rangle$ for Rayleigh-distributed noise can be obtained as follows: (1) Introduce the identity

$$\begin{aligned} 1 &= \int_0^\infty dz\, \delta\left(z - \sum X_n^2\right) \\ &= \frac{1}{2\pi} \int_0^\infty dz \int_{-\infty}^\infty dq \exp\left[i\left(z - \sum X_n^2\right)\right] \end{aligned} \tag{A10}$$

into Eq. (A5); (2) perform the integrals over $X_1,...,X_n$ which are implicit in Eq. (A5); (3) perform the integral over q by contour integration; (4) perform the integral over z, which is elementary. The result is

$$\langle W \rangle = 3 - 3/(N+1). \tag{A11}$$

The approximation in Eq. (A9) agrees with this exact result to within 1% for $N \geqslant 10$.

APPENDIX B: WAVEFORM AND ENVELOPE FOURTH MOMENTS IN THE NARROW-BAND LIMIT

Equation (9) says that for narrow-band signals the waveform fourth moment and the envelope fourth moment are related by a simple factor. This can be proved by writing an expression for the waveform in a way which makes the envelope apparent.

The first step is to extract the mean frequency by redefining the summation variables. Taking N to be odd for simplicity, we rewrite Eq. (1) as

$$x(t) = \frac{1}{\sqrt{N}} \sum_{j=-J}^{J} X_j \cos\left(\frac{2\pi(j+M)t}{T} + \phi_j\right), \quad \text{(B1)}$$

where the mean frequency is given by

$$M/T = (N_1 + N_2)/(2T) \quad \text{(B2)}$$

and

$$J = (N_2 - N_1)/2. \quad \text{(B3)}$$

Then by expanding the cosine in Eq. (B1) we have

$$x(t) = F(t)\cos(2\pi Mt/T) - G(t)\sin(2\pi Mt/T), \quad \text{(B4)}$$

where

$$F(t) = \frac{1}{\sqrt{N}} \sum_{j=-J}^{J} X_j \cos(2\pi jt + \phi_j) \quad \text{(B5)}$$

and

$$G(t) = \frac{1}{\sqrt{N}} \sum_{j=-J}^{J} X_j \sin(2\pi jt + \phi_j), \quad \text{(B6)}$$

and we have relabeled $\{X_n\}$ and $\{\phi_n\}$. The envelope power can be written in terms of the variables F and G:

$$E^2 = F^2 + G^2. \quad \text{(B7)}$$

The fourth power of the waveform, obtained from Eq. (B1), is given by

$$\begin{aligned} x^4(t) = {} & F^4[\tfrac{3}{8} + \tfrac{1}{2}\cos(2\psi) + \tfrac{1}{8}\cos(4\psi)] \\ & - 4F^3G\,[\tfrac{1}{4}\sin(2\psi) + \tfrac{1}{8}\sin(4\psi)] \\ & + 6F^2G^2[\tfrac{1}{8} - \tfrac{1}{8}\cos(4\psi)] \\ & - 4FG^3[\tfrac{1}{4}\sin(2\psi) - \tfrac{1}{8}\sin(4\psi)] \\ & + G^4[\tfrac{3}{8} - \tfrac{1}{2}\cos(2\psi) + \tfrac{1}{8}\cos(4\psi)], \end{aligned} \quad \text{(B8)}$$

where we have defined $\psi = 2\pi Mt/T$.

Each term in Eq. (B8) is the product of an envelope factor, for example, F^4, and a factor in square brackets which comes from the mean frequency oscillations. The basis of our proof is that if these two factors contain no frequencies in common, then their product makes no contribution to the time integral which is $\overline{x^4}$. The highest frequency in F or G is J/T; thus the highest frequency in envelope factors F^4, F^3G, etc., is $4J/T$. The lowest frequency in the square brackets in Eq. (B8) (apart from the constant terms) is $2M/T$. Thus if $J < M/2$ then the oscillating factors in Eq. (B8) have no frequencies in common. In that case, only the constant term in any square bracket contributes to the time integral and

$$\overline{x^4} = \tfrac{3}{8}\,\overline{F^4} + \tfrac{6}{8}\,\overline{F^2G^2} + \tfrac{3}{8}\,\overline{G^4}$$

$$= \tfrac{3}{8}\,\overline{(F^2+G^2)^2} = \tfrac{3}{8}\,\overline{E^4}. \quad \text{(B9)}$$

The narrow-band condition $J < M/2$ is actually a rather weak one: It requires only that the total bandwidth be smaller than the mean frequency, or, equivalently,

$$N_2 < 3N_1. \quad \text{(B10)}$$

Equation (B9) says that for this narrow-band condition, the fourth moment of the waveform is $\frac{3}{8}$ times the fourth moment of the envelope, or, in normalized form,

$$\overline{E^4}/\overline{(E^2)}^2 = \tfrac{4}{3}\,\overline{x^4}/\overline{(x^2)}^2. \quad \text{(B11)}$$

APPENDIX C: THE DISTRIBUTION OF CREST FACTORS

Table I shows that for narrow bands, the crest factor depends only on the number of components N and not upon the center frequency. This can be understood as follows: When the band of frequencies is narrow compared to the mean frequency, the cosine and sine factors in Eq. (B4) vary rapidly compared to factors F and G. Hence, for any given time t there is a domain of nearby values of t over which F and G are essentially constant while the oscillations at the mean frequency execute an entire cycle. The maximum and minimum values over this cycle are $\pm\sqrt{(F^2+G^2)}$, namely, the envelope values. Hence, in this limit the crest factor is independent of the band center frequency because the envelope is independent of the band center frequency.

Table I also shows that crest factors for different waveforms tend to be rather similar in value, i.e., that they are narrowly distributed about the mean value and that the mean value itself increases only slowly with the number of components N. This can be understood from the following heuristic argument in which we imagine $x(t)$ at different times t to be independent random variables.

We consider only the narrow-band case, so that the crest factor is essentially given by the peak value of the envelope. Then, if the number of components N is large, the quantities F and G in Eqs. (B5) and (B6) are, to a good approximation, normally distributed. Then the envelope E is Rayleigh distributed. If all the values of X_n are unity, then F and G have variance $\frac{1}{2}$, and the probability density for envelope E is given by

$$\frac{dQ}{dE} = 2Ee^{-E^2}. \quad \text{(C1)}$$

The cumulative distribution is

$$Q(E) = \int_0^E dQ = 1 - \exp(-E^2). \quad \text{(C2)}$$

The probability density for the largest of K independent samples from the distribution of Eq. (C1), $z = \max[E_1, E_2, \ldots, E_K]$ is f, given by

$$f(z) = \frac{d}{dz} Q^K(z), \quad \text{(C3)}$$

because Q is the probability that a single E will be less than z. Computing moments by numerical integration, we find a mean peak value of

$$\langle z\rangle \approx [\ln(1.7K)]^{1/2} \quad \text{(C4)}$$

and a standard deviation of

$$(\langle z^2\rangle - \langle z\rangle^2)^{1/2} \approx 0.648/[\ln(3.4K)]^{1/2}. \quad \text{(C5)}$$

These approximations are accurate to 1% for all K greater than 5.

The median can be calculated analytically from Q^K. It is given by

$$z_{\text{median}} = \sqrt{\ln[K/\ln(2)]}. \tag{C6}$$

By differentiating f we find the most probable value, where the maximum of density f occurs. It is given by

$$z_{\text{mode}} = [\ln(K)]^{1/2}. \tag{C7}$$

The middle 68% range of the distribution for peak values of x is defined by upper and lower limits,

$$z_{\text{lower}} \approx [\ln(0.546\text{K})]^{1/2}, \quad z_{\text{upper}} \approx [\ln(5.736\text{K})]^{1/2}. \tag{C8}$$

For large K, we can write

$$z_{\text{upper}} - z_{\text{lower}} \approx 0.588/[\ln(1.18\text{K})]^{1/2}, \tag{C9}$$

which is similar to Eq. (C5), but not identical because the distribution is not Gaussian. The results of Eqs. (C4)–(C9) apply to the peak values of the waveform. Because the crest factor is normalized by the rms value, the above values for z must be multiplied by $\sqrt{2}$ to obtain the corresponding values for the crest factor. The results also depend upon the number of components N, but that dependence does not appear in Eqs. (C4)–(C9) because of the initial normalization of the waveform in Eq. (1). The dependence on N can be made explicit by dropping the normalization in Eq. (1), whereupon one must simply multiply all the results in Eqs. (C4)–(C9) by $\sqrt{N}$, and not by $\sqrt{2}$.

To use the above formulas we must estimate the number of independent samples K. It is evident that in order to track the waveform envelope with some accuracy, the rate of taking samples, K/T, must be somewhat larger than the maximum frequency in the envelope, which is $N/2T$. Therefore, a lower limit for K/T should be about N/T. If the noise is not periodic, then the value of K itself depends upon the duration of the noise record (Rice, 1954; Beranek, 1954), but for the noise described by Eq. (1), which ultimately repeats, there is an upper limit to K. Because a sample rate of N/T approximates the envelope on the average, the upper bound for K cannot be much larger than N, the number of sine components. A value between $2N$ and $3N$ appears to be a reasonable choice. Fortunately, it is not necessary to specify K precisely because the statistical properties [Eq. (C4)–(C9)] depend only weakly on K. A value of $K = \frac{8}{3}N$, accounts for the median and mean crest factors and for the s.d. in Table I.

APPENDIX D: FITTING A SINE CURVE TO DATA

The fitting procedure begins with a set of data points $\{L_p(\phi_n)\}$ $(1 \leqslant p \leqslant P)$ which represent threshold levels as a function of signal phases ϕ_n $(1 \leqslant n \leqslant N)$.

The number of data points, P, must, of course, be at least as large as the number of phases, N. Of practical interest is the case where P is several times as large as N. The goal is to fit the data points with the best function of the form

$$F_n = c + a\sin(\phi_n + \phi_0). \tag{D1}$$

This requires that optimum values of c, a, and ϕ_0 be chosen to minimize the error,

$$E = \sum_{p=1}^{P} [L_p(\phi_n) - F_n]^2. \tag{D2}$$

Such a minimization can be done in general with numerical search procedures. This Appendix shows that the best-fitting parameters can be obtained directly if three conditions are met:

(1) The number of data points L_p at each value of ϕ_n must be the same, namely, P/N.

(2) There must be more than two values of ϕ_n (i.e., $N \geqslant 3$).

(3) The N values of ϕ_n must divide the circle into equal parts (i.e., $\phi_n - \phi_{n-1} = 360/N$ degrees; $n = 2$ to N).

If these three requirements are met then the following three important properties follow, for all values of ϕ_0.

P1. Zero mean:

$$\sum_{n=1}^{N} \sin(\phi_n + \phi_0) = 0, \tag{D3}$$

P2. Normality:

$$\sum_{n=1}^{N} \sin^2(\phi_n + \phi_0) = \frac{N}{2} \tag{D4}$$

P3. Orthogonality:

$$\sum_{n=1}^{N} \sin(\phi_n + \phi_0)\cos(\phi_n + \phi_0) = 0. \tag{D5}$$

Minimizing E with respect to c and using property P1 gives the constant term

$$c = \frac{1}{P}\sum_{p=1}^{P} L_p(\phi_n), \tag{D6}$$

i.e., the constant term is the average threshold.

Minimizing E with respect to amplitude a, and using properties P1 and P2 gives the amplitude

$$a = \frac{2}{P}\sum_{p=1}^{P} L_p(\phi_n)\sin(\phi_n + \phi_0). \tag{D7}$$

Minimizing E with respect to ϕ_0 and using properties P1 and P3 give

$$\sum_{p=1}^{P} L_p(\phi_n)\cos(\phi_n + \phi_0) = 0, \tag{D8}$$

or

$$\tan(\phi_0) = \frac{\Sigma_{p=1}^{P} L_p(\phi_n)\cos(\phi_n)}{\Sigma_{p=1}^{P} L_p(\phi_n)\sin(\phi_n)}. \tag{D9}$$

Equation (D9) can be used to solve for the best phase parameter ϕ_0 except for an ambiguity of 180° in the arctangent function. The two possible choices of ϕ_0 correspond to different signs for $\cos(\phi_0)$. Differentiating E twice with respect to ϕ_0 shows that the error E is minimized when $\cos(\phi_0)$ has the same sign as the numerator on the right-hand side of Eq. (D9). The other choice of ϕ_0 actually corresponds to the worst-fitting phase parameter. The procedure used in the text above was to solve for c using Eq. (D6) and then for ϕ_0 using Eq. (D9). That value of ϕ_0, used in Eq. (D7), gave a value for amplitude a.

[1]Smith *et al.* (1986) have recently done masking experiments employing Schroeder's algorithm for minimum crest factor noise. To create a "reasonable" masker they used sine components with a course spacing in frequency so that the repetition rate of the frequency sweep was very rapid, having a period shorter than auditory integration times.

[2]The distributions of instantaneous values of the waveforms of the random-phase noises and the low-noise noises were also different. These distributions were measured by sampling the electrical signal sent to the headphones at a sample rate of 65 455 Hz and with a 16-bit resolution. The distributions for the random-phase noises were peaked around zero and resembled the normal distribution. The distributions for the low-noise noises NL and WL were bimodal and resembled the distribution for a sine wave except that the discontinuities at the extremes of the sine wave distribution were replaced by tails.

[3]An estimate of improvement with experience was obtained by comparing the first two runs for a given subject (6) and noise (6) and phase angle (6) with the second two runs. The average improvement was 0.93 dB ($N = 216$).

Ahumada, A., and Lovell, J. (**1971**). "Stimulus features in signal detection," J. Acoust. Soc. Am. **53**, 1751–1756.

Ahumada, A., Marken, R., and Sandusky, A. (**1975**). "Time and frequency analyses of auditory signal detection," J. Acoust. Soc. Am. **57**, 385–390.

Beranek, L. L. (**1954**). *Acoustics Measurements* (Wiley, New York).

Bos, C. E., and de Boer E. (**1966**). "Masking and discrimination," J. Acoust. Soc. Am. **39**, 708–715.

de Boer, E. (**1966**). "Intensity discrimination of fluctuating signals," J. Acoust. Soc. Am. **40**, 552–560.

Dolan, T. R., Hirsh, I. R., and Yost, W. A. (**1981**). "Detection of binaural signals partially masked by a reproducible noise masker: The influence of the phase angle alpha," J. Acoust. Soc. Am. **70**, 886–888.

Feth, L. L. (**1970**). "Detection in non-Gaussian noise," J. Acoust. Soc. Am. **48**, 77 (abstract).

Gilkey, R. H. (**1981**)."Molecular psychophysics and models of auditory signal detectability," Ph.D. thesis, Indiana University (unpublished).

Gilkey, R. H., Robinson, D. E., and Hanna, T. E. (**1985**). "Effects of masker waveform and signal-to-masker phase relation on diotic and dichotic masking by reproducible noise," J. Acoust. Soc. Am. **78**, 1207–1219.

Green, D. M. (**1960**). "Psychoacoustics and detection theory," J. Acoust. Soc. Am. **32**, 1189–1203.

Green, D. M. (**1964**). "Consistency of auditory detection judgments," Psychol. Rev. **71**, 392–407.

Green, D. M., and Swets, J. A. (**1966**). *Signal Detection and Psychophysics* (Wiley, New York).

Hanna, T. E., and Robinson, D. E. (**1985**). "Phase effects for a sine wave masked by reproducible noise," J. Acoust. Soc. Am. **77**, 1129–1140.

Hartmann, W. M. (**1987**). "Temporal fluctuations and the discrimination of spectrally dense signals by human listeners," in *Auditory Processing of Complex Sound*, edited by W. A. Yost and C. S. Watson (Erlbaum, Hillsdale, NJ).

Hartmann, W. M., McAdams, S., Gerzso, A., and Boulez, P. (**1986**). "Discrimination of spectral density," J. Acoust. Soc. Am. **79**, 1915–1925.

Jeffress, L. A. (**1968**). "Mathematical and electrical models of signal detection," J. Acoust. Soc. Am. **44**, 187–203.

Levitt, H. (**1971**). "Transformed up–down methods in psychoacoustics," J. Acoust. Soc. Am. **49**, 467–477.

Margolis, R. H., and Small, A. M., Jr. (**1974**). "Masking with narrow-band FM noise," J. Acoust. Soc. Am. **56**, 692–694.

Moore, B. C. J., and Glasberg, B. R. (**1983**). "Suggested formulas for calculating auditory-filter bandwidths and excitation patterns," J. Acoust. Soc. Am. **74**, 750–753.

Pfafflin, S. M. (**1968**). "Detection of auditory signals in restricted sets of reproducible noise," J. Acoust. Soc. Am. **43**, 487–490.

Pfafflin, S. M., and Mathews, M. V. (**1962**). "Energy detection model for monaural auditory detection," J. Acoust. Soc. Am. **34**, 1842–1853.

Pfafflin, S. M., and Mathews, M. V. (**1966**). "Detection of auditory signals in reproducible noise," J. Acoust. Soc. Am. **39**, 340–345.

Pumplin, J. (**1985**). "Low-noise noise," J. Acoust. Soc. Am. **78**, 100–104.

Raab, D. H., and Goldberg, I. A. (**1975**). "Auditory intensity discrimination with bursts of reproducible noise," J. Acoust. Soc. Am. **57**, 437–447.

Rice, S. O. (**1954**). "Mathematical analysis of random noise," in *Selected Papers on Noise and Stochastic Processes*, edited by Nelson Wax (Dover, New York).

Scharf, B. (**1970**). "Critical bands," in *Foundations of Modern Auditory Theory*, edited by J. Tobias (Academic, New York), Vol. 1, pp. 159–202.

Schroeder, M. R. (**1970**). "Synthesis of low-peak-factor signals and binary sequences with low autocorrelation," IEEE Trans. Inf. Theory **IT-16**, 85–89.

Smith, B. K., Sieben, U. K., Kohlrausch, A., and Schroeder, M. R. (**1986**). "Phase effects in masking related to dispersion in the inner ear," J. Acoust. Soc. Am. **80**, 1631–1637.

Sorkin, R. D., Woods, D. D., and Boggs, G. J. (**1979**). "Signal detection in computer-synthesized noise," J. Acoust. Soc. Am. **66**, 1351–1355.

Zwicker, E. (**1961**). "Subdivision of the audible frequency range into critical bands (Frequenzgruppen)," J. Acoust. Soc. Am. **33**, 248.